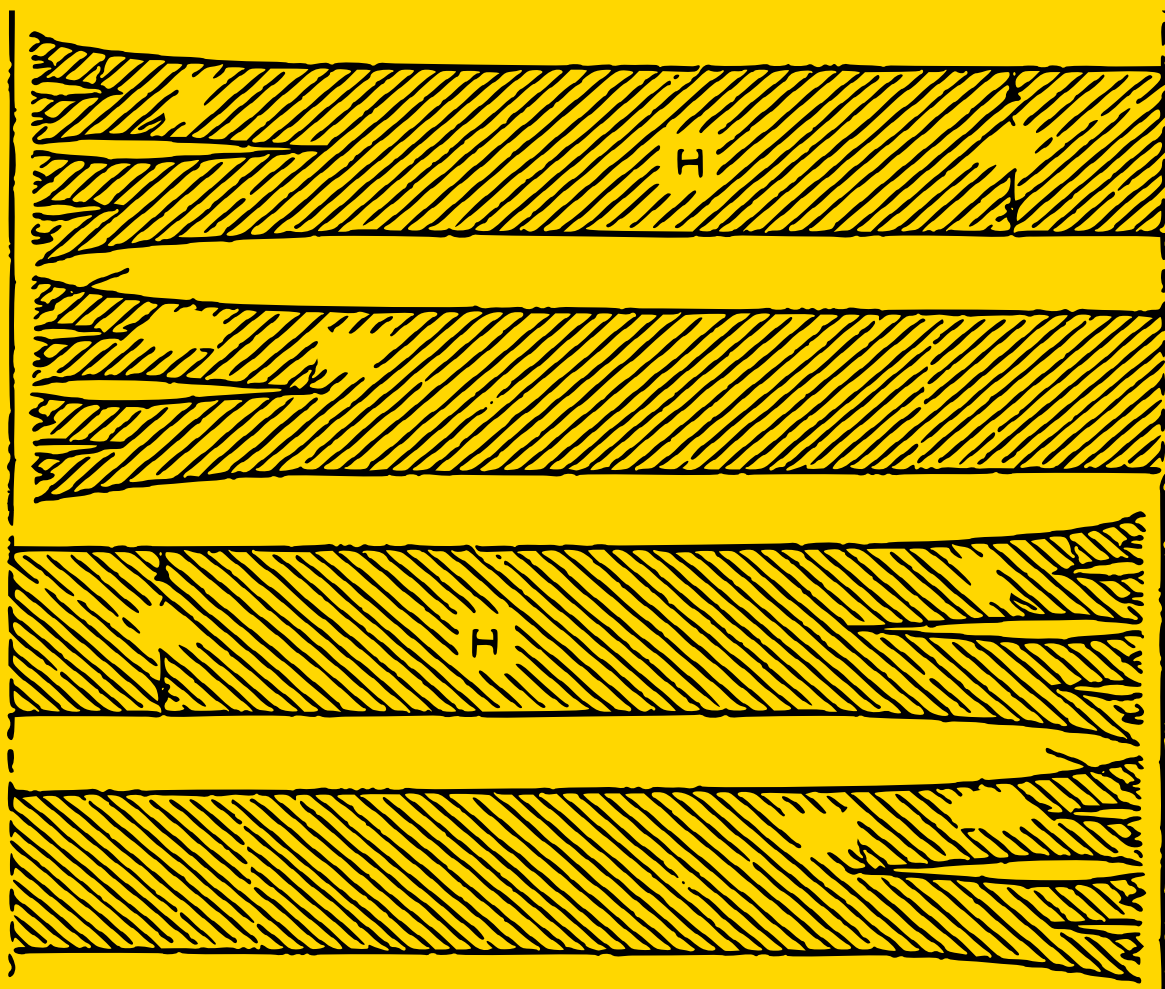


D. Ter Haar (Editor)

Collected Papers of L.D. LANDAU



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EDITED AND WITH AN INTRODUCTION

BY

D. TER HAAR

GORDON AND BREACH, SCIENCE PUBLISHERS

NEW YORK • LONDON • PARIS

Gordon and Breach, Science Publishers, Inc., 150 Fifth Avenue, New York 11, N.Y.

Gordon and Breach, Science Publishers Ltd., 171 Strand, London W.C. 2

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Pergamon Press Ltd. and

Gordon and Breach, Science Publishers, Inc.

First edition 1965

This book is published jointly by

Gordon and Breach, Science Publishers Inc.

and Pergamon Press Ltd.

Library of Congress Catalog Card No. 64-17191

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* *JETP* = *Journal of Experimental and Theoretical Physics of the U.S.S.R.*

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types of elementary excitations. It would be more correct to speak simply of the long wave and short wave excitations". In that second paper he had fitted the energy spectrum to agree with Peshkov's data on second sound propagation, and found that the two branches of the spectrum should continuously merge into one another—a feature since then abundantly corroborated by microscopic theories. In the earlier paper, Landau calculates the specific heat of liquid helium following from the energy spectrum, and shows that superfluidity is also a consequence of the spectrum inasmuch as the fluid, if moving with sufficiently low velocity, will be unable to slow down by the excitation of a single elementary excitation. He also evaluates the velocity of sound in liquid helium, and finds that apart from the ordinary sound waves, temperature waves may be propagated with a velocity different from that of ordinary sound. These waves are the so-called second-sound waves. Finally, he started the discussion of hydrodynamics of helium II, a subject further developed by him in a later paper^{53, 66}. In a paper with Pomeranchuk⁶⁴ he shows that impurities in helium II move with the normal and not with the superfluid part of the liquid. Landau's other contribution to the theory of liquid helium has been in two papers with Khalatnikov^{69, 70} in which they gave an extensive discussion of viscosity phenomena in liquid helium, based upon the idea that transport phenomena can be described in terms of collisions between the elementary excitations. Together with Lifshitz⁸⁵ he has considered the problem of rotation in liquid helium; although the picture given there is not in agreement with the now generally accepted Feynman-Onsager theory it may be correct under certain conditions.

It is nowadays generally accepted that the peculiar properties of liquid ^4He are due to the fact that the ^4He -atoms are bosons. In three—by now classical—papers^{90, 91, 95} Landau developed a theory of Fermi liquids, that is, of a system of interacting fermions, the most important of which are the conduction electrons in a metal and liquid ^3He . As in the case of ^4He , the first problem is to find the energy spectrum for the elementary excitations. Landau⁹⁰ found this spectrum by assuming that in switching on the interaction between the fermions, the classification of the levels remains invariant, "dressed" fermions taking the place of the original "bare" fermions. We find then that the energy of the excitations is a functional of the distribution function of the particles—and is thus temperature dependent. In the second paper⁹¹ Landau discusses the propagation of waves in a Fermi liquid and finds a new kind of "sound": zero sound or "high frequency" sound. While ordinary "low-frequency" sound with its rarefactions and compressions corresponds to the oscillation of the radius of the Fermi surface which remains spherical about its centre, zero sound corresponds to a periodic oscillation of the shape of the Fermi surface. In the third paper⁹⁵ Landau discusses the forward scattering of the "dressed" fermions, which determines the general properties of a Fermi liquid.

2. SOLID STATE PHYSICS

Among Landau's contributions to solid state physics we should first of all mention his paper on the diamagnetism of metals⁴. Classically one knows that a system of charged particles will have a vanishing magnetic susceptibility.

However, quantum mechanically a magnetic field introduces a quantisation of the levels—the so-called Landau levels—and this discreteness of the levels leads to a non-vanishing diamagnetism. This effect becomes especially noticeable at low temperatures, and the susceptibility shows a periodic variation with the magnetic field (de Haas-van Alphen effect) due to the changing relative position of the highest occupied level and the Fermi level. This effect has been studied experimentally, notably by Shoenberg, and was theoretically treated by Landau³⁹.

Landau discussed¹² the field-dependence of the low-temperature magnetic susceptibility of such anti-ferromagnetic substances as chromium chloride and with Lifshitz¹⁸ gave a thermodynamic theory of domain structure as well as the basic equation of motion for the magnetisation in ferromagnetics and a first theory of ferromagnetic resonance.

In two papers in 1937²⁹ Landau developed a theory of second-order phase transitions, pointing out the close connexion between such transitions and symmetry properties. In a paper with Khalatnikov³² he showed that near a second-order phase transition point anomalous sound absorption occurs, while he has also considered³² the scattering of X-rays near such a transition point. In an earlier paper on a related topic¹⁷ Landau had considered the specific heat anomalies near a critical point and found below the critical point a $(T_c - T)^{1/2}$ law. Zeldovich and Landau⁵⁰ discussed the relations between the liquid, dielectric, and metallic forms of a substance, while Landau also considered the scattering of X-rays by crystals with a variable lamellar structure³³ and the equilibrium shape of crystals⁷². Together with Rumer³⁸ the absorption of short wavelength sound was investigated, and the absorption was considered to be due to sound-wave-lattice-wave collisions.

In 1933 Landau¹⁰ suggested the possibility that an electron might dig its own hole in a crystal through its polarising action. This idea is essential for polaron theory, and in a later paper with Pekar⁶⁷ the effective mass of a polaron was evaluated. With Lifshitz³⁰ photoconductivity in semi-conductors was studied.

We finally mention a paper with Pomeranchuk²⁵ on the electrical conductivity and thermo-electric power of metals at low temperatures and a paper²⁰ on the theory of the accommodation coefficient both at low temperatures where quantum effects become important and at high temperatures where the classical theory holds.

3. PLASMA PHYSICS

In plasma physics Landau has made two important contributions. The first one²⁴ was his derivation of a transport equation for a system of charged-particles, where the long range of the Coulomb forces makes it impossible to use the normal Boltzmann equation. The second one⁶¹ was a paper where he discusses in some detail plasma oscillations and shows that these are always damped (Landau damping).

4. HYDRODYNAMICS

Apart from providing physics with one of the best existing text books on hydrodynamics, Landau has also from time to time contributed short notes on various problems in hydrodynamics and aerodynamics. With Levich⁴⁸ he studied the formation of a thin liquid layer upon the surface of a solid which is dragged through the liquid, and in a later note⁵¹ he showed that the motion in an axially symmetric jet can be determined for arbitrary Reynolds numbers by solving the Navier-Stokes equations rigorously. Landau investigated⁶⁰ the behaviour of shock waves at large distances from their point of origin, pointing to the existence of two (rather than one) shock waves. With Lifshitz⁸³ he used the Euler-Tricomi equations to study weak discontinuities at the sonic line.

The onset of turbulence as an eigenvalue problem was studied⁵² in order to further the understanding of turbulence, while with Lifshitz⁹³ hydrodynamic fluctuations were considered.

We finally mention three papers with Staniukovich^{57, 58, 59} on the hydrodynamics of detonating gaseous mixtures and condensed explosives.

5. ASTROPHYSICS

Landau has also made a few brief excursions into astrophysics, and again as one is bound to expect, with interesting results. In his first paper on this subject⁸ he discusses in general terms stellar equilibrium. He points out the importance of quantum statistics and relativity for the equation of state of matter at the centre of a star, and comes to the conclusion that for stars heavier than about 1.5 solar masses, quantum mechanics can no longer hold for parts of such stars. He discards the idea that "some mysterious process of mutual annihilation of protons and electrons" would supply stellar radiation, but suggests that rather the law of conservation of energy should no longer be valid in the relativistic region⁶. However, a year later he considers with Gamow¹³ the consequences of nuclear reactions for the temperature at the centre of a star and concludes that either lithium will not be present in any appreciable amount, or the central temperatures can not exceed a few million degrees. In his final paper on stellar energy²⁷ Landau suggests that this energy is released by transforming ordinary matter into "neutronic" matter, that is, by pushing all electrons into the nuclei. He shows that this would be energetically favourable for systems with masses more than about one thousandth of a solar mass.

6. NUCLEAR PHYSICS AND COSMIC RAYS

In 1937 Landau applied³¹ Bohr's concept of a statistical theory of nuclei to find a relation between the level density and the neutron scattering width. He has also³⁵ pointed out the importance of selection rules—especially the forbiddenness of transitions between states of different parity—for the

stability of nuclei against α -particle disintegration. With Tamm⁴⁰ he sketched a theory of nuclear forces.

Landau has at various times studied scattering processes (see also the next section), for instance, scattering of mesons by nuclear forces⁴², scattering of light by mesons (with Smorodinski⁴⁵), and proton-proton scattering (also with Smorodinski⁴⁵). With Lifshitz⁶⁸ he has considered the energy transfer during neutron-deuteron and proton-deuteron scattering*, while with Pomeranchuk⁷⁷ he discussed the emission of gamma-quanta when fast pions are scattered by nucleons.

With Rumer, Landau has considered various processes leading to cosmic ray showers^{34, 36} especially the cascade theory of electronic showers. Following these calculations Landau evaluated the angular distribution in a shower⁴³ and also has developed a theory of secondary showers produced by mesons⁴⁴. He has also studied the ionisation energy loss of fast particles⁵⁶. One of his most widely quoted contributions to the theory of cosmic ray phenomena is the theory of multiple production of particles by fast incoming particles⁷⁴. This theory is an extension of ideas, first proposed by Fermi. The essential features of the production process are that when two nucleons collide, a compound system is produced and within a small volume, Lorentz-contracted because of the high velocity of the centre-of-mass system, a large number of particles is created. The density in that small volume is so high and the mean free path so small that statistical equilibrium is established. After that the system expands, at first "hydrodynamically", that is, in such a way that the mean free path of the constituents remains small, and finally in such a way that the individual products of the collision-process can be distinguished. This theory was further developed by Belen'kii and Landau⁸⁸. With Pomeranchuk⁷⁶ Landau applied their considerations of Bremsstrahlung processes⁷⁵ to electron-cascade processes at very high energies.

7. QUANTUM MECHANICS

Among the many contributions by Landau to quantum mechanics we must first of all mention a fundamental paper with Peierls⁶ in which the applicability of the uncertainty relations to relativistic quantum mechanics is studied. This attempt to establish in relativity restrictions on measuring processes which are not included in the uncertainty relations led to the well-known papers by Bohr and Rosenfeld on the measurement of electromagnetic fields. Landau also discussed¹⁹ the relativistic corrections to the many-body Schrödinger equation, the so-called Breit Hamiltonian. The problem of the angular momentum of a system of two photons which is of importance in the theory of electron-positron annihilation processes was discussed by Landau in 1948⁶⁵.

* It is interesting to note the title of this paper. Instead of a title such as "Neutron-deuteron and proton-deuteron scattering," the authors gave the paper a title so that it looked like a continuation of refs. 7 and 9 in order to be able to publish this paper at a time when nuclear physics papers seldom obtained permission to be published.

Apart from the scattering processes mentioned in the previous section, Landau has also made a general study of inelastic collisions^{7,9} and applied this theory, for instance, to excitation of nuclei, and excitation of vibrations during optical transitions. Bremsstrahlung emitted by fast electrons was evaluated by a method slightly simpler than Heitler's original one¹⁵, and the limitations of the applicability of the theory were considered. With Pomeranchuk, Landau returned to this problem⁷⁵ in a paper in which pair-production was also taken into account, a topic also treated in a paper with Lifschitz¹⁶. The cross-section for photon-photon scattering was calculated for the case of very high energies²⁶ with Akhiezer and Pomeranchuk. In a short note in the *Physical Review*⁸⁹ Landau pointed out that the failure to observe the theoretically predicted polarisation of electrons through scattering was likely to be caused by the fact that most electrons underwent multiple rather than single scattering. Berestetskii and Landau⁷¹ considered the second-quantised wave equation of the electron-positron system.

Among Landau's earlier papers there are a few dealing with spectra. In the first paper he wrote alone¹ Landau evaluated the band spectrum of diatomic molecules, the intensity distribution in the spectrum, and the Zeeman and Stark effect in the band spectrum. In a paper on damping in wave mechanics Landau introduced the density matrix to discuss coupled systems. In a short note in the *Naturwissenschaften*⁵ Landau pointed out the importance of sum rules for estimating the intensity of spectral lines. Together with Placzek, Landau studied the fine structure of Rayleigh scattering lines, but a detailed paper on this topic never saw the light of day, and only the main conclusions of this study were published¹⁴.

S. QUANTUM FIELD THEORY

With Peierls, Landau derived³ the Heisenberg-Pauli quantum electrodynamics in a slightly different manner while Landau⁴¹ has considered the limits beyond which electrodynamics loses its validity in quantum mechanics, and applied these considerations to electrons and mesons.

With Abrikosov and Khalatnikov, Landau has considered the elimination of infinities in quantum electrodynamics⁷⁸, the propagators or Green functions for photons⁸⁰ and electrons⁷⁹, and the electron mass⁸¹, while Landau and Pomeranchuk⁸⁶ discussed point interactions in quantum electrodynamics. A survey of the results of these papers were presented in the *Nuovo Cimento* supplement dealing with Russian physics⁸⁹ and in the *Bohr Festschrift*⁸⁴. The gauge transformation of charged particle propagators and vertex operators were considered by Landau and Khalatnikov⁸⁷. Landau⁹⁴ also studied some spectral properties of the temperature-dependent single-particle Green functions.

Abrikosov, Galanin, Gorkov, Landau, Pomeranchuk and Ter-Martirosyan⁹⁶ have shown the impossibility of constructing a consistent strong-coupling fermion theory. The difficulty here is that as one increases the so-called cut-off limit, the physical interaction tends to zero independent of how large the bare coupling constant is.

Recent developments in quantum field theory are mainly concerned with constructing theories without Hamiltonians and wave functions in which dispersion relations and diagram methods play the main role. One of the tasks to be accomplished is to find the singularities of the various quantities occurring in the theory. Landau⁹⁸ (see also ref. 99 and a general discussion in ref. 100) has developed a general method to find such singularities.

In connection with the developments following Lee and Yang's theory of non-conservation of parity, Landau proposed⁹² the hypothesis of conservation of combined parity and also discussed in that connexion the properties of the neutrino.

9. MISCELLANEOUS

In this last section we have collected all those papers which do not fall easily into the earlier categories. In an interesting paper with Bronstein¹¹ Landau discussed the connection with the second law of thermodynamics and the universe as a whole. Discussions with Vavilov led Landau to a study of the thermodynamics of photo-luminescence⁶² and the limitations imposed by thermodynamics upon its yield. The temperature dependence of the dispersion of sound in gases was studied together with Teller²².

Landau has also made some contributions to theoretical chemistry. He studied the pressure dependence of the decomposition rate of large molecules²³, the theory of slow combustion⁸⁴, and the stability of sols and the coalescence of charged particles in electrolytic solutions under the action of van der Waals forces⁴⁷.

Finally, Landau has studied with Meiman and Khalatnikov⁹⁷ finite difference methods for solving differential equations.

1. ON THE THEORY OF THE SPECTRA OF DIATOMIC MOLECULES

The model of the diatomic molecule as a rotator with internal momentum is treated by means of the new quantum mechanics. The familiar band theory is obtained for the frequencies. All intensities are calculated. The behaviour of the molecule in electric and magnetic fields (Stark and Zeeman effects in the bands) has been examined.

THE Hamiltonian for a diatomic molecule is†

$$H = U + \frac{1}{2m} \sum p^2 + \frac{1}{2M_1} P_1^2 + \frac{1}{2M_2} P_2^2, \quad (1)$$

where the small letters refer to electrons and the capital letters to the nuclei; U denotes the potential energy.

In order to separate the translatory motion of the molecule as a whole, we make the following transformation of co-ordinates:

$$\left. \begin{aligned} \bar{r} &= r - \frac{M_1 R_1 + M_2 R_2}{M_1 + M_2}, & R &= R_2 - R_1, \\ C &= \frac{m \sum r + M_1 R_1 + M_2 R_2}{\sum m + M_1 + M_2}; \end{aligned} \right\} \quad (2)$$

the new co-ordinates of the electrons are their vector distances from the centre of mass of the nuclei, R is the distance between the nuclei, and C is the radius vector of the centre of mass of the molecule. After this transformation the Hamiltonian becomes

$$\begin{aligned} H = U + \frac{1}{2m} \sum \bar{p}^2 + \frac{1}{2(M_1 + M_2)} (\sum \bar{p})^2 \\ + \frac{1}{2} \left(\frac{1}{M_1} + \frac{1}{M_2} \right) P^2 + \frac{1}{2(\sum m + M_1 + M_2)} \gamma^2, \end{aligned} \quad (3)$$

where γ is the momentum corresponding to the co-ordinate C . It is seen that only the last term in equation (3) relates to the translatory motion; it is therefore of no further interest here, and we shall ignore it. We shall also omit the bar from r and p . The angular momentum is then

$$M = \sum [r \wedge p] + [R \wedge P]. \quad (4)$$

L. Landau, Zur Theorie der Spektren der zweiatomigen Moleküle, *Z. Phys.* 40, 621 (1926).

† All co-ordinates and momenta which occur are treated as matrices.

If finally we use polar co-ordinates R, θ, φ for \mathbf{R} , the result is

$$H = U + \frac{1}{2m} \sum p^2 + \frac{1}{2(M_1 + M_2)} (\sum \mathbf{p})^2 + \frac{1}{2} \left(\frac{1}{M_1} + \frac{1}{M_2} \right) p_R^2 + \frac{1}{2} \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \frac{1}{R^2} \left\{ p_\theta^2 - \frac{1}{4} \hbar^2 + \frac{1}{\sin^2 \theta} \left(p_\varphi^2 - \frac{1}{4} \hbar^2 \right) \right\}; \quad (5)$$

$$\left. \begin{aligned} M_x &= -\sin \varphi p_\theta - \frac{1}{2} \cot \theta (\cos \varphi p_\varphi + p_\varphi \cos \varphi) + D_x, \\ M_y &= \cos \varphi p_\theta - \frac{1}{2} \cot \theta (\sin \varphi p_\varphi + p_\varphi \sin \varphi) + D_y, \\ M_z &= p_\varphi + D_z, \end{aligned} \right\} \quad (6)$$

where D_x, D_y, D_z denote the components of the electron angular momentum

$$\mathfrak{D} = \sum [\mathbf{r} \wedge \mathbf{p}].$$

θ and φ occur explicitly in (5) in the potential energy also; in order to isolate them, another co-ordinate transformation is necessary. Let the new z' axis have the direction of the vector \mathbf{R} , and the x' axis be normal to the z and z' axes. Then the result of the transformation is

$$\left. \begin{aligned} H &= H_0 + H_1, \\ H_0 &= U + \frac{1}{2m} \sum p'^2 + \frac{1}{2(M_1 + M_2)} (\sum \mathbf{p}')^2 + \frac{1}{2} \left(\frac{1}{M_1} + \frac{1}{M_2} \right) p_R^2, \\ H_1 &= \frac{1}{2J} \left\{ (p_\theta + D_x)^2 - \frac{1}{4} \hbar^2 \right. \\ &\quad \left. + \frac{1}{\sin^2 \theta} \left[(p'_\varphi + \sin \theta D'_y - \cos \theta D_z)^2 - \frac{1}{4} \hbar^2 \right] \right\}, \end{aligned} \right\} \quad (7)$$

where

$$\frac{1}{J} = \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \frac{1}{R^2},$$

$$D_x = \sum (y' p_z - z' p_y), \text{ etc.}$$

The components of the angular momentum along the old co-ordinate axes are

$$\left. \begin{aligned} M_x &= -\sin \varphi p_\theta - \frac{1}{2} \cot \theta (\cos \varphi p_\varphi + p_\varphi \cos \varphi) + \frac{\cos \varphi}{\sin \theta} D_{z'}, \\ M_y &= \cos \varphi p_\theta - \frac{1}{2} \cot \theta (\sin \varphi p_\varphi + p_\varphi \sin \varphi) + \frac{\sin \varphi}{\cos \theta} D_{z'}, \\ M_z &= p_{\varphi'}. \end{aligned} \right\} \quad (8)$$

Since H_1 is small compared with H_0 (on account of the relative magnitude of J), we can first of all neglect H_1 . This corresponds to a motion of the molecule in which the two nuclei are fixed on the z' axis. It is easily seen that the z' -component of the electron angular momentum is then constant, and D_z is therefore a diagonal matrix. Born, Heisenberg and Jordan¹ have shown that in this case

$$\left. \begin{aligned} D_z^n &= k \hbar, \\ z'^n &= R_{n_1}^n = p_{R_{n_1}}^n = 0 \quad \text{when } k \neq k_1, \\ x'^n &= y'^n = D_x^n = D_y^n = 0 \quad \text{when } k \neq k_1 \pm 1, \end{aligned} \right\} \quad (9)$$

where n is the set of all quantum numbers and k is one of these.

H_1 introduces two further quantum numbers, on account of the new degrees of freedom. The H_0 problem is degenerate with respect to these. A canonical transformation is therefore perhaps necessary. The transformation matrix must be such that¹

$$S_{n_1 l_1}^n = 0 \quad \text{when } n \neq n_1, \quad (10)$$

l denoting these new quantum numbers. Since the solution already found for any quantity can be put in the form

$$A_{(0)n_1 l_1}^n = A_{n_1}^n \delta_{l_1}^l,$$

we can write

$$A = S A_0 S^{-1}, \quad A_{n_1 l_1}^n = A_{n_1}^n [S_n S_{n_1}^{-1}]_{l_1}^l = A_{n_1}^n s_{n_1 l_1}^n, \quad S_{(n)l_1}^n = S_{n_1 l_1}^n, \quad (11)$$

where s is independent of the nature of the quantity A . If $n = n_1$, then

$$s_{n_1 l_1}^{n_1} = \delta_{l_1}^l, \quad A_{n_1 l_1}^{n_1} = A_{n_1}^{n_1}. \quad (12)$$

For any function of θ , φ , p_θ , p_φ alone, since H_1 is small, we have to a first approximation $dB/dt = 0$, and hence it follows directly that

$$B_{n_1 l_1}^n = 0 \quad \text{when } n \neq n_1. \quad (13)$$

The above discussion shows that¹

$$\begin{aligned} E_{(1)n, l} &= H_{(1)n l}^{n l} \\ &= \left[\frac{1}{2J} \left(p_\theta^2 - \frac{1}{4} \hbar^2 + \frac{p_\varphi^2 - 2 \cos \theta p_\varphi D_z + D_z^2 - \frac{1}{4} \hbar^2}{\sin^2 \theta} \right. \right. \\ &\quad \left. \left. + D_x^2 + D_y^2 - D_z^2 \right) \right]_{n l}^{n l} \\ &= \left[\frac{1}{2J} (M_x^2 + M_y^2 + M_z^2 + D_x^2 + D_y^2 - D_z^2) \right]_{n l}^{n l}, \end{aligned} \quad (14)$$

since from (9) $D_x^n = D_y^n = 0$.

Born, Heisenberg and Jordan have shown that

$$M_{n,l}^2 = (M_x^2 + M_y^2 + M_z^2)_{n,l} = \hbar^2 j(j+1); \quad (15)$$

let j be one of the two quantum numbers 1. From (14), (15) and (12) we find

$$E_{(1)n,l} = a_n j(j+1) + b_n,$$

or

$$E_{n,l} = E_{(0)n,l} + E_{(1)n,l} = A_n + a_n j(j+1), \quad (16)$$

where A_n and a_n depend only on n and not on l . By the j selection rule¹ the well-known formula (16) gives the following frequencies:

$$\left. \begin{aligned} \nu_{n',j}^n &= \nu_n^n + \beta_n^n j(j+1), \\ \nu_{n',j-1}^n &= \nu_n^n + j(\alpha_n^n + \beta_n^n j), \\ \nu_{n',j}^{n-1} &= \nu_n^n + j(-\alpha_n^n + \beta_n^n j); \end{aligned} \right\} \quad (17)$$

α_n^n and β_n^n are constants characterising the n band system.

Let us now calculate the intensities. According to Born, Heisenberg and Jordan¹

$$\left. \begin{aligned} (\xi + i\eta)_{jm-1}^{jm} &= f_j^j \sqrt{(j+m)(j-m+1)}, \\ (\xi - i\eta)_{jm-1}^{jm-1} &= (\xi + i\eta)_{jm-1}^{jm}, \quad \zeta_{jm}^{jm} = f_j^j m, \\ (\xi + i\eta)_{j-1,m-1}^j &= -f_{j-1}^j \sqrt{(j+m)(j+m-1)}, \\ (\xi - i\eta)_{j-1,m}^{j-1} &= f_{j-1}^j \sqrt{(j-m)(j-m+1)}, \\ \zeta_{j-1,m}^{j-1} &= f_{j-1}^j \sqrt{(j+m)(j-m)}, \\ (\xi + i\eta)_{j-1,m-1}^{j-1} &= (\xi - i\eta)_{j-1,m}^{j-1}, \\ (\xi - i\eta)_{j-1,m}^{j-1} &= (\xi + i\eta)_{j-1,m-1}^{j-1}, \quad \zeta_{j-1,m}^{j-1} = \zeta_{j-1,m}^j, \end{aligned} \right\} \quad (18)$$

where ξ, η, ζ are the same combinations of x and X, y and Y, z and Z , respectively, and the f are independent of m . It is easily seen that similar formulae are valid for the components of the unit vector along R , viz. $\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta$. Elimination of p_θ and p_φ from (8) gives

$$\sin \theta \cos \varphi M_x + \sin \theta \sin \varphi M_y + \cos \theta M_z = D_z, \quad (19)$$

or

$$\frac{1}{2} \sin \theta e^{i\varphi} (M_x - i M_y) + \frac{1}{2} \sin \theta e^{-i\varphi} (M_x + i M_y) + \cos \theta M_z = D_z. \quad (20)$$

The diagonal terms of (20) give

$$u_{kj}^{kj} = \frac{k}{j(j+1)} \quad (21)$$

(where u_j^i is the value of f_j^i for $\sin\theta \cos\varphi$, $\sin\theta \sin\varphi$, $\cos\theta$), since¹

$$(M_x + i M_y)_{jm-1}^{jm} = (M_x - i M_y)_{jm}^{j, m-1} = \hbar \sqrt{(j+m)(j-m+1)}, \quad (22)$$

$$(M_z)_{jm}^{jm} = \hbar m.$$

The relations

$$\sin\theta e^{i\varphi} \cos\theta - \cos\theta \sin\theta e^{i\varphi} = 0,$$

$$\sin\theta e^{i\varphi} \sin\theta e^{-i\varphi} + \cos^2\theta = 1$$

give finally

$$u_{kj-1}^{kj} = u_{kj}^{kj-1} = \frac{1}{j} \sqrt{\frac{(j+k)(j-k)}{(2j+1)(2j-1)}}. \quad (23)$$

Formula (23) shows that we must have

$$j \geq |k|. \quad (24)$$

If we recall that the $\frac{k}{j}$ terms of x' and y' are zero (equation (9)), we can put

$$x_k^k = z_k^k \sin\theta \cos\varphi, \quad y_k^k = z_k^k \sin\theta \sin\varphi, \quad z_k^k = z_k^k \cos\theta.$$

Since also for the nucleus co-ordinates $X_k^k = R_k^k \sin\theta \cos\varphi$, etc., it follows that similar relations hold for all possible ξ , η , ζ . This gives

$$f_{kj}^{kj} = f_k^k \frac{k}{j(j+1)}, \quad f_{kj-1}^{kj} = f_{kj}^{kj-1} = f_k^k \frac{1}{j} \sqrt{\frac{(j+k)(j-k)}{(2j+1)(2j-1)}}, \quad (25)$$

with f_k^k independent of j . In order to calculate the f_{k-1}^k , we use the $\frac{k}{j-1}$ terms of the commutation relation.

$$z \cos\theta - \cos\theta z = 0$$

Then

$$\left. \begin{aligned} f_{k-1j}^k &= f_{k-1}^k \frac{\sqrt{(j+k)(j-k+1)}}{j(j+1)}, \\ f_{k-1j-1}^k &= -f_{k-1}^k \frac{1}{j} \sqrt{\frac{(j+k)(j+k-1)}{(2j+1)(2j-1)}}, \\ f_{k-1j}^{k-1} &= f_{k-1}^{k-1} \frac{1}{j} \sqrt{\frac{(j-k)(j-k+1)}{(2j+1)(2j-1)}}. \end{aligned} \right\} \quad (26)$$

Similar formulae are obtained for the f_k^{k-1} .

Since ν_{nk}^k is approximately equal to ν_n^k , on account of the smallness of E_1 , and therefore is independent of j and m (to a first approximation), we can use formulae (25), (26) also for time derivatives of ξ , η , ζ of any order. They therefore give the radiation amplitudes also.

In the *field-free case* the system is degenerate with respect to m . It is easily seen that we then have for the intensity of (unpolarised) radiation

$$J = 3 \sum_{m=-j}^j (A_m)^2, \quad (27)$$

where A_m is the amplitude of the radiation polarised in the direction of the z axis (in the non-degenerate system). Substitution of A_m from (18) gives

$$\left. \begin{aligned} J_j^j &= (f_j^j)^2 j(j+1)(2j+1), \\ J_{j-1}^j &= (f_{j-1}^j)^2 j(2j-1)(2j+1), \\ J_j^{j-1} &= J_{j-1}^j, \end{aligned} \right\} \quad (28)$$

or

$$\left. \begin{aligned} J_{kj}^{kj} &= J_k^k \left(\frac{1}{j} + \frac{1}{j+1} \right) k^2, \\ J_{k,j-1}^{kj} &= J_k^k \frac{(j+k)(j-k)}{j}, \\ J_{kj}^{kj-1} &= J_{k,j-1}^{kj}, \\ J_{k-1,j}^{kj} &= J_{k-1}^k \left(\frac{1}{j} + \frac{1}{j+1} \right) (j+k)(j-k+1), \\ J_{k-1,j-1}^{kj} &= J_{k-1}^k \frac{(j+k)(j+k-1)}{j}, \\ J_{k-1,j}^{k,j-1} &= J_{k-1}^k \frac{(j-k)(j-k+1)}{j}, \\ J_k^{k-1,j} &= J_{k-1,j}^k, \quad J_{k-1,j-1}^{k-1,j} = J_{k-1,j}^{k,j-1}, \quad J_k^{k-1,j-1} = J_{k-1,j-1}^k. \end{aligned} \right\} \quad (29)$$

A magnetic field along the z -axis gives in a first approximation the following addition to the Hamiltonian†

$$\Delta H = \frac{e|\mathbf{H}|}{2\mu c} D_z. \quad (30)$$

Hence

$$\Delta E = \frac{e|\mathbf{H}|}{2\mu c} D_z \cos \theta = \frac{e\hbar}{2\mu c} |\mathbf{H}| \frac{k^2 m}{j(j+1)}; \quad (31)$$

$(D_x)_n^n = (D_y)_n^n = 0$ from (9). This completes the investigation of the Zeeman effect for the bands: the splitting is obtained from (31), the intensities and polarisations from (18), and (25), (26). An n -band system for which $k = k' = 0$ shows, according to (31), no splitting in the first approximation. For sufficiently strong fields we must carry the approximation to (7) one step further and also take into account the quadratic terms in the field perturbation. The calculation

† Here and below we denote the electron mass by μ to avoid confusion with the quantum number m .

is somewhat involved; the result is

$$(\Delta E)_{k=0} = |H| \gamma_n m + |H|^2 \kappa_n \frac{j(j+1) + (m-1)(m+1)}{(2j-1)(2j+3)}; \quad (32)$$

γ_n is of the order of $1/M$ (M being the nuclear masses).

The solution is quite similar for an *electric field*, with

$$\Delta H = |E| C_z, \quad (33)$$

where C_z denotes the z component of the polarisation vector, and from (18) and (25)

$$\Delta E = |E| \varepsilon_n k \frac{m}{j(j+1)}. \quad (34)$$

If $k = 0$, we must distinguish between molecules where the atoms are the same and those where they are different. In the former case the formula for the Stark effect is similar to (32), but if the molecule is polar, the second term in that equation must be replaced by

$$|E|^2 \lambda_n \frac{3m^2 - j(j+1)}{j(j+1)(2j-1)(2j+3)},$$

where λ_n is of the order of M .

REFERENCE

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2. THE DAMPING PROBLEM IN WAVE MECHANICS

A formula is derived for the wave-mechanical treatment of damping, and is used to investigate some related problems; coherence phenomena also are explained. An expression is obtained for spontaneous emission, and the problem of the intensity of spectral lines is solved in this way.

1. COUPLED SYSTEMS IN WAVE MECHANICS

A system cannot be uniquely defined in wave mechanics; we always have a probability ensemble (statistical treatment).† If the system is coupled with another, there is a double uncertainty in its behaviour.

Let the state of the first system be described by the quantities a_n in

$$\psi = \sum a_n \psi_n, \quad (1)$$

and for the second system let

$$\psi' = \sum b_r \psi'_r. \quad (2)$$

The Schrödinger function for the two systems together is then

$$\Psi = \psi \psi' = \sum_n \sum_r a_n b_r \psi_n \psi'_r = \sum_n \sum_r c_{nr} \psi_n \psi'_r, \quad (3a)$$

where

$$c_{nr} = a_n b_r. \quad (3b)$$

If there is coupling, then c_{nr} is a function of time and can no longer be resolved as in equation (3b). Thus a_n and b_r can no longer be used separately.

For a function f of co-ordinates (and momenta) of the first system alone we have‡

$$\bar{f} = \int \Psi f \Psi^* d\tau d\tau' = \sum_n \sum_m \sum_r c_{nr}^* c_{mr} \int \psi_n^* f \psi_m d\tau = \sum_n \sum_m \alpha_{nm} f_{nm}, \quad (4a)$$

where

$$\alpha_{nm} = \sum_r c_{nr}^* c_{mr}, \quad (4b)$$

and

$$f_{nm} = \int \psi_n^* f \psi_m d\tau$$

L. Landau, Das Dämpfungsproblem in der Wellenmechanik, *Z. Phys.* 45, 430 (1927).

† It has recently been shown by Heisenberg¹ that the reason for this is in the nature of the problem; this finally resolves the model problem in wave mechanics. The relationship with classical mechanics is discussed by Ivanenko and Landau².

‡ \bar{f} denotes the probability average value of f .

are the matrix components of f . Thus the "state" of the system can now be described by the quantities α_{nm} . For a system defined by the quantities a_n it is well known that

$$\bar{f} = \int \psi^* f \psi d\tau = \sum_n \sum_m a_n^* a_m f_{nm}. \quad (5)$$

In this particular case, therefore, $\alpha_{nm} = a_n^* a_m$; in general such a representation is not possible, and α_{nm} must then be regarded as a certain mean value of $a_n^* a_m$.

2. CAVITY RADIATION IN WAVE MECHANICS

Wave mechanics usually deals with objects having a limited number of degrees of freedom. The general problem of quantisation of the electromagnetic field (quantum electrodynamics) still presents insuperable difficulties. If the structure of the field is not involved, however, but only the properties of the radiation as a whole, the problem is considerably simpler.

If we imagine the radiation enclosed in a vessel of any shape, it is known to consist of independent eigen-oscillations, in general of different frequencies. Since each eigen-oscillation corresponds to a separate degree of freedom, such a system has a discrete, though infinite, series of degrees of freedom. Since they are independent, the eigen-oscillations can be quantised individually.

As the co-ordinate of an eigen-oscillation we take the phase of the corresponding electric (or magnetic) force for a particular point†. We have

$$\frac{d\varphi}{dt} = \omega, \quad (6)$$

where φ is the phase, and ω the frequency of the eigen-oscillation multiplied by 2π . If we now use the energy E , equation (6) may be written

$$\frac{d\varphi}{dt} = \frac{dE}{d(E/\omega)} = \frac{\partial H}{\partial p}, \quad (7)$$

where $H = E$ is the Hamiltonian and

$$p = \frac{E}{\omega} \quad (8)$$

must be regarded as the momentum corresponding to the co-ordinate φ ; the second Hamiltonian equation gives an identity in consequence of the relation $dE/dt = 0$.

In order now to derive Schrödinger's equation, we use the operator method. From

$$\omega p - E = 0$$

we have

$$\frac{\hbar}{i} \omega \frac{\partial \psi}{\partial \varphi} + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0$$

† Here I follow the ideas of Dirac³. The introduction of light quanta is, however, arbitrary and not necessary.

or

$$\omega \frac{\partial \psi}{\partial \varphi} + \frac{\partial \psi}{\partial t} = 0. \quad (9)$$

Since φ is a cyclic co-ordinate, the permissible solutions of (9) must have the period 2π ; the quantity E , being the energy, must be positive for every combination of such solutions. Separation of the variables gives

$$\psi_r = \frac{1}{\sqrt{2\pi}} e^{i r (\varphi - \omega t)}, \quad (10)$$

where r is a constant which, by the first condition, takes only integral values. The energy is

$$E_r = r \hbar \omega, \quad (11)$$

and r must therefore be positive†. We shall also use the quantities $e^{i\varphi}$ and $e^{-i\varphi}$; in the matrices corresponding to them, all elements are zero except

$$|e^{i\varphi}|_{r, r-1} = e^{i\omega t} \quad \text{and} \quad |e^{-i\varphi}|_{r-1, r} = e^{-i\omega t}. \quad (12)$$

3. DAMPING IN ATOMIC STRUCTURES

In view of the results of section 2 the methods of ordinary perturbation theory are applicable here, since the damping phenomena are caused by the reaction of the atomic radiation field.

The perturbation function is known to be

$$\eta = (C \cdot E), \quad (13)$$

where C denotes the polarisation vector and E the electric force at the centre of the atom. The latter can be represented as a sum of electric forces corresponding to the individual eigen-oscillations:

$$E = \sum_x E_x, \quad \eta = (C \cdot \sum_x E_x). \quad (14)$$

If we take the centre as the point which determines the phase, then

$$E_x = e_x \cos \varphi_x, \quad (15)$$

with e_x independent of time. The total energy of an eigen-oscillation is‡

$$E_x = \int \frac{1}{8\pi} (\mathbf{E}_x^2 + \mathbf{H}_x^2) dV = \int \frac{1}{4\pi} \mathbf{E}_x^2 dV = \int \frac{1}{4\pi} e_x^2 \overline{\cos^2 \varphi_x} dV = \frac{1}{8\pi} \int e_x^2 dV, \quad (16)$$

and the space average of e_x^2 is given by

$$E_x = \frac{1}{8\pi} V \overline{e_x^2}, \quad \overline{e_x^2} = 8\pi \frac{E_x}{V},$$

† From equation (11) Planck's radiation formula can be obtained by direct application of Planck statistics (see also ref. 4).

‡ Here we average over the time.

so that

$$e_x = \sqrt{8\pi \frac{E_x}{V}} n_x, \quad (17)$$

where n_x depends only on the position of the centre and satisfies the condition

$$\overline{n_x^2} = 1. \quad (18)$$

Substitution of (17) in (15) gives

$$E_x = \sqrt{8\pi \frac{E_x}{V}} \cos \varphi_x n_x. \quad (19)$$

If now we regard E_x and $\cos \varphi_x$ as matrices, η and therefore E_x must be symmetrised:

$$E_x = \sqrt{\frac{2\pi}{V}} (\sqrt{E_x} e^{i\varphi_x} + e^{-i\varphi_x} \sqrt{E_x}) n_x \quad (20)$$

According to general perturbation theory, the coefficients c_N in $\varphi = \sum c_N \varphi_N$ are such that

$$\frac{dc_N}{dt} = \frac{i}{\hbar} \sum_M \eta_{NM} c_M. \quad (21)$$

If we use radiation quantum numbers r_x as well as the atomic quantum numbers $n(m, k, \dots)$, we have in our case from (14) and (19), and (11) and (12),

$$\begin{aligned} \frac{d}{dt} c(r_x, n) = i \sum_y \sum_m \left\{ c(r_x - \delta_{xy}, m) \sqrt{\frac{2\pi r_y \omega_y}{\hbar V}} e^{i\omega_y t} \right. \\ \left. + c(r_x + \delta_{xy}, m) \sqrt{\frac{2\pi (r_y + 1) \omega_y}{\hbar V}} e^{-i\omega_y t} \right\} (n_y \cdot C_{nm}), \end{aligned} \quad (22)$$

where

$$\begin{aligned} \delta_{xy} &= 0 \quad \text{for } x \neq y, \\ &= 1 \quad \text{for } x = y. \end{aligned}$$

For simplicity we take as the initial state a "state" of the atom determined in the usual way (see section I); the final formulae can be immediately applied to the general case. Since the previously emitted field has no effect on the atom, we also suppose that no radiation field is present at the initial instant. Then all the coefficients c except the $c(0, n)$ are zero. At the next instant the $c(\delta_{xy}, n)$ must be brought in, since (22) shows that only they have a non-zero time derivative:

$$\left. \begin{aligned} \frac{d}{dt} c(\delta_{xy}, n) &= i \sqrt{\frac{2\pi \omega_y}{\hbar V}} e^{i\omega_y t} \sum_m c(0, m) (n_y \cdot C_{nm}), \\ \frac{d}{dt} c(0, n) &= i \sum_y \sum_m c(\delta_{xy}, m) \sqrt{\frac{2\pi \omega_y}{\hbar V}} e^{-i\omega_y t} (n_y \cdot C_{nm}). \end{aligned} \right\} \quad (23)$$

The state of the atom is described, according to (4), by the quantities

$$\alpha_{nm} = c^*(0, n) c(0, m) + \sum_y c^*(\delta_{xy}, n) c(\delta_{xy}, m). \quad (24)$$

The time derivative of α_{nm} is

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} = i \sum_k \sum_y \left\{ c^*(0, n) c(\delta_{xy}, k) \sqrt{\frac{2\pi\omega_y}{\hbar V}} e^{-i\omega_y t} (n_y \cdot C_{mk}) \right. \\ - c^*(\delta_{xy}, k) c(0, m) \sqrt{\frac{2\pi\omega_y}{\hbar V}} e^{i\omega_y t} (n_y \cdot C_{kn}) \\ + c^*(\delta_{xy}, n) c(0, k) \sqrt{\frac{2\pi\omega_y}{\hbar V}} e^{i\omega_y t} (n_y \cdot C_{mk}) \\ \left. - c^*(0, k) c(\delta_{xy}, m) \sqrt{\frac{2\pi\omega_y}{\hbar V}} e^{-i\omega_y t} (n_y \cdot C_{kn}) \right\}. \quad (25) \end{aligned}$$

If we substitute directly in (25) the initial conditions $c_0(\delta_{xy}, n) = 0$, we arrive at the absurd conclusion that $d\alpha_{nm}/dt = 0$. The reason for this is evidently that the infinite frequency sums \sum_y are divergent. In order to obtain a more plausible result, we apply (25) not to the initial instant but to a later instant T . In the same approximation we have

$$\left. \begin{aligned} c(\delta_{xy}, n) &= i \sqrt{\frac{2\pi\omega_y}{\hbar V}} \sum_m c_0(0, m) \int_0^T (n_y \cdot C_{nm}) e^{i\omega_y t} dt, \\ c(0, n) &= c_0(0, n), \end{aligned} \right\} \quad (26)$$

and so, using the relation $\alpha_{nm}^0 = c_0^*(0, n) c_0(0, m)$,

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} = \sum_k \sum_e \sum_y \frac{2\pi\omega_y}{\hbar V} \left\{ -\alpha_{ne}^{(0)} (n_y \cdot C_{mk}) e^{-i\omega_y T} \int_0^T (n_y \cdot C_{ke}) e^{i\omega_y t} dt \right. \\ - \alpha_{em}^{(0)} (n_y \cdot C_{kn}) e^{i\omega_y T} \int_0^T (n_y \cdot C_{ek}) e^{-i\omega_y t} dt \\ + \alpha_{ek}^{(0)} (n_y \cdot C_{mk}) e^{i\omega_y T} \int_0^T (n_y \cdot C_{en}) e^{-i\omega_y t} dt \\ \left. + \alpha_{kn}^{(0)} (n_y \cdot C_{kn}) e^{-i\omega_y T} \int_0^T (n_y \cdot C_{me}) e^{i\omega_y t} dt \right\}. \quad (27) \end{aligned}$$

Since the volume V is regarded as infinite, the frequency sums can be replaced by integrals. The number of eigen-oscillations between ω and $\omega + d\omega$ is known to be

$$N_\omega d\omega = V \frac{\omega^2}{\pi^2 c^3} d\omega. \quad (28)$$

Instead of each expression of the type $(n \cdot A)(n \cdot B)$ we must put

$$\frac{1}{3} (A \cdot B) \quad (29)$$

in consequence of the averaging, having regard to (18). Equation (27) may therefore be written

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} = \sum_k \sum_e \int_0^\infty \left\{ \alpha_{ek}^{(0)} \left(C_{mk} e^{i\omega T} \cdot \int_0^T C_{en} e^{i\omega t} dt \right. \right. \\ + \alpha_{ke}^{(0)} \left(C_{kn} e^{-i\omega T} \cdot \int_0^T C_{me} e^{i\omega t} dt \right. \\ - \alpha_{ne}^{(0)} \left(C_{mk} e^{-i\omega T} \cdot \int_0^T C_{ke} e^{i\omega t} dt \right. \\ \left. \left. - \alpha_{em}^{(0)} \left(C_{kn} e^{i\omega T} \cdot \int_0^T C_{ek} e^{-i\omega t} dt \right) \right\} \frac{2\omega^3}{\pi \hbar c^3} d\omega. \quad (30) \end{aligned}$$

A number of considerations which will not be given here show that the double integrals occurring in this formula (which are divergent) must be independent of the upper limit T . If we raise the latter to infinity, apply the theory of the Fourier integral, and return to the initial instant, the result is

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} = \sum_k \sum_e \frac{2i}{3\hbar c^3} \left\{ \alpha_{ek} \left(C_{mk} \cdot \frac{d^3}{dt^3} C_{en}^+ \right) - \alpha_{ke} \left(C_{kn} \cdot \frac{d^3}{dt^3} C_{me}^- \right) \right. \\ \left. + \alpha_{ne} \left(C_{mk} \cdot \frac{d^3}{dt^3} C_{ke}^- \right) - \alpha_{em} \left(C_{kn} \cdot \frac{d^3}{dt^3} C_{ek}^+ \right) \right\} \end{aligned}$$

or

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} = \sum_k \sum_e \frac{2i}{3\hbar c^3} \left\{ \alpha_{ke} \left(C_{me} \cdot \frac{d^3}{dt^3} C_{kn}^+ \right) - \left(C_{kn} \cdot \frac{d^3}{dt^3} C_{me}^- \right) \right. \\ \left. + \alpha_{nk} \left(C_{me} \cdot \frac{d^3}{dt^3} C_{ek}^- \right) - \alpha_{em} \left(C_{kn} \cdot \frac{d^3}{dt^3} C_{ek}^+ \right) \right\}; \quad (31) \end{aligned}$$

the signs + and - signify that only the part containing positive and negative frequencies respectively is taken. This is because the integration is only over positive ω .

Formula (31) is of fundamental importance in the theory of damping phenomena. The quantities a_n cannot be used in this problem; the "state of the atom" must be described by means of the quantities α_{nm} .

If the external influences on the atom are quasi-periodic, the quantities α_{nm} change in a definite manner under the action of damping and tend (in the first approximation) asymptotically to definite values. The "state of the atom" corresponding to such values of α_{nm} will be called the *zero state*. In some cases there are several such states, of which some may be metastable.

For any quantity f with matrix f_{nm} we obtain from (4b)

$$\begin{aligned}\frac{d\tilde{f}}{dt} &= \frac{d}{dt} \sum_n \sum_m \alpha_{nm} f_{nm} = \sum_n \sum_m \left(\alpha_{nm} \frac{df_{nm}}{dt} + \frac{d\alpha_{nm}}{dt} f_{nm} \right) \\ &= \frac{\tilde{d}_0 f}{dt} + \sum_n \sum_m \frac{d\alpha_{nm}}{dt} f_{nm};\end{aligned}\quad (32)$$

$d_0 f/dt$ denotes the time derivative of f without allowance for damping. Substitution of (31), gives, in matrix notation,

$$\begin{aligned}\frac{d\hat{f}}{dt} &= \frac{d_0 \hat{f}}{dt} + \frac{2i}{3\hbar c^3} \left[\left(\frac{d^3 \hat{C}^+}{dt^3} \cdot \hat{f} \hat{C} \right) - \left(\hat{C} \hat{f} \cdot \frac{d^3 \hat{C}^-}{dt^3} \right) \right. \\ &\quad \left. + \left(\hat{f} \hat{C} \cdot \frac{d^3 \hat{C}^-}{dt^3} \right) - \left(\frac{d^3 \hat{C}^+}{dt^3} \cdot \hat{C} \hat{f} \right) \right] \\ &= \frac{d_0 \hat{f}}{dt} + \frac{2}{3c^3} \left[\left(\frac{d^3 \hat{C}^+}{dt^3} \cdot \hat{F} \right) + \left(\hat{F} \cdot \frac{d^3 \hat{C}^-}{dt^3} \right) \right],\end{aligned}\quad (33)$$

where $\hat{f} \hat{C} - \hat{C} \hat{f} = \frac{\hbar}{i} \hat{F}$.

For $f = \text{constant}$, the damping term in (33) is zero, as we should expect. The same is true of any function of co-ordinates only, on account of the commutation relations. For an electron moment \hat{H} we have

$$\frac{d\hat{H}}{dt} = \frac{d_0 \hat{H}}{dt} - \frac{2e}{3c^3} \left(\frac{d^3 \hat{C}^+}{dt^3} + \frac{d^3 \hat{C}^-}{dt^3} \right) = \frac{d_0 \hat{H}}{dt} - \frac{2e}{3c^3} \frac{d^3 \hat{C}}{dt^3}, \quad (34)$$

where e is the electron charge. This equation agrees entirely with the classical theory. Since linear relations always appear in wave mechanics unchanged, this conclusion can be used to justify (31). If f is taken as the energy ε of the atomic system, then

$$\begin{aligned}\hat{F} &= \frac{d\hat{C}}{dt}, \\ \frac{d\varepsilon}{dt} &= \frac{2}{3c^3} \left[\left(\frac{d^3 \hat{C}^+}{dt^3} \cdot \frac{d\hat{C}}{dt} \right) + \left(\frac{d\hat{C}}{dt} \cdot \frac{d^3 \hat{C}^-}{dt^3} \right) \right].\end{aligned}\quad (35)$$

In order to derive from this formula an expression for the energy radiated, we follow the classical theory; elimination of a complete derivative gives

$$\dot{J} = -\frac{d\hat{E}}{dt} = \frac{2}{3c^3} \left[\left(\frac{d^2 \hat{C}^+}{dt^2} \cdot \frac{d^2 \hat{C}}{dt^2} \right) + \left(\frac{d^2 \hat{C}}{dt^2} \cdot \frac{d^2 \hat{C}^-}{dt^2} \right) \right]. \quad (36)$$

This formula is of importance not only in damping theory but also in the wave mechanics representation of the radiation intensity, which can not be derived from the expression for the "mean" field (in the sense of the comment following equation (5)).†

In the continuous "spectrum" the sums in (31) must be replaced by integrals.

4. SOME PARTICULAR APPLICATIONS

Let us now apply the above results to systems which, apart from the damping, are conservative. In such systems, as we know, for any quantity f which does not depend explicitly on time,

$$f_{nm} = f^{nm} e^{i\omega_{nm}t}, \quad (37a)$$

$$\tilde{f} = \sum_n \sum_m \alpha_{nm} f_{nm} = \sum_n \sum_m \alpha_{nm} f^{nm} e^{i\omega_{nm}t}, \quad (37b)$$

where $\omega_{nm} = (E_n - E_m)/\hbar$ and the f^{nm} are independent of time.

In order to obtain an expression for the intensity of spectral lines, we use formula (36):

$$\left. \begin{aligned} J^{nm} &= \frac{2}{3c^3} \sum_k^{E_k < E_n} \omega_{nk}^2 \omega_{km}^2 (C^{nk} \cdot C^{kn}) + \sum_k^{E_k < E_m} \omega_{nk}^2 \omega_{km}^2 (C^{nk} \cdot C^{km}) \\ \tilde{J} &= \sum_n \sum_m a_n^* a_m J^{nm} e^{i\omega_{nm}t} \end{aligned} \right\} \quad (38)$$

or, after averaging over time,

$$\bar{J} = \sum_n |a_n|^2 J^{nn} = \sum_n \sum_k^{E_k < E_n} \frac{2}{3c^3} \omega_{nk}^4 |C_{nk}|^2 |a_n|^2. \quad (39)$$

Each term in this formula is to be regarded as the intensity of the radiation of the corresponding frequency. The characteristic properties of intensity (including, of course, polarisation etc.) of a spectral line thus correspond entirely to the amplitude coefficient of the radiation field (matrix component of $d^2 C/dt^2$). In addition, the intensity of radiation for a frequency ω_{nk} is, as we should expect, proportional to the quantity $|a_n|^2$ or, in the usual terminology, to the number of atoms in the initial state (the state with the higher energy value).

† The expression for the square of a quantity is, in wave mechanics, not equal to the square of the expression for that quantity; the quantities a_n must always appear homogeneously in the second degree. The results of Joos⁵ are therefore unsatisfactory.

In order to analyse the damping process, we use the basic formula (31):

$$\begin{aligned}
 \frac{d\alpha_{nm}}{dt} = & \sum_k \sum_e^{E_k > E_n} \frac{2\omega_{kn}^3}{3\hbar c^3} (C^{kn} \cdot C^{me}) e^{i(\omega_{kn} + \omega_{mn})t} \alpha_{ke} \\
 & + \sum_k \sum_e^{E_e > E_m} \frac{2\omega_{em}^3}{3\hbar c^3} (C^{kn} \cdot C^{me}) e^{i(\omega_{kn} + \omega_{mn})t} \alpha_{ke} \\
 & - \sum_k \sum_e^{E_k > E_e} \frac{2\omega_{ke}^3}{3\hbar c^3} (C^{me} \cdot C^{ek}) e^{i\omega_{mn}t} \alpha_{nk} \\
 & - \sum_k \sum_e^{E_k < E_e} \frac{2\omega_{ek}^3}{3\hbar c^3} (C^{kn} \cdot C^{ek}) e^{i\omega_{mn}t} \alpha_{em}, \quad (40a)
 \end{aligned}$$

and for $m = n$

$$\begin{aligned}
 \frac{d\alpha_{nn}}{dt} = & \sum_k \sum_e^{E_k > E_n} \frac{2\omega_{kn}^3}{3\hbar c^3} (C^{kn} \cdot C^{ne}) e^{i\omega_{kn}t} \alpha_{ke} \\
 & + \sum_k \sum_e^{E_e > E_n} \frac{2\omega_{en}^3}{3\hbar c^3} (C^{kn} \cdot C^{ne}) e^{i\omega_{kn}t} \alpha_{ke} \\
 & - \sum_k \sum_e^{E_k > E_e} \frac{2\omega_{ke}^3}{3\hbar c^3} (C^{ne} \cdot C^{ek}) e^{i\omega_{kn}t} \alpha_{nk} \\
 & - \sum_k \sum_e^{E_k < E_e} \frac{2\omega_{ek}^3}{3\hbar c^3} (C^{kn} \cdot C^{ek}) e^{i\omega_{kn}t} \alpha_{en}. \quad (40b)
 \end{aligned}$$

If all the frequencies ω_{nm} are different†, averaging over time gives

$$\frac{d\overline{\alpha_{nm}}}{dt} = - \left\{ \sum_e^{E_m > E_e} \frac{2\omega_{me}^3}{\hbar c^3} |C^{me}|^2 + \sum_k^{E_n > E_k} \frac{2\omega_{nk}^3}{3\hbar c^3} |C^{nk}|^2 \right\} \alpha_{nm}, \quad (41a)$$

$$\frac{d\overline{\alpha_{nn}}}{dt} = \sum_k^{E_k < E_n} \frac{4\omega_{kn}^3}{3\hbar c^3} |C^{kn}|^2 \alpha_{kk} - \alpha_{nn} \sum_k^{E_k < E_n} \frac{4\omega_{nk}^3}{3\hbar c^3} |C^{nk}|^2. \quad (41b)$$

The latter formula also follows from elementary arguments (α_{nm} corresponds to the quantity $|a_n|^2$). The relation

$$J(\omega_{nm}) = |a_n|^2 A_{nm} \hbar \omega_{nm},$$

where

$$A_{nm} = \frac{4\omega_{nm}^3}{3\hbar c^3} |C^{nm}|^2 \quad (42)$$

is the Einstein transition probability for spontaneous emission, is also in agreement with this. The time spent in the n th state is given from (41b) as

$$\frac{1}{\tau_n} = \sum_k^{E_k < E_n} A_{nk}. \quad (43)$$

† The majority of the following results are valid also for ordinary cases of eigenvalue degeneracy (directional degeneracy, the many-body problem).

The state of minimum energy (the ground state of the atom) here evidently represents the zero state. States of higher energy which show no transitions to the zero state are, of course, only metastable; a further approximation in equation (31) gives the corresponding time for these states also.

Equation (40a) shows that the quantities α_{nm} always decrease on average in the course of time. The reason is that the intensity of the radiation behaves quite differently according to (39) and (41b), evidently, in the increasing uncertainty of the phase, which causes a reduced coherence of the radiation.† The "duration" of coherence τ_{nm} is given by equation (41a) as

$$\frac{1}{\tau_{nm}} = \lambda_{nm} = \frac{1}{2} \left(\frac{1}{\tau_n} + \frac{1}{\tau_m} \right). \quad (44)$$

Instead of the quantities α_{nm} which correspond to the unperturbed system and are now functions of time, we can also use quantities independent of time:

$$\tilde{f} = \sum_n \sum_m \alpha'_{nm} f'_{nm}, \quad (45)$$

where α'_{nm} are the arbitrary constants. If new quantities

$$\gamma_{nm} = \alpha_{nm} e^{i\omega_{nm}t} \quad (46)$$

are introduced in order to eliminate the time from (37), then (40) becomes

$$\begin{aligned} \frac{d\gamma_{nm}}{dt} = & i\omega_{nm}\gamma_{nm} + \sum_k \sum_e \frac{2\omega_{kn}^3}{3\hbar c^3} (C^{kn} \cdot C^{me}) \gamma_{ke} \\ & + \sum_k \sum_e \frac{2\omega_{em}^3}{3\hbar c^3} (C^{kn} \cdot C^{me}) \gamma_{ke} \\ & - \sum_k \sum_e \frac{2\omega_{ke}^3}{3\hbar c^3} (C^{me} \cdot C^{ek}) \gamma_{nk} \\ & - \sum_k \sum_e \frac{2\omega_{ek}^3}{3\hbar c^3} (C^{kn} \cdot C^{ek}) \gamma_{km}. \end{aligned} \quad (47)$$

Exponential solutions of this linear differential equation with constant coefficients correspond to certain relations between the γ and therefore also certain expressions f'_{nm} for \tilde{f} . In the first approximation we have, with f'^{nm} a constant,

$$\left. \begin{aligned} f'_{nm} &= e^{(i\omega_{nm} - j_{nm})t} f'^{nm} = e^{[i(E_n - E_m) - j_n - j_m]t/\hbar} f'^{nm}, \\ \text{where} \quad j_n &= \frac{1}{2} \hbar \sum_k \frac{E_k < E_n}{A_{nk}} = \sum_k \frac{E_k < E_n}{2} \frac{2\omega_{nk}^3}{3c^3} |C^{nk}|^2. \end{aligned} \right\} \quad (48)$$

† This phenomenon is only partly related to the damping process; the agency which keeps the emission in a steady state (e.g. black-body radiation, optical excitation, etc.) also causes a perturbation of coherence.

The quantities f'^{nm} are evidently linear combinations of the f^{nm} . They can easily be derived from equation (46). For the zero state we evidently have

$$f'^{00} = f^{00} \quad (49)$$

and

$$\alpha'_{00} = 1, \quad (50)$$

replacing the usual condition $\sum_n \alpha_{nm} = 1$.

For the line width (to which the preceding remark applies also) we find

$$\Delta \omega_{nm} = \pi \gamma_{nm} = \frac{1}{\hbar} (\pi j_n + \pi j_m), \quad (51)$$

and so we can define a "term width" $\Delta E_n = \pi j_n$.

Finally, I should like to express my most sincere thanks to my friend and colleague D. Ivanenko for helpful discussions and many suggestions.

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3. QUANTUM ELECTRODYNAMICS IN CONFIGURATION SPACE

The electromagnetic field and its interaction with matter are described by a Schrödinger equation in the configuration space of the light quanta. The results are identical with those of Heisenberg and Pauli.

1. INTRODUCTION

Heisenberg and Pauli¹ (henceforward cited as I and II) have derived a quantum theory of the electromagnetic field and its interaction with matter, using the method of quantised waves. It seems desirable to make use of the configuration space of light quanta, by analogy with ordinary quantum mechanics. It has been found that the form of the equations can be deduced from a small number of physically plausible assumptions. The discussion by Dirac and others² indicates that the equations are equivalent, and we shall confirm this by direct calculation in section 4.

Our treatment does not, however, avoid the difficulty that a particle has an infinite interaction with itself. The equations are therefore certainly not yet physically correct, and we do not believe that this defect can be removed by any purely formal modification.

2. THE WAVE EQUATION FOR A LIGHT QUANTUM

The opinion has sometimes been expressed that it is impossible to set up a wave equation for light quanta, because there is no conservation of their number. Pauli and Heisenberg (II, p. 190) first pointed out that this difficulty can easily be overcome by considering a system of functions in 0, 3, 6, . . . -dimensional space, of which the N th function gives the probability that N particles are present *and* that the configuration of these particles is given by the relevant point in $3N$ -dimensional space. These functions have then to be related by a set of simultaneous equations.

Let us first consider the case where no matter is present. Then the functions are independent, and only the "one-quantum" function need be considered. The three-dimensional equations must be such that their solutions correspond to all possible states of motion and polarisation of a light quantum, i. e. to all solutions of Maxwell's equations:

$$\dot{\mathbf{E}} = c \operatorname{curl} \mathbf{H}, \quad (1a)$$

$$\dot{\mathbf{H}} = -c \operatorname{curl} \mathbf{E}, \quad (1b)$$

$$\operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0. \quad (2)$$

L. Landau und R. Peierls, Quantenelektrodynamik im Konfigurationsraum, *Z. Phys.* 62, 188 (1930).

The functions must, however, differ from those solutions in having a time dependence $e^{i\omega t}$ for a light quantum of frequency ω , so that the law of conservation of energy should hold in interaction with matter. Thus we cannot require that E and H should be real. Instead we must impose another subsidiary condition, which expresses the fact that only quanta of positive energy are present. Let us examine this condition for a linearly polarised plane wave:

$$E = E_0 e^{i[\omega t + (k \cdot r)]}, \quad H = H_0 e^{i[\omega t + (k \cdot r)]}.$$

If ω is negative, it can be replaced by $-c|k|$. (In the usual notation, a positive energy corresponds to a negative frequency.) Maxwell's equations give for this case

$$-i k E = \text{curl } H, \quad k = |k|. \quad (3)$$

For any other solution, (3) must hold for each Fourier component of E and H . For brevity we define an operator $|\nabla|$ such that it transforms a function

$$\Phi(r) = \int \varphi(k) e^{i(k \cdot r)} d\Omega, \quad d\Omega = dk_x dk_y dk_z,$$

into

$$|\nabla| \Phi = \int i k \varphi(k) e^{i(k \cdot r)} d\Omega. \quad (4)$$

The notation $|\nabla|$ is justified by the fact that the repetition of this operation evidently gives the Laplacian operator. Then the subsidiary condition must be that

$$-\text{curl } H = |\nabla| E. \quad (5)$$

The compatibility of this condition with Maxwell's equations is trivial, but now H is uniquely determined by E , so that only one of the two functions need be found. Very little now remains of the relation with the field quantities, and so we shall call the wave function F and not E . Then (1a), (2) and (5) give

$$\frac{1}{c} \dot{F} = -|\nabla| F, \quad (6a)$$

$$\text{div } F = 0. \quad (6b)$$

We must also, however, include a normalisation condition which expresses the fact that one quantum is involved. This may plausibly be done as follows. Again assuming a monochromatic wave, the classical Maxwellian theory gives the total energy as

$$\frac{1}{2} \int (E^2 + H^2) dV,$$

and thus the number of light quanta as

$$\frac{1}{2\hbar|\omega|} \int (E^2 + H^2) dV.$$

Hence we call

$$\frac{1}{2\hbar|\omega|} \int (\mathbf{F} \cdot \mathbf{F}^*) dV \quad (7)$$

the number of light quanta for a monochromatic wave.† On account of the subsidiary condition we can again replace $|\omega|$ by ck and so calculate for any field function, which we take to have the form

$$\mathbf{F}(\mathbf{r}) = \int \mathbf{f}(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r})} d\Omega$$

at some given time, the number of quanta

$$N = \frac{1}{2\hbar} \int dV \int \frac{(\mathbf{f}(\mathbf{k}) \cdot \mathbf{f}^*(\mathbf{k}'))}{ck} e^{i(\mathbf{k} - \mathbf{k}' \cdot \mathbf{r})} d\Omega d\Omega'. \quad (8)$$

Corresponding to (4) we define the operator

$$\frac{1}{|\nabla|} \Phi(\mathbf{r}) = \int \frac{1}{ik} \varphi(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r})} d\Omega, \quad (9)$$

which, by Fourier's theorem, may also be written as an integral operator:

$$\frac{1}{|\nabla|} \Phi(\mathbf{r}) = \frac{1}{2\pi^2 i} \int \frac{\Phi(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} dV'. \quad (9a)$$

With this definition, (8) gives

$$\begin{aligned} N &= \frac{i}{2\hbar c} \int \left(\mathbf{F}^*(\mathbf{r}) \cdot \frac{1}{|\nabla|} \mathbf{F}(\mathbf{r}) \right) dV \\ &= \frac{1}{4\pi^2 \hbar c} \int \frac{(\mathbf{F}^*(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^2} dV dV', \end{aligned} \quad (10)$$

and the normalisation condition states that the integral (10) has the value unity.

The quantity

$$\left(\mathbf{F}^* \cdot \frac{1}{|\nabla|} \mathbf{F} \right)$$

cannot, however, be defined as a probability density, since it is not positive definite. The correct expression for the probability density has not yet been found.

3. SEVERAL PARTICLES WITHOUT INTERACTION

Let us now consider the case of an arbitrary number of light quanta and electrons. Here it is sufficient to write the equations for the case of one electron, since that of several electrons is a simple generalisation. Let N be the

† Equation (7) is permissible, since from equations (5) and (2)

$$\int (\mathbf{E} \cdot \mathbf{E}^*) dV = \int (\mathbf{H} \cdot \mathbf{H}^*) dV.$$

number of quanta; then an equation in $(3N + 3)$ -dimensional space is needed. The solutions must be products of solutions of equations (6) and a solution of the force-free Dirac equation. Thus a quantity with 4×3^N components is required. This is denoted by

$$F_{m_1 m_2 \dots m_N \rho}^N(q_1, q_2, \dots, q_N, Q, t). \quad (11)$$

The suffix ρ relates to the Dirac function, m_ν denotes the three directions in the space of the ν th quantum, $Q = Q_1, Q_2, Q_3$ are the co-ordinates of the electron, and $q_\nu = q_1^\nu, q_2^\nu, q_3^\nu$ those of the ν th quantum. For clarity, however, we shall often omit the affixes and write simply F^N . It is then understood that a differential operator which acts on the co-ordinates of the ν th particle also relates to the suffix m_ν , e.g.

$$\text{div}_\nu = \sum_{m_\nu} \frac{\partial}{\partial q_{m_\nu}^\nu}.$$

Differential operators without suffix relate to the co-ordinates of the electron. Likewise, the Dirac matrices $\alpha^1, \alpha^2, \alpha^3, \alpha^4 = \alpha$, the first three of which may also be written as a vector α , will be used without indices, and will then always act on the suffix ρ of the function following.

By definition, therefore, the function F satisfies, from equations (6a and b), the equation

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + (\alpha \cdot \nabla) + \frac{i m c}{\hbar} \alpha \right) F^N + \sum_\nu |\nabla_\nu| F^N = 0. \quad (12)$$

In the familiar manner we choose, of course, not the simple product functions but those linear combinations which remain unaltered when the points q_ν and the corresponding suffixes m_ν are permuted in the same way (principle of symmetry).

For clarity we may write these equations explicitly for $N = 0$ and $N = 1$:

$$\frac{1}{c} \frac{\partial}{\partial t} F_\rho^0 + (\alpha_{e\sigma} \cdot \nabla) F_\sigma^0 + \frac{i m c}{\hbar} \alpha_{e\sigma} F_\sigma^0 = 0, \quad (13)$$

$$\frac{1}{c} \frac{\partial}{\partial t} F_\rho^1 + (\alpha_{e\sigma} \cdot \nabla) F_\sigma^1 + \frac{i m c}{\hbar} \alpha_{e\sigma} F_\sigma^1 + |\nabla_1| F_\rho^1 = 0. \quad (14)$$

In addition, we have, of course, the subsidiary condition

$$\text{div}_\nu F^N = 0, \quad (12a)$$

that is, for instance,

$$\text{div}_1 F^1(q_1, Q) = 0. \quad (14a)$$

The normalisation condition is that

$$J_N = \left(\frac{i}{2\hbar c} \right)^N \int F^{N*} \frac{1}{|\nabla_1|} \dots \frac{1}{|\nabla_N|} F^N dW dV_1 \dots dV_N = 1, \quad (15)$$

$$dW = dQ_1 dQ_2 dQ_3,$$

$$dV_\nu = dq_1^\nu dq_2^\nu dq_3^\nu.$$

The derivation of (12) shows that both (12a) and (15) are compatible with the equations, i.e., are automatically satisfied at all times if they hold good at time zero.

4. INTERACTION

In order to take account of the mutual interaction of the particles, we must modify the equations in several ways. First of all, the law of conservation of number of light quanta is certainly no longer valid. The integral J_N in (15) can therefore no longer be regarded as constant; instead, it gives the probability that just N quanta are present. But we must still require the fulfilment of a condition of the form

$$\sum_{N=0}^{\infty} J = 1. \quad (16)$$

This condition must, of course, hold for all times, and we shall see that the possibilities in setting up the equations are thereby greatly restricted.

We must further notice that the classical equation (2) is valid only for the matter-free case and is otherwise to be replaced by

$$\operatorname{div} \mathbf{E} = \rho. \quad (17)$$

For the present, we shall leave open the question of how this equation is to be generally formulated in our theory, i.e., what must be the general equation replacing (12a), but for the special case $N = 1$, i.e. for (14a), this question is easily answered. For this expression represents the charge density at the point density at the point q on the assumption that there is an electron at the point Q . It is scarcely possible to find any plausible expression for this other than

$$\operatorname{div}_1 \mathbf{F}_q^1(q, Q) = e \delta(q - Q) \mathbf{F}_q^0(Q), \quad (18)$$

where $\delta(r) = \delta(x) \delta(y) \delta(z)$ is the three-dimensional singular function of Dirac³.

It must now be noted that the relation (5) applied only to the divergence-free part of \mathbf{E} . If \mathbf{E} is divided into divergence-free and curl-free (transverse and longitudinal) parts, it is easily seen that the former is

$$\frac{-\operatorname{curl} \operatorname{curl}}{\nabla^2} \mathbf{E}.$$

Thus the operator $|\nabla_r|$ in (12) must be replaced by

$$\frac{-\operatorname{curl}_r \operatorname{curl}_r}{|\nabla_r|}. \quad (19)$$

We can likewise require only that the normalisation integral for the transverse part of \mathbf{F} should retain the form (10). There is still an arbitrary factor in the longitudinal part. The equations acquire their simplest form, however, if the integral (10) is retained for the total \mathbf{F} . This has the intuitive meaning that a longitudinal quantum corresponds to an electric field of the same amplitude as a transverse quantum of the same wavelength.

We also have to express the effect of the field on the electron. A single electron must execute a force-free motion so long as no quanta are present, i.e. so long as only F^0 is not zero, but it will in general deviate from this motion as soon as a quantum is present. In equation (13), therefore, we must add terms depending on F^1 . These terms must, of course, be analogous to the potential terms in Dirac's equation, but with a characteristic difference; for, if we consider an electron under the influence of a given constant external electric field, its classical energy is eV , but if we regard the whole field as produced by electrons, the energy of each electron must be taken as $\frac{1}{2}eV$ to give the correct interaction, because otherwise the same energy is counted twice. This circumstance, which is quite well known in the classical theory, has the consequence that only half the potentials are to be used in the Dirac equation. Hitherto only the field strengths have been used, and the potentials must now be expressed in terms of these. This procedure is unambiguous only if the potentials are subject to a subsidiary condition, for which a convenient form is

$$\operatorname{div} A = 0. \quad (20)$$

With this condition and the equations

$$E = -\nabla \varphi - \frac{1}{c} \dot{A}, \quad H = \operatorname{curl} A$$

the potentials can be written†

$$A = -\frac{\operatorname{curl}}{\nabla^2} H; \quad \varphi = -\frac{\operatorname{div}}{\nabla^2} E. \quad (21)$$

The quantities to be used in the Dirac equation instead of φ , ψ_e and A , ψ_e are therefore, using equations (5), (19) and (21),

$$-\frac{1}{2} \frac{\operatorname{div}_1}{\nabla_1^2} F_e^1(q, Q); \quad -\frac{1}{2} \frac{\operatorname{curl}_1 \operatorname{curl}_1}{|\nabla_1|^3} F_o^1(q, Q), \quad (22)$$

so that equation (13) becomes

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} F^0 + (\alpha \cdot \nabla) F^0 + \frac{i m c}{\hbar} \alpha F^0 \\ + \frac{i e}{2 \hbar c} \left[\frac{\operatorname{div}_1}{\nabla_1^2} F^1(Q, Q) + \left(\alpha \cdot \frac{\operatorname{curl}_1 \operatorname{curl}_1}{|\nabla_1|^3} F^1(Q, Q) \right) \right] = 0. \end{aligned} \quad (23)$$

These added terms in the expression $\partial F_0 / \partial t$ also make a contribution to $\partial J_0 / \partial t$. On account of the conservation of the normalisation, this change must be compensated by a corresponding change in J_1 . We must therefore include additional terms in (14) which depend on F^0 . (The physical significance of this is that the existence of an effect of the field on the electron necessarily implies a corresponding effect of the electron on the field.) In order to derive the form of these terms, we calculate $\partial J_0 / \partial t$. It can easily be seen, however, that the longitudinal terms in (22), on account of (18), make no contribution,

† The significance of the operator $1/\nabla^2$ will be evident from the foregoing.

so that the first term in (22) may be omitted and the second replaced by

$$\frac{i e}{2 \hbar c} \left(\mathbf{a} \cdot \frac{1}{|\nabla_1|} \mathbf{F}^1(\mathbf{Q}, \mathbf{Q}) \right).$$

Finally we obtain

$$\begin{aligned} \frac{1}{c} \frac{\partial J_0}{\partial t} = & + \int F^{0*} \frac{i e}{2 \hbar c} \left(\mathbf{a} \cdot \frac{1}{|\nabla_1|} \mathbf{F}^1(\mathbf{Q}, \mathbf{Q}) \right) dW \\ & + \int F^0(\mathbf{Q}) \frac{i e}{2 \hbar c} \left(\mathbf{a}^* \cdot \frac{1}{|\nabla_1|} \mathbf{F}^{*1}(\mathbf{Q}, \mathbf{Q}) \right) dW. \end{aligned}$$

This can be transformed to

$$+ \frac{i}{2 \hbar c} \int \frac{1}{|\nabla_1|} (F^{1*}(\mathbf{q}, \mathbf{Q}) \delta(\mathbf{q} - \mathbf{Q}) \cdot e \mathbf{a}) F^0(\mathbf{Q}) dV_1 dW + \dots$$

Thus we see that, for $\sum J_N$ to be constant, we must add to (14) a term

$$e \mathbf{a} \delta(\mathbf{q} - \mathbf{Q}) F^0(\mathbf{Q}). \quad (24)$$

This term is precisely what would be expected, since $e \mathbf{a} \delta(\mathbf{q} - \mathbf{Q})$ is the expression for the current density at the point \mathbf{q} in Dirac's theory, and the term corresponds exactly to the current-density term in Maxwell's equations. This provides further confirmation of the factor $\frac{1}{2}$ in the potential terms in the zero-order equation.

The form of this added term could also have been derived from the compatibility of the divergence condition with the equations. If we form the divergence of equation (14) with the added term (24) and use (18), we obtain (13) multiplied by $\delta(\mathbf{q} - \mathbf{Q})$. This demonstrates the compatibility of the equations apart from the added terms by which (23) differs from (13). Thus two changes are necessary: firstly, exactly similar additional terms must be added to (14), and secondly, a further subsidiary condition between F^2 and F^1 must be assumed, analogous to (18). It then again follows from the constancy of the normalisation that terms analogous to (24) must be added in the equation for F^3 . The general equations are therefore

$$\begin{aligned} \left(\frac{1}{c} \frac{\partial}{\partial t} + (\mathbf{a} \cdot \nabla) + \frac{i m c}{\hbar} \alpha \right) \mathbf{F}^N - \sum_{\nu} \frac{\text{curl}_{\nu} \text{curl}_{\nu}}{|\nabla_{\nu}|} \mathbf{F}^N \\ + \frac{i e}{2 \hbar c} \left[\frac{\text{div}_{N+1}}{\nabla_{N+1}^2} \mathbf{F}^{N+1}(\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{Q}, \mathbf{Q}) \right. \\ \left. + \left(\frac{\mathbf{a} \cdot \text{curl}_{N+1} \text{curl}_{N+1}}{|\nabla_{N+1}|^3} \mathbf{F}^{N+1}(\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{Q}, \mathbf{Q}) \right) \right] \\ + e \left(\mathbf{a} \cdot \sum_{\nu} \mathbf{F}^{N-1}(\mathbf{q}_1, \dots, \mathbf{q}_{\nu-1}, \mathbf{q}_{\nu+1}, \dots, \mathbf{q}_N, \mathbf{Q}) \right) \delta(\mathbf{q}_{\nu} - \mathbf{Q}) = 0, \quad (25) \end{aligned}$$

with the subsidiary conditions

$$\text{div}_{\nu} \mathbf{F}^N = e \delta(\mathbf{q}_{\nu} - \mathbf{Q}) \mathbf{F}^{N-1}(\mathbf{q}_1, \dots, \mathbf{q}_{\nu-1}, \mathbf{q}_{\nu+1}, \dots, \mathbf{q}_N, \mathbf{Q}). \quad (25a)$$

We can also immediately state how the equations are to be generalised to the case of several electrons. No interactions between the electrons need be assumed the interaction terms between the individual electrons and the light quanta can be added. The number of equations also is not increased, since we assume a conservation of number of particles of matter. For this case we obtain†

$$\begin{aligned} & \left(\frac{1}{c} \frac{\partial}{\partial t} + \sum_p (\alpha_p \cdot \nabla_p) + \frac{i m c}{\hbar} \alpha_p \right) F_N(q_1, \dots, q_N, Q_1, \dots, Q_n) \\ & - \sum_p \text{curl}_p \text{curl}_p \frac{1}{|\nabla_p|} F^N \\ & + \frac{i e}{2 \hbar c} \sum_p \left[\frac{\text{div}_{N+1}}{\nabla_{N+1}^2} F^{N+1}(q_1, \dots, q_N, Q_p, Q_1, \dots, Q_n) \right. \\ & + \left(\alpha_p \cdot \frac{\text{curl}_{N+1} \text{curl}_{N+1}}{|\nabla_{N+1}|^3} F^{N+1}(q_1, \dots, q_N, Q_p, Q_1, \dots, Q_n) \right. \\ & \left. \left. + e \sum_{p, \nu} (\alpha_p \cdot \delta(q - Q_p)) F^{N-1}(q_1, \dots, q_{\nu-1}, q_{\nu+1}, \dots, q_N, Q_1, \dots, Q_n) \right] \right] \\ & = 0 \end{aligned} \quad (26)$$

and

$$\begin{aligned} & \text{div}_p F^N(q_1, \dots, q_N, Q_1, \dots, Q_n) \\ & = e \sum_p \delta(q_p - Q_p) F^{N-1}(q_1, \dots, q_{p-1}, q_{p+1}, \dots, q_N, Q_1, \dots, Q_n). \end{aligned} \quad (26a)$$

5. THE RELATION TO THE HEISENBERG-PAULI THEORY

In this section we shall show that the equations derived here are entirely equivalent in content to those of Heisenberg and Pauli. This shows at the same time the relativistic invariance of our equations, which does not follow rigorously from their derivation in the three-dimensional notation where preference is given to one system of co-ordinates. The invariance could, of course, also be verified directly by converting for our equations the operator Δ pertaining to a Lorentz transformation (cf. II).

To obtain a direct comparison of the two theories, it is best to transform the Heisenberg-Pauli equations, since for our purposes it is more convenient to use travelling waves instead of the standing eigen-oscillations of a cavity, and to impose a periodic boundary condition on these waves so as to obtain discrete eigen-oscillations.

It should also be noted that we write the field strengths in Heaviside units. Then the Hamiltonian function is (apart from the contribution due to matter, which remains unchanged)

$$H = \int dV \left[\frac{1}{4} \left(\frac{\partial \Phi_i}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_i} \right)^2 + \frac{1}{2} c^2 \Pi_k^2 + e \Phi_k \alpha_{es}^k \psi_s^* \psi_s \right]; \quad (27)$$

† The matrices α_p, α_p act on the suffix q_p of the function following, which belongs to the p th electron.

cf. II, equation (42). We expand Φ and Π as Fourier series:

$$\Phi_i = \frac{1}{\sqrt{4\pi}} L^{-3/2} \sum_k e^{2\pi i(k \cdot r)/L} \sqrt{\frac{cL}{k}} [e_i^{k1} (A_{k1} + A_{-k-1}) + e_i^{k2} (A_{k2} + A_{-k-2}) + e_i^{k3} q_{k3} \sqrt{2}], \quad (28)$$

$$\Pi_i = L^{-3/2} \sum_k e^{2\pi i(k \cdot r)/L} \sqrt{\frac{\pi k}{Lc}} [-i e_i^{k1} (A_{k1} - A_{-k-1}) - i e_i^{k2} (A_{k2} - A_{-k-2}) + e_i^{k3} p_{k3} \sqrt{2}]. \quad (29)$$

Here e^{k1} , e^{k2} are unit vectors perpendicular to k , and e^{k3} is parallel to k . Also $e^{k1} = e^{-k1}$, $e^{k2} = e^{-k2}$, but $e^{k3} = -e^{-k3}$. Then the following commutation relations must hold:

$$[A_{k\lambda}, A_{-k-\lambda}] = 0; \quad [A_{k\lambda}, A_{-k\lambda}] = 0;$$

$$[A_{k\lambda}, A_{-k-\lambda}] = \hbar \quad (\lambda = 1, 2), \quad (30)$$

$$[p_{k3}, q_{-k3}] = -i\hbar. \quad (30a)$$

These relations, together with the remark that Φ and Π are real quantities, and so their operators are Hermitian, justify the assumption that

$$\left. \begin{aligned} A_{k\lambda} &= \sqrt{(\hbar)} e^{-i\phi_{k\lambda}/\hbar} \cdot N_{k\lambda}^{1/2}, \\ A_{-k-\lambda} &= \sqrt{(\hbar)} N_{k\lambda}^{1/2} e^{i\phi_{k\lambda}/\hbar} \end{aligned} \right\}. \quad (31)$$

For the longitudinal part, however, another procedure is convenient: we have the auxiliary condition, which in our variables is

$$\left(p_{k3} + \frac{ie}{\sqrt{8\pi^3 c k^3}} e^{-2\pi i(k \cdot Q)/L} \right) \varphi(N_{k\lambda}, P_{k3}, Q) = 0. \quad (32)$$

The functionals which satisfy this subsidiary condition form a sub-system which does not combine with others, and for all q numbers only those matrix elements are physically significant which belong to two states compatible with the subsidiary condition.

Instead of equation (32) we now use the condition

$$\left(B_{k3} + \frac{ie}{\sqrt{8\pi^3 c k^3}} e^{-2\pi i(k \cdot Q)/L} \right) \varphi = 0, \quad B_{k3} = p_{k3} + i q_{-k3}, \quad (32a)$$

and accordingly replace p_{k3} by B_{k3} in the Hamiltonian. This leads to different functionals, but the relations between all gauge-invariant quantities remain unchanged, since in the sub-system selected by (32a) all matrix elements of q_{k3} are zero; this is easily seen by writing (32a) for $-\kappa$ and taking the complex conjugate of both sides. This possibility arises because the condition for the longitudinal part of the field strength to be real is not new, but follows from $\text{div } E = \rho$. The quantities B_{k3} satisfy commutation relations analogous to

(30), so that we can put

$$B_{k3} = -i\sqrt{\hbar} e^{-i\Theta_{k3}/\hbar} N_{k3}^{1/2} \quad (33)$$

In the term in $B_{k3} B_{-k3}$ in the Hamiltonian, it is convenient also to express one factor by means of (32a). Thus, instead of the equation in II, p. 69, we have finally

$$\begin{aligned} & [(-E + \sum N_{k\lambda} \hbar \omega_{k\lambda}) \delta_{q\sigma} - i k c (a_{q\sigma} \cdot \nabla) + m c^2 \alpha_{q\sigma}] \varphi_{\sigma} \{N_{k\lambda}, Q\} \\ & - \frac{1}{4\pi} \sum_k \sqrt{\frac{\hbar}{\pi c k^3}} \omega_k (N_{k3} + 1)^{1/2} e^{2\pi i(k \cdot Q)/L} \varphi_{\sigma}(\dots, N_{k3} + 1, \dots) \\ & + \frac{1}{\sqrt{4\pi}} \sum_{k, \lambda=1,2} e^{\sqrt{\frac{\hbar c}{k}}} \cdot \frac{1}{L} (a_{q\sigma} \cdot e_{k\lambda}) [N_{k\lambda}^{1/2} e^{-2\pi i(k \cdot Q)/L} \\ & \times \varphi_{\sigma}(\dots, N_{k\lambda} - 1, \dots, Q) + (N_{k\lambda} + 1)^{1/2} e^{2\pi i(k \cdot Q)/L} \varphi_{\sigma}(\dots, N_{k\lambda} + 1, \dots, Q)] \\ & + \frac{1}{\sqrt{4\pi}} \sum_k e^{\sqrt{\frac{\hbar c}{k}}} \frac{1}{L} (a_{q\sigma} \cdot e_{k3}) e^{-2\pi i(k \cdot Q)/L} N_{k3}^{1/2} \varphi_{\sigma}(\dots, N_{k3} - 1, \dots, Q) = 0, \end{aligned} \quad (34)$$

with the subsidiary condition

$$\sqrt{\hbar} (N_{k3} + 1)^{1/2} \cdot \varphi(N_{k3} + 1, \dots, Q) = e^{-\frac{e^{-2\pi i(k \cdot Q)/L}}{\sqrt{8\pi^3 c k^3}}} \varphi(N_{k3}, \dots, Q) \quad (34a)$$

We now form, in the familiar manner (see, e.g. Dirac²), the functions

$$F_q^0 = \varphi_q(0, 0, \dots, 0, Q),$$

$$F_q^1(q, Q) = \sqrt{4\pi} i c L^{-3/2} \sum_{k, \lambda} e_{k\lambda} \sqrt{\frac{k \hbar}{L c}} e^{2\pi i(k \cdot q)/L} \varphi_q(0, \dots, 1_{k\lambda}, \dots, Q),$$

$$F_q^2(q, q', Q) = -4\pi c^2 L^{-3}$$

$$\begin{aligned} & \times \left\{ \sum'_{k, k', \lambda, \lambda'} \frac{1}{\sqrt{2}} \sqrt{k k'} \frac{\hbar}{L c} e_{k\lambda} e_{k'\lambda'} e^{2\pi i[(k \cdot q) + (k' \cdot q')]/L} \varphi_q(0, \dots, 1_{k\lambda}, \dots, 1_{k'\lambda'}, \dots) \right. \\ & \left. + \sum_{k\lambda} k \cdot \frac{\hbar}{L c} e_{k\lambda} e_{k\lambda} e^{2\pi i(k \cdot q + q')/L} \varphi_q(0, \dots, 2_{k\lambda}, \dots), \text{ etc.} \right\} \end{aligned}$$

We now take (34) for the case where all $N = 0$; the $\varphi_q(0, \dots, 1, 0, \dots, Q)$ occurring in this equation are expressed in terms of F^1 . This gives (25) for $N = 0$. We also obtain (25) if in (33) we put all $N = 0$ except for a certain $k\lambda$ for which $N_{k\lambda} = 1$, multiply by

$$i\sqrt{4\pi} c L^{-3/2} \sqrt{\frac{k \hbar}{L c}} e_{k\lambda} e^{2\pi i(k \cdot q)/L}$$

and sum over all k and λ .

It can also easily be seen in the same way that (33a) is identical with (25a) and that the normalisation condition

$$\sum_{N_1, N_2} \dots \sum_{N_{k, \lambda}} \int \varphi_q \{N_{k, \lambda}, Q\} \varphi_q^* \{N_{k, \lambda}, \dots, Q\} dW = 1$$

leads to (16). This proves that the two systems of equations are identical.

6. MOMENTUM AND FIELD OPERATORS

It would be expected that the total momentum operator is here given by

$$-i\hbar \sum_n \nabla_n \quad (35)$$

the summation being extended over all particles (matter and light quanta). It is now immediately seen that (35) is indeed an integral of the equations of motion, since only co-ordinate differences occur explicitly in (25). It is further evident that (35) is the only quantity available for this purpose.

It is also of interest to derive the operator of the electric and magnetic field strengths at a point r . For this purpose we note that for these operators Maxwell's equations must be satisfied as q -number relations. In particular this must apply to the equation $\text{div } E = \rho$, so that the longitudinal part of the electric field must be represented by the operator

$$E_{\text{long}}(r) F^N(q_1, \dots, q_N, Q) = F_{\text{long}}^{N+1}(q_1, \dots, q_N, r, Q).$$

It would be plausible to put for the field operator in general

$$F^N \rightarrow F^{N+1}(q_1, \dots, q_N, r, Q). \quad (36)$$

This is not permissible, however, because, although the longitudinal part of (36) is Hermitian on account of the subsidiary condition (25a), the transverse part is not. We must therefore render the transverse part of (36) Hermitian, and this leads to the operator†

$$\begin{aligned} E_L(r) F_m^N \varrho(q_1, \dots, q_N, Q) \\ = \frac{1}{2} \left\{ F_{m_1, \dots, m_N}^{N+1} \varrho(q_1, \dots, q_N, r, Q) \right. \\ \left. + \frac{\nabla \text{div}_{N+1}}{\nabla_{N+1}^2} F_{m_1, \dots, m_N}^{N+1} \varrho(q_1, \dots, q_N, r, Q) \right\} \\ + i\hbar \sum_\nu \left[|\nabla_\nu| \delta_{m_\nu} - \frac{1}{|\nabla_\nu|} \frac{\partial^2}{\partial x_i \partial x_j} \right] \delta(q_\nu - r) \\ \times F_{m_1 \dots m_{\nu-1} m_{\nu+1} \dots m_N}^{N-1} \varrho(q_1, \dots, q_{\nu-1}, q_{\nu+1}, \dots, q_N, Q). \quad (37) \end{aligned}$$

† On account of the circumstance discussed in section 4, the longitudinal part of the field operator is not uniquely determined; an operator could also be defined which is identically Hermitian but satisfied Maxwell's equations only in consequence of the subsidiary condition.

It may be verified that (37) in fact satisfies Maxwell's equations, and that the conversion of Heisenberg and Pauli's operators also leads to (37).

The magnetic field strength operator can be defined similarly if it is noted that our function F is, on account of (5), simultaneously representative of the electric and magnetic field strengths, and if we again render the operator Hermitian. These operators naturally obey the same commutation rules as those of Heisenberg and Pauli.

The authors are deeply grateful to Professor Pauli for much helpful criticism.

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4. DIAMAGNETISM OF METALS

It is shown that in quantum theory even free electrons, besides spin-paramagnetism, have a non-vanishing diamagnetism originating from their orbits, which is due to the limitation of the electron orbits in the magnetic field. A few further possible inferences concerning this orbit limitation are indicated.

1. Up to now, it has been more or less quietly assumed that the magnetic properties of electrons, other than spin, are due exclusively to the binding of electrons in atoms. For free electrons, the classical zero-result is assumed for the orbital effect, on the basis that the Fermi integral of the corresponding Hamiltonian, just as the Boltzmann function, is independent of the magnetic field. However, a quantum phenomenon is thereby allowed to be neglected. In the presence of a magnetic field, the motion of the electron is obviously finite in the plane perpendicular to the field. This leads, of necessity, to a partial discreteness (corresponding to the motion in this plane) of the eigenvalues of the system, which gives rise, as will be shown, to a non-vanishing orbital magnetism.

The Hamiltonian of a free electron in a magnetic field can be written in the familiar form:

$$E = \frac{m v_1^2}{2} + \frac{m v_2^2}{2} + \frac{m v_3^2}{2} \quad (1)$$

where

$$v_1 = \frac{1}{m} \left(p_1 - \frac{e H}{2c} y \right), \quad v_2 = \frac{1}{m} \left(p_2 + \frac{e H}{2c} x \right), \quad v_3 = \frac{1}{m} p_3 \quad (2)$$

are the velocities of the systems (H is the absolute value of the magnetic field in the direction of the z -axis). The motion in the direction of the field is independent of the field and of other components of motion, and can be split off by putting p_3 simply equal to a constant, which corresponds to the Schrödinger function

$$\psi(x, y, z) = f(x, y) e^{i p_3 z / \hbar}. \quad (3)$$

The energy value of the system will then be represented as the sum of two independent terms. Now, instead of having to solve the two corresponding Schrödinger equations for the xy -motion, we can use an artificial method for deriving the energy values by writing down the commutation relationships of the component velocities v_1 and v_2 . From equation (2) it follows directly

$$[v_1, v_2] \equiv v_1 v_2 - v_2 v_1 = \frac{\hbar}{i} \frac{e H}{c m^2}, \quad (4)$$

since, as is well known $[x, y] = [p_1, p_2] = 0$, $[p_1, x] = [p_2, y] = \hbar/i$. The constant on the right-hand side of equation (4) is reminiscent of the usual p, q -commutation relation. In order to come back to that case, we can now temporarily introduce the co-ordinates P and Q by means of

$$v_1 = \frac{P}{\sqrt{m}}, \quad v_2 = \frac{e H}{c m \sqrt{m}} Q. \quad (5)$$

The commutation relation reduces into the usual form $[P, Q] = \hbar/i$. The equation referring to the energy can now be written in the form:

$$E = \frac{P^2 + \left(\frac{e H}{m c}\right)^2 Q^2}{2}. \quad (6)$$

This, however, is none other than the Hamiltonian of a linear oscillator with mass m and frequency $\omega = e H/m c$. The eigenvalues of such a system are, as is well-known, equal to

$$E = \left(n + \frac{1}{2}\right) \hbar \omega = \left(n + \frac{1}{2}\right) \frac{e \hbar}{m c} H, \quad (7)$$

where n can assume all positive integral values. Together with the z -motion this gives

$$E = \left(n + \frac{1}{2}\right) \frac{e \hbar}{m c} H + \frac{p_z^2}{2m}, \quad (8)$$

for the eigenvalue of the translational motion of the electron.

The eigenfunctions can also be determined in a simple manner. For this purpose we eliminate one of the co-ordinates, for example x , from the velocity operators (and thus also from the energy operator), by putting

$$\psi = e^{-\frac{i e H x y}{2 \hbar c}} \chi. \quad (9)$$

This gives

$$\begin{aligned} v_1 \psi &= \frac{\hbar}{i} \frac{\partial \psi}{\partial x} - \frac{e H}{2c} y \psi = e^{-\frac{i e H x y}{2 \hbar c}} \left(\frac{\hbar}{i} \frac{\partial \chi}{\partial x} - \frac{e H}{c} y \chi \right), \\ v_2 \psi &= \frac{\hbar}{i} \frac{\partial \psi}{\partial y} + \frac{e H}{2c} x \psi = e^{-\frac{i e H x y}{2 \hbar c}} \frac{\hbar}{i} \frac{\partial \chi}{\partial y}. \end{aligned} \quad (10)$$

The Schrödinger equation corresponding to this is written

$$\left\{ \left(\frac{\hbar}{i} \frac{\partial}{\partial y} \right)^2 + \left(\frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{e H}{\hbar c} y \right)^2 - 2m E \right\} \chi = 0. \quad (11)$$

This equation does not contain x explicitly; thus, its solutions can be written in the exponential form

$$\chi = e^{\frac{i}{\hbar} \sigma x} \varphi(y), \quad (12)$$

where σ is a constant and φ is no longer dependent on x . If we substitute equation (12) in equation (11), we obtain immediately for φ an oscillator equation

$$\frac{d^2\varphi}{dy^2} + \frac{2m}{\hbar^2} \left[E - \frac{m}{2} \left(\frac{eH}{mc} \right)^2 \left(y - \frac{c}{eH} \sigma \right)^2 \right] \varphi = 0, \quad (13)$$

which is just as we should expect from what has previously been said. The "equilibrium point" of this oscillator is at the point $\eta = c \sigma / e H$. Thus, we obtain finally for the complete eigenfunction of the system

$$\psi = e^{\frac{i}{\hbar} \left(p_z z + \sigma x - \frac{eH}{2c} xy \right)} \varphi_n \left[\sqrt{\frac{eH}{\hbar c}} \left(y - \frac{c}{eH} \sigma \right) \right], \quad (14)$$

where φ_n denotes the eigenfunction of the equation

$$\frac{d^2\varphi_n}{du^2} + (2n + 1 - u^2) \varphi_n = 0. \quad (15)$$

The quantity σ does not enter into the eigenvalue. Since it can assume an arbitrary value, then our problem is still degenerate in a continuous way. In order to determine the density of the eigenvalues we replace, as usual, the infinite space by a finite container with the linear dimensions A , B and C in the x -, y - and z -directions. In the z -direction the number of possible p_z -values in the interval Δp is well known, and is equal to

$$R_{\Delta p} = \frac{C}{2\pi \hbar} \Delta p. \quad (16)$$

In a quite similar manner, we obtain for the x -direction

$$R_{\Delta \sigma} = \frac{A}{2\pi \hbar} \Delta \sigma. \quad (17)$$

In the y -direction we require that the trajectory always lies in the container at a sufficient distance from the walls. Then we need not consider the influence of the " y "-walls, because of the rapid damping of φ_n with range. Since the number of trajectories colliding at the walls can be considered as small, with an adequately large container, then we can assume that this requirement gives us practically all the existing trajectories. On account of the large container dimensions, we can thus neglect also the radius of the trajectory and we can simply write

$$0 < \frac{c}{eH} \sigma < B$$

or

$$0 < \sigma < \frac{eB}{c} H. \quad (18)$$

If, now, we wish to obtain the total number of eigenvalues corresponding to the given non-degenerate quantum number n , then we have to substitute $\Delta\sigma = (eB/c)H$ in equation (17). This gives

$$R_n = \frac{eH}{2\pi\hbar c} AB = \frac{eH}{2\pi\hbar c} S,$$

where S is the area of the container sides. Altogether we have

$$R_{\Delta p, n} = R_{\Delta p} R_n = \frac{eH}{4\pi^2\hbar^2 c} V \Delta p, \quad (19)$$

thus, as was to be expected, proportional to the volume. It can be easily checked that equation (19), as a result of the limiting transition $H \rightarrow 0$, converts into the usual eigenvalue distribution of free motion. Together with the spin, we have

$$E' = E \pm \frac{e\hbar}{2mc} H, \quad (20)$$

that is to say

$$E = \frac{e\hbar H}{mc} n + \frac{p_z^2}{2m}, \quad (21)$$

so that to every $n > 0$, the double degeneracy

$$R_{n, \Delta p} = \frac{eH}{2\pi^2\hbar^2 c} V \Delta p \quad (22a)$$

corresponds, and with $n = 0$, we have

$$R_{0, \Delta p} = \frac{eH}{4\pi^2\hbar^2 c} V \Delta p. \quad (22b)$$

2. In order to obtain the magnetic properties of the system, we require, as is well known, only to evaluate the summation

$$\Omega = -kT \sum \ln \left(1 + e^{\frac{\omega - E}{kT}} \right) \quad (23)$$

over all eigenvalues; ω denotes the so-called chemical potential. The number of particles N is linked with ω through the expression

$$N = -\frac{\partial \Omega}{\partial \omega}, \quad (24)$$

and the magnetic moment through

$$M = -\frac{\partial \Omega}{\partial H}. \quad (25)$$

In our case, we have a continuous and a discrete parameter, so that the summation in equation (23) can be represented by a sum of integrals. Thus, in

order to resolve the effect more clearly, we shall start from the orbital energies of equation (8) and consider to begin with the spin only in the multiplicity. If we put

$$\frac{e \hbar}{m c} = \mu, \quad (26)$$

then we have

$$\Omega = -k T \sum_{n=0}^{\infty} \int \ln \left[1 + e^{\frac{\omega - (n+1/2)\mu H}{kT} - \frac{p^2}{2mkT}} \right] \frac{e H}{2\pi^2 \hbar^2 c} V dp_3. \quad (27)$$

If, now, for the sake of brevity we write

$$-k T \int \ln \left(1 + e^{\frac{\omega}{kT} - \frac{p^2}{2mkT}} \right) \frac{m V}{2\pi^2 \hbar^3} dp_3 = f(\omega), \quad (28)$$

then Ω assumes the form

$$\Omega = \mu H \sum_0^{\infty} f \left[\omega - \left(n + \frac{1}{2} \right) \mu H \right]. \quad (29)$$

In order to determine this sum, we can use the familiar series expansion

$$\sum_a^b f \left(x + \frac{1}{2} \right) = \int_a^b f(x) dx - \frac{1}{24} |f'(x)|_a^b. \quad (30)$$

Its admissibility requires, in general, that

$$\frac{f_{x+1} - f_x}{f_x} \ll 1. \quad (31)$$

It can easily be seen in our case that this corresponds to

$$\mu H \ll k T. \quad (32)$$

This condition is no longer fulfilled at very low temperatures and in strong fields. On account of this, the latter case should lead to a complicated, no longer linear dependence of the magnetic moment on H , which should have a very strong periodicity in the field. Because of this periodicity, it should be hardly possible to observe this phenomenon experimentally, since on account of the inhomogeneity of the existing field, an averaging will occur. If, however, we average the series in equation (29) over an interval ΔH , the condition for equation (31) will again be fulfilled, if in the "dangerous" part near $\omega - [n + (1/2)] \mu H = 0$, the change of argument is considerably larger than the difference between the two successive arguments, i. e.

$$n \mu \Delta H \gg \mu H, \quad \omega \frac{\Delta H}{H} \gg \mu H,$$

or

$$\frac{\Delta H}{H} \gg \frac{\mu H}{\omega}. \quad (33)$$

Even with the strongest possible fields ($H = 3 \times 10^5$ gauss), the right-hand side gives only 0.1 per cent with $\omega = 3$ eV.

If we now use the summation formula (30) explicitly, we obtain

$$\begin{aligned}\Omega &= \mu H \int_0^\infty f(\omega - n \mu H) dn + \frac{1}{24} \mu^2 H^2 \left| \frac{\partial f(\omega - n \mu H)}{\partial \omega} \right|_0^\infty \\ &= \int_{-\infty}^\infty f(x) dx - \frac{\mu^2 H^2}{24} \frac{\partial}{\partial \omega} f(\omega)\end{aligned}\quad (34)$$

$[f(\infty) = 0]$. The first term of this summation is independent of the magnetic field. It represents the summation in the field-free state, so that in place of equation (34) we can write

$$\Omega = \Omega_0 - \frac{\mu^2 H^2}{24} \frac{\partial^2 \Omega_0}{\partial \omega^2}.$$

From this, we obtain

$$M = - \frac{\partial \Omega}{\partial H} = \frac{\mu^2}{12} \frac{\partial^2 \Omega_0}{\partial \omega^2} H. \quad (35)$$

If we now put

$$\frac{\partial \Omega}{\partial \omega} = -N, \quad \omega = \frac{\partial F}{\partial N},$$

where $F = \Omega - \omega(\partial \Omega / \partial \omega)$ is the free energy of the system, then equation (35) becomes

$$M = - \frac{\mu^2 H}{12 \frac{\partial \omega}{\partial N}} = - \frac{\mu^2 H}{12 \frac{\partial^2 F}{\partial N^2}}. \quad (36)$$

We have thus in reality a diamagnetism, which is exactly equal to one third of the Pauli spin paramagnetism¹, for which we have, in the familiar form

$$\Omega = \frac{1}{2} \Omega_0 \left(\omega + \frac{\mu H}{2} \right) + \frac{1}{2} \Omega_0 \left(\omega - \frac{\mu H}{2} \right) = \Omega_0 + \frac{\mu^2 H^2}{8} \frac{\partial^2 \Omega_0}{\partial \omega^2} + \dots \quad (37)$$

Thus, free electrons are altogether still paramagnetic.

If the electrons are found in the periodic field of a lattice, then it is well known² that their motion can still be considered as free, in a certain sense. The principal characteristic of the effect of the magnetic field therefore remains unchanged, although the above calculation is, of course, no longer quantitatively applicable. In particular, the ratio of para- and diamagnetism changes, and it is quite possible that in actual cases the latter can also exceed the former, so that we obtain a diamagnetic substance like bismuth. However, this is only possible with a relatively powerful lattice effect, so that a quantitative theory of this phenomenon should scarcely be possible. Another effect of the

interaction consists in the fact that the diamagnetism loses its symmetry and now depends on direction, a property in which this type of diamagnetism differs from the normal atomic diamagnetism as well as from the necessarily symmetrical spin-paramagnetism.

A similar phenomenon can also take place in non-conducting substances and indeed with paramagnetic substances, where we also have a continuous eigenvalue spectrum. Here, we also get discrete eigenvalues in the magnetic field and, as a result of this—diamagnetism. This diamagnetism is quite small compared with the paramagnetism which is present, but differs from it by its asymmetry, so that perhaps it forms the main basis of the observed asymmetry in paramagnetic crystals (another reason is the so-called magnetic or relativistic interaction between spins). On this account, it is of interest to estimate the order of magnitude of the effect. This is done in the simplest manner dimensionally. The susceptibility is first of all proportional to $(e/c)^2$, since the action of the magnetic field is always introduced by eH/c . The mass of the electron m does not occur explicitly in the calculation, in this case. It plays its role in the exchange-integral, which characterises the exchange phenomena in the lattice. Moreover, only \hbar and the density N/V can still make their appearance. Clearly this leads to the expression

$$\chi \sim \frac{e^2}{\hbar^2 c^2} \left(\frac{V}{N} \right)^{1/3} J. \quad (38)$$

The exchange integral J determines, as is well-known, the Curie temperature Θ , and $k\Theta$ is of the order of magnitude of J , so that in place of equation (38) we can write

$$\chi \sim \frac{e^2}{\hbar^2 c^2} \left(\frac{V}{N} \right)^{1/3} k\Theta. \quad (39)$$

The phenomena turn out to be quite different if the external effects are of a non-periodic nature. Such effects destroy the direction-degeneracy of the motion and consequently, if they cannot be assumed to be small, the possibility that the field produces an effect of the type investigated here. This requires that the "mean free path" corresponding to this effect is small compared with the diameter of the electron orbits in the magnetic field. Since this diameter in normal fields is of the order of magnitude of a tenth of a millimetre, then even very small impurities or even powdering of the substance can suffice. Such changes of susceptibility have been detected in bismuth, and for the first case in a whole range of substances. It would be of great interest to be able to observe in these cases a change of susceptibility with field, which ought to take place according to the present theory, when $r_H \gg \lambda$ (r_H is the radius of the orbit in the magnetic field, λ is the mean free path or the dimensions of the crystal) changes to $r_H \ll \lambda$.

In conclusion, I should like to make the supposition that the phenomena which have been investigated might explain also the Kapitza effect of linear resistance changes in a magnetic field. For the admissibility of the presumed approximation of free electrons in a magnetic field, it is not necessary that r_H

be smaller than the mean free path corresponding to the lattice (which would be impossible at normal temperatures), because the interaction with the lattice oscillations involves, apart from momentum transfer, also energy transfer. However, according to the foregoing remarks, it is probably essential that r_H be considerably smaller than the mean free path of the lattice distortions, which leads, after short calculations, to the expression

$$H \gg e c \frac{N}{V} R, \quad (40)$$

where R is the specific resistance (in electrostatic units) of the crystal. If inequality (40) is not fulfilled, then the method considered here is not applicable and it can be seen quite easily that all the effects of the field must be necessarily quadratic. The field in expression (40) is in good agreement with the critical field of the Kapitza experiments, which should lend support to the theory. I have not yet succeeded in presenting a quantitative development of the theory.

At this stage, I should like to thank sincerely Mr P. Kapitza for discussions of the experimental results and for the communication of certain unpublished data.

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2. F. BLOCH, *Z. Phys.* **52**, 555 (1928).

5. NOTE ON THE SCATTERING OF HARD GAMMA-RAYS

In a communication by G. Beck under the same title (*Naturwiss.* 18, 896 (1930)) the intensity of nuclear scattering by hard γ -rays is estimated. In it, the author purports to show that this can, in heavy nuclei, represent a considerable portion of the total scattering. However, a somewhat closer observation shows that this result is based on an erroneous assumption concerning the matrix elements of the γ -transition considered. From the commutation relation follows that the equation

$$\sum \frac{2m\omega|r|^2}{\hbar} = 1$$

must always be exactly fulfilled (m is the mass of the radiating particle), where ω and r represent the frequency ($2\pi\nu$) and the matrix element of the transition, and the sum is over all possible transitions to the ground state. If we put in the Beck formulae $\hbar\omega = 3 \times 10^{-6}$ erg, $r = 10^{-12}$ cm, then the left-hand side becomes equal to 40. Consequently, a strong decrease is necessary of the scattering intensity, which varies as the square of $|r|^2$, which would result in a scattering which is at least a factor of 10,000 smaller than that of Compton scattering.

L. Landau, Bemerkung zur Streuung harter γ -Strahlen, *Naturwiss.* 18, 1112 (1930).

6. EXTENSION OF THE UNCERTAINTY PRINCIPLE TO RELATIVISTIC QUANTUM THEORY

It is shown by considering possible methods of measurement that all the physical quantities occurring in wave mechanics can in general no longer be defined in the relativistic range. This is related to the well-known failure of the methods of wave mechanics in that range.

1. INTRODUCTION

It is known that the application of the methods of wave mechanics to problems in which the speed of light cannot be regarded as infinite leads to absurd results. In the first place, states with negative mass appear in Dirac's relativistic equation¹. This difficulty arises because the relation between momentum and energy in relativity theory is quadratic, so that two energy states are possible for a given momentum. In contrast to classical ($\hbar = 0$) relativity theory, where the continuous change of all quantities means that transitions between the two kinds of state are impossible, such transitions cannot reasonably be forbidden in quantum theory.

In the second place, the interaction of a charged particle with the field produced by itself is inevitably divergent².

The infinite zero-point energy of the radiation field which occurs on quantisation of the field³ can be avoided by the use of suitable variables⁴, but it still has the effect that the energy-density matrix elements become infinite. This is very closely related to the self-energy difficulty mentioned above (see also ref. 5).

This complete failure of the theory suggests that in the range considered the physical requirements for the applicability of the methods of wave mechanics are no longer satisfied. The present paper investigates this problem.†

2. THE CONCEPT OF MEASUREMENT IN WAVE MECHANICS

The significance of any physical theory is to derive from the result of an experiment conclusions regarding the results of subsequent experiments. Thus the relations between measurements and the physical states of a system are

L. Landau und R. Peierls, Erweiterung des Unbestimmtheitsprinzips für die relativistische Quantentheorie, *Z. Phys.* 69, 56 (1931).

† The uncertainty relations on which our conclusions are based are derived mainly from discussions in Copenhagen. Professor Bohr's attitude to these relations will be described in an article to appear shortly in *Nature*.

‡ This section is essentially a development of ideas put forward by N. Bohr in his lecture at Como⁶.

of two kinds. Firstly, the measurement determines the state of the system *after* the measurement is made, and secondly it examines the state of the system which existed *before* the measurement. In classical ($\hbar = 0$) mechanics this distinction is of no importance, since the states of the system before and after the measurement can be regarded as identical.

In wave mechanics, however, the situation is quite different, since the measurement always causes a change in the state of the system, and this change is in principle impossible to determine. If the measurement had no other property, the wave-mechanics description would be neither possible nor meaningful. It is necessary to make use of another physical property of measurements, which is usually described as repeatability. This signifies that, when the same measurement is immediately repeated, the same result is certainly obtained. In this form, however, the hypothesis is physically incorrect in most cases, as will be shown in more detail below; and in this strict form it is not necessary for wave mechanics. The important point is that for any system there should exist *predictable* measurements. This means measurements such that for every result there is a state of the system in which this measurement *certainly* gives that result. For, if this requirement were not fulfilled, the state of the system after a measurement could not be described by a ψ function. This may be seen as follows. We can describe the state of the system and the measuring apparatus together by a wave function which, before the measurement, consists of a product $\psi \varphi_0$. Here ψ is the initially arbitrary wave function of the system, and φ_0 the known wave function of the measuring apparatus. After the interaction, the wave function will in general no longer be a product. If we expand it in terms of the eigenfunctions of the measuring apparatus, in the form $\sum \psi_n \varphi_n$, then ψ_n describes the state in which the system remains after the measurement. In general, the form of ψ_n depends on that of ψ . If the wave function of the system is to be deduced from an observation of the measuring apparatus, ψ_n must be independent of ψ apart from a constant factor, i.e. $\psi_n = a_n u_n$, with u_n normalised to unity. From the linearity of the wave equation it follows that a_n depends linearly on ψ , i.e. can be written in the form $\int \psi v_n^* d\tau$, with v_n any function dependent on the process of measurement. Then $|a_n|^2$ is the probability that the measurement gives the n th result. The sum of all these probabilities must be equal to unity, i.e. $\sum |a_n|^2 = 1$ independent of ψ (provided that ψ is normalised):

$$\sum a_n a_n^* = \int \sum a_n v_n \psi^* d\tau.$$

This expression must therefore always equal unity if $\int \psi \psi^* d\tau = 1$, i.e. we must have

$$\sum a_n v_n = \psi.$$

(The v_n form a complete orthogonal system). From this, however, it follows that the measurement is predictable if we take ψ to have the particular value of one of the v_n ; then only one of the a_n is not zero. The repeatability of the

measurement would signify that the v_n and the u_n are identical, and this is not in general true.†

If, however, the wave function of the system can not be determined by any measurement, it can have no physical meaning. The use of wave functions would then be as pointless as, for example, the use of the concept of paths in quantum mechanics. Thus the existence of predictable measurements is an absolutely necessary condition for the validity of wave mechanics.

The condition of repeatability cannot in general be satisfied. This is particularly seen if the time necessary for the measurement is taken into account. This time is restricted by the relation $\Delta E \Delta t > \hbar$, which has very often been stated, but which has been correctly interpreted only by Bohr⁶. Clearly it does *not* signify that the energy can not be known exactly at a given time (for in that case the concept of energy would have no meaning), nor does it mean that the energy can not be measured with arbitrary accuracy within a short time. We must take into account the change caused by the process of measurement even in the case of a predictable measurement, i.e. of the difference between the result of the measurement (v_n) and the state after the measurement (u_n). The relation then signifies that this difference causes an energy uncertainty of the order of $\hbar/\Delta t$, so that in time Δt no measurement can be performed for which the energy uncertainty in *both* states is less than $\hbar/\Delta t$.

This follows from a consideration of the time evolution of the interaction process. The method of the variation of constants shows that transitions within short times occur not only between states which satisfy the condition $E + \varepsilon = E' + \varepsilon'$ (E and E' being the energy of the system before and after the transition, ε and ε' that of the apparatus). These states are given preference by resonance only after a long time, the corresponding transition probabilities increasing greatly with time. In practice, after a time Δt , only transitions for which $|E + \varepsilon - E' - \varepsilon'| \lesssim \hbar/\Delta t$ are of importance. This fact does not, of course, contradict the strict validity of the law of conservation of energy in wave mechanics, but the energy of interaction between the system and the apparatus is also indeterminate by the same amount. In the most favourable case, where ε and ε' are precisely known, the uncertainty must be $\Delta(E - E') > \hbar/\Delta t$.

This relation has important consequences as regards the measurement of momentum. Any measurement of momentum is made by allowing the body to collide with another. In measuring a component of momentum (most simply done by collision with a plane mirror) the law of conservation of momentum is to be applied rigorously, but that of conservation of energy applies only to within $\hbar/\Delta t$, because of the unknown interaction energy. Thus to determine the

† In a measurement which occupies a short time it can easily be shown that the u_n are identical with the v_n only when the corresponding operator commutes with the energy of interaction between the system and the apparatus. In wave mechanics (neglecting relativity) this interaction energy is always a function of the co-ordinate. The only quantity for which a repeatable measurement is possible is therefore the co-ordinate. Measurements of the co-ordinate always actually have this property. It is also seen that the u_n need not in general be orthogonal, i.e. the measurement does not in general diagonalise an operator. This physical circumstance also is usually overlooked in the presentation of transformation theory.

particle momentum P we have the equations

$$p + P - p' - P' = 0,$$

$$|\varepsilon + E - \varepsilon' - E'| \gtrsim \frac{\hbar}{\Delta t};$$

$p, p', \varepsilon, \varepsilon'$, i.e. the motion of the measuring apparatus before and after the collision, may be regarded as known. Then $\Delta P = \Delta P'$ and, since $\Delta E = v \Delta P$,

$$(v - v') \Delta P > \frac{\hbar}{\Delta t}. \quad (1)$$

Thus any measurement of momentum necessarily involves a *definite* change of momentum (in addition to the unknown change which restricts the accuracy of the measurement).† This fact was first recognised by Bohr⁶. The non-repeatability of the momentum measurement in a short time is thus shown with particular clarity. Momentum measurements which last a long time, on the other hand, are meaningful only for free particles.

3. MOMENTUM MEASUREMENT IN THE RELATIVISTIC CASE

We now wish to make use of relativity, i.e. of the finite speed of propagation. There exists as yet no satisfactory relativistic quantum theory, but it is clear that here also we certainly cannot go beyond the limits imposed on the accuracy of measurement by the general principles of wave mechanics.

The scope of the relation just derived for momentum measurement is considerably extended by relativity. In the non-relativistic theory, the definite change of velocity could be made arbitrarily large, and so the momentum could be measured with arbitrary accuracy even in a short time. If, however, we take into account the fact that the velocity cannot exceed c , then $v - v'$ can be at most of the order of c , so that equation (1) gives

$$\Delta P \Delta t > \frac{\hbar}{c}. \quad (2)$$

The inequality (2) is particularly easy to derive for the state *after* the measurements. If we assume that the particle had a definite position before the measurement, then after a time Δt , on account of the finite velocity limit, the position is still known with accuracy $c \Delta t$. If the momentum after this time were determined more accurately than as given by (2), this would contradict the result $\Delta P \Delta q > \hbar$.

† Here an important point is that not every Hamilton's function can actually occur in nature; as already mentioned, the interaction function is always a function of the co-ordinates and so does not commute with the momentum. If the form of the Hamilton's function could be chosen arbitrarily, the momentum could be measured in an arbitrarily short time without change of velocity; this is a trivial deduction from the fact that co-ordinates and momenta are then equivalent.

On account of (2) the concept of momentum has a precise significance only over long times. Thus, in cases where the momentum changes appreciably within such times, the use of the concept of momentum is purposeless.

In the measurement of momentum of a charged body, in addition to the above-mentioned inaccuracy, a further perturbation of the measurement arises because the body will emit radiation in the necessary change of velocity. We shall consider only the case where the velocity of the body before the measurement is certainly small compared with c . In this case it is favourable to conduct the measurement so that after the measurement the velocity is again considerably less than c . For, if the velocity approaches c , the relation (1) gives very little benefit, while the accuracy is greatly reduced by the emission of radiation. Thus the non-relativistic formula for radiation damping can be used. The energy emitted is then

$$\frac{e^2}{c^3} \int \dot{v}^2 dt,$$

where e is the charge on the body. This energy evidently has its least value for uniform acceleration, i.e. for $\dot{v} = (v' - v)/\Delta t$, so that the energy emitted is

$$\frac{e^2}{c^3} \frac{(v' - v)^2}{\Delta t}.$$

This unknown change of energy has to be taken into account in the energy balance, and there thus arises in the momentum a further inaccuracy:

$$(v' - v) \Delta P > \frac{e^2}{c^3} \frac{(v' - v)^2}{\Delta t},$$

or

$$\Delta P \Delta t > \frac{e^2}{c^3} (v' - v). \quad (3)$$

For electrons this inequality gives no new information, since even in the most unfavourable case where $v \sim v' + c$ it gives only $\Delta P \Delta t > e^2/c^2$, and this is weaker than (2), since $e^2 < \hbar c$. For macroscopic bodies, however, the relation (3) is significant. Multiplication by (1) gives

$$\Delta P \Delta t > \frac{\hbar}{c} \sqrt{\frac{e^2}{\hbar c}}, \quad (4)$$

and in this form we shall make use of it later. The inequality (4) is, of course, valid independently of the method of measurement used, and in particular when the measurement is made by means of the charge on the body, as in the case of the Compton effect, where, in addition to the Compton scattered radiation used in the measurement, there is a further radiation corresponding to that discussed above, obtained when higher approximations are taken into account in the perturbation calculation for the interaction between the radiation and the particle⁷. (In the ordinary Compton effect with electrons this effect is of no importance, on account of the smallness of $e^2/\hbar c$.)

4. FIELD MEASUREMENT

The simplest method of measuring an electric field is to observe the acceleration of a charged test body. In order to avoid interference by magnetic fields, we use a body of very large mass and very small velocity. Let the momentum of the body before the measurement be known, and let the momentum afterwards be measured, again with accuracy ΔP . From this we can deduce the electric field strength with accuracy such that

$$e \Delta \mathcal{E} \Delta t > \Delta P. \quad (5)$$

In addition, however, the condition (4) must be satisfied in the momentum measurement. Multiplication of equations (4) and (5) gives

$$\Delta \mathcal{E} > \frac{\sqrt{\hbar c}}{(c \Delta t)^2}. \quad (6)$$

For the magnetic field strength we easily obtain the same result by considering the motion of a magnetic needle:

$$\Delta \mathcal{H} > \frac{\sqrt{\hbar c}}{(c \Delta t)^2}. \quad (6a)$$

If it is desired to measure the electric and magnetic field strengths simultaneously, then, in addition to the effects already discussed, we have to take into account the effect on the needle of the magnetic field due to the charged body and vice versa. This magnetic field is, in order of magnitude,

$$\Delta \mathcal{H} > \frac{e}{(\Delta l)^2} \cdot \frac{v'}{c}, \quad (7)$$

where Δl is the distance between the test body and the needle. If we multiply this inequality by equations (5) and (1), then (with $v = 0$) we have

$$\Delta \mathcal{E} \Delta \mathcal{H} > \frac{\hbar c}{(c \Delta t)^2} \cdot \frac{1}{(\Delta e)^2}. \quad (6b)$$

This condition differs from the product of (6) and (6a) in that $c \Delta t$ in the denominator is partly replaced by Δe .

It follows from (6), (6a) and (6b) that for $\Delta t = \infty$ the measurement can be made arbitrarily accurate for both \mathcal{E} and \mathcal{H} . Thus static fields can be completely defined in the classical sense.†

In wave fields (that is, field which are further than $c/v = \lambda$ from the bodies which produce them), it is sufficient to use (6) and (6a), because as a result of the coupling of the space and time variation nothing is discovered about the

† Our thanks are due to Professor Bohr for pointing out this situation and the significance of time in general.

field if the region of measurement has an extent less than $c \Delta t$ for a given Δt . Thus here also the measurements of \mathcal{E} and \mathcal{H} do not interfere, and to the extent that the field strengths can be measured in accordance with (6) and (6a) they can be measured simultaneously. Thus the field strengths are in accordance with the classical theory insofar as they can be defined at all. In the quantum range, on the other hand, the field strengths are not measurable quantities.†

5. MEASUREMENTS ON LIGHT QUANTA

We shall now show that in a radiation field no measurements can be carried out with certainty within a short time, i. e. measurements for which every possible result gives information about the state of the system. (Thus we do not consider such measurements as, for example, a measurement of position by means of a collision which does not occur with probability unity within the period of observation, so that, although a deflection of the test body shows that the body under measurement was at the point considered, the absence of such a deflection shows nothing.) The time necessary for the measurement depends on the state of the system. If the energy of the radiation field is approximately determined as E , we shall show that this time is greater than \hbar/E . Since the field consists of light quanta, the greatest frequency occurring in the Fourier resolution of the field can be at most E/\hbar ; if we carry out the measurements in times small compared with \hbar/E , we remain within the period of oscillation, and so the field strength may be regarded as constant during the measurements. All measurements in such short times are therefore field measurements and are subject to the inaccuracy (6). Thus, in order that an effect should be detectable, the field strength must considerably exceed $\sqrt{\hbar c/(c \Delta t)^2}$. The smallest wavelength occurring, on the other hand, is $\hbar c/E$, and so the field strength, if non-zero at any point, must be non-zero in a region of at least this extent. Consequently, the total field energy must be at least of the order $E > \mathcal{E}^2 (\hbar c/E)^3 > (\hbar c)^4/E^3 (c \Delta t)^4$, i. e.

$$\Delta t > \frac{\hbar}{E}, \quad (8)$$

in contradiction with our hypothesis. Thus measurements which do not satisfy equation (8) are impossible.

This result applies in particular, of course, when the radiation field consists of a single quantum of light. Within the time given by (8), a quantum of light is therefore undetectable, and in particular its position cannot be determined with any accuracy. In a measurement of position, the time to which that position refers is therefore indeterminate by more than \hbar/E . If the measurement of position is to be used to investigate a state, as discussed in section 2, we are interested in the position at a time up to which the state under investigation

† The inaccuracy for the field measurement with an electron found by Jordan and Pock³ is greater than (6) and therefore proves only that the electron is not a suitable means of measuring the field.

(i.e. the state whose energy was of the order of E) existed. The measurement of position indicates such a quantity with inaccuracy, at best $\Delta q > \hbar c/E$.

It might be thought that the accuracy could be improved by measuring momentum at the same time as position (within the limits given by $\Delta P \Delta q > \hbar$, of course), and seeking to deduce how far and in which direction the light quantum has meanwhile travelled. A closer examination, however, shows that the resulting accuracy can be no better than $\hbar c/E$. Thus in every state it is meaningful to give the probability of presence of the light quantum only for regions large compared with the wavelength, and the position of the light quantum is a meaningful concept only in geometrical optics.

If the number of light quanta varies appreciably within the period of oscillation, the concept of light quanta itself is meaningless.

6. MEASUREMENTS ON MATERIAL PARTICLES

Let us now investigate the corresponding relations for material particles. (We shall always speak of electrons, but the arguments of course apply for any kind of material particle.) Such particles can best be detected by means of collision processes, for example using the Compton effect. Thus the presence of an electron is demonstrated by making two momentum measurements on the light quantum and deducing from the change in momentum that a collision has occurred in between. Here, however, the course of the process depends considerably on the length of the time interval between the two measurements. Over long times the Compton effect is obtained, i.e. the momentum of the light quantum changes either not at all or by an amount determined by the initial momenta, which can be made arbitrarily large by using very hard radiation. Over very short times, however, any changes of momentum may occur, provided that the law of conservation of momentum remains valid; the sum of the energies of the electron and the light quantum need be conserved only to within $\hbar/\Delta t$, as shown in section 2. For the same reasons as in measurements on light quanta, the small momentum changes have much the greatest probability. An elementary calculation shows that the second behaviour begins when the time interval is no longer large compared with \hbar/E , where E is the approximate energy of the electron before the measurement.

Thus, if the duration of the process of measurement is made shorter than \hbar/E , the momentum of the light quantum (and therefore also that of the electron) changes by an arbitrary amount. Hence, from the fact that no measurable change of momentum has occurred, we cannot conclude that no collision has taken place. Physically this signifies that the measurement of the momentum of the light quantum destroys the initial state of the electron. We cannot ensure that the electron is found with probability unity at the first measurement: if it was in a volume δq before the measurement, a time $\delta q/c$ is necessary before we can be sure that the light has reached the electron. Since $\delta q/c > \hbar/c \delta P > \hbar/c P > \hbar/E$, we should therefore have to make several measurements before being able to detect the electron, and thus completely destroy its state before we find it. Measurements in times less than \hbar/E are therefore useless.

We now again ask how accurately this measurement can be used to derive the position of the electron at a time up to which it was in its former state. To do this, of course, only the knowledge about the velocity which is compatible with the position measurement can be used, and not, for example, the velocity in the state before the measurement. If an exact measurement of position is made, no information is obtained as regards the velocity, which thus remains indeterminate within c . The co-ordinate can therefore be derived only with an error $\Delta q > \hbar c/E$. Elementary considerations show that no higher accuracy can be achieved by measuring the momentum and co-ordinate simultaneously with any accuracies compatible with $\Delta P \Delta q > \hbar$. Thus

$$\Delta q > \frac{\hbar c}{E} \quad (9)$$

represents the limit of the accuracy with which the position of the electron can reasonably be defined. If in particular, if the velocity of the electron is not very close to c , this becomes

$$\Delta q > \frac{\hbar}{m c} \quad (10a)$$

The derivation of (10a) shows that it is valid only for electrons which are not moving too rapidly. The statement frequently found in the literature that \hbar/mc is a general limit for the accuracy of measurements of position is based on incorrect arguments.

A superficial consideration might suggest that the uncertainty relations derived above are not relativistically invariant. In reality, of course, there can be no contradiction with relativity, which has been taken into account throughout the derivations. The explanation is that the inequalities themselves need not transform in a relativistically invariant manner, since the most favourable possible measurements of a quantity need not be so when they are viewed from a moving system of co-ordinates. Thus we have only to require that the limit of accuracy should not be exceeded when such a measurement is viewed from a moving system of co-ordinates. This requirement is, of course, always satisfied.

Particular care is needed in this respect with position measurements, for here the statement of the problem is itself not relativistically invariant, but distinguishes a time axis, since we ask for the co-ordinate at a *time* up to which the unperturbed state existed.

7. MATHEMATICAL FAILURE OF THE METHODS OF WAVE MECHANICS

The above-stated unmeasurability of all wave-mechanical quantities also appears, of course, in the formalism which results when we attempt to apply the methods of wave mechanics to the relativistic case.

The most fundamental quantity in the theory, both for electrons and for light quanta, is the momentum; this is of course due to the fact that if it remains constant in time it can be defined with arbitrary accuracy, although very long times are needed for its measurement. This latter fact does not, of course, appear in the wave-mechanics formalism, and in consequence the statements of the theory regarding short times have no meaning.

The unmeasurability of the position, on the other hand, is directly expressed in the formalism. For electrons this is because the Dirac equation also allows the physically meaningless solutions with negative energy. The result of a measurement can, of course, in reality only be a wave function composed only of states with positive energy. Such states cannot, however, form an arbitrary wave packet. It is easily seen that the dimensions of a wave packet in general cannot be less than \hbar/mc . There are, it is true, special wave packets of smaller size (namely those whose centre moves at almost the speed of light[†]), but the corresponding wave functions do not form a complete system, and the state before the measurement cannot in general be expanded in terms of them. This corresponds to the result shown earlier that in short times a determination of position may sometimes chance to be possible, but a measurement cannot be carried out with certainty.

The conditions for light quanta are still more extreme in that no mathematical expression can be given for the probability density. This is seen from the fact that, on account of polarisation properties, the wave function for a light quantum must be a tensor of rank two⁴. The probability density and current must form a four-vector, which is impossible, because they depend quadratically on the wave function. In geometrical optics it is, of course, possible to construct wave packets in which all effects vanish outside a certain region. But here also these wave functions do not form a complete system.

The unmeasurability of the field strength is shown by the fact that in empty space (no light quanta) the field strength operator^{2,4} is not zero, but even the expectation of the square of the field strength is infinite. This is related to the fact that for $\Delta t = 0$ we have from (6) an infinite indeterminacy of the field strength.

8. CONCLUSIONS

We have seen that no predictable measurements can exist for the fundamental quantities of wave mechanics (except when these quantities are constant in time, and then an infinitely long time is needed for an exactly predictable measurement). It cannot, of course, be formally demonstrated that there are not in nature some particularly complicated quantities for which predictable measurements are possible, but such a speculation need not be discussed. The assumptions of wave mechanics which have been shown to be necessary in section 2 are therefore not fulfilled in the relativistic range, and the application of wave mechanics methods to this range goes beyond their scope. It is

[†] Our thanks are due to Professor O. Klein for pointing this out.

therefore not surprising that the formalism leads to various infinities; it would be surprising if the formalism bore any resemblance to reality.

The applicability of wave mechanics is restricted to processes where the state of the system varies sufficiently slowly. In cases where the ordinary Schrödinger's equation is applicable this is of course not always true. For radiation alone wave mechanics is never meaningful, since the limit $c = \infty$ has no meaning.

In the correct relativistic quantum theory (which does not yet exist), there will therefore be no physical quantities and no measurements in the sense of wave mechanics. One can, of course, cause the system to interact with some apparatus and ask what happens to the latter. The theory will give a probability for the result of this experiment, but this cannot be interpreted as the probability of a parameter of the system under investigation, since it can in no way be ensured that the probability of a given result is unity and that of all other results is zero. In addition, it is in principle impossible to make the duration of such an experiment arbitrarily short.

This view is confirmed by the known fact that the β -spectra of radioactive nuclei are continuous, although the uniform lifetime indicates that the nuclei are not in different states. For, if all the β -particles had the same energy, the process could be regarded as a predictable measurement.

This fact presents an insuperable difficulty in wave mechanics because, as Bohr has emphasised, it means that the law of conservation of energy is probably invalid for nuclear electrons. This law is indissolubly connected with the foundations of wave mechanics. In relativistic quantum theory, however, the preceding discussion shows that the concept of energy need not be mechanically definable. It is of course definable in a certain sense in terms of the total mass of the nucleus, because the nucleus in its motion as a whole satisfies wave mechanics, but this does not imply a predictable measurement of quantities related to the internal state of the nucleus.

If the law of conservation of energy is not valid, then in radioactive processes the mass of the whole system will of course change, but this change cannot be followed in the course of time, since the mass cannot be measured in an arbitrarily short time. If we consider the process of measurement of the mass as in section 3, the time needed for the measurement is such that

$$\Delta m \Delta t > \frac{\hbar}{c^2}$$

The preceding discussion is not contradicted by the fact that the spectra of protons and α -particles are discrete. On account of their large mass (low velocity) these particles obey wave mechanics even in the nucleus, rather as the nuclei in a molecule can be essentially described in classical terms despite their strong interaction with the electrons, for which classical mechanics fails completely.

One of us (Landau) thanks the Rockefeller Foundation for making it possible for him to work in Copenhagen and in Zürich.

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7. A THEORY OF ENERGY TRANSFER ON COLLISIONS

A theory of "adiabatic", inelastic collisions is developed. The application of this theory to atomic collisions of the second kind leads to the result that the azimuthal quantum number of the entire system always changes by ± 1 on collision, forbidding a transition from two S -states to two other S -states. The corresponding reaction cross-section is proportional to $(E - U)^{3/2}/E$ where U is the energy at the intersection of the two terms, at which the transition always occurs. The case of polyatomic molecules is discussed generally. The process of nuclear excitation and splitting without alpha capture is considered.

1. Electron collisions for the theoretically most important case of fast incident electrons were treated by Bethe. The problem of collisions of the second kind has, however, till now given rise merely to false results. Kallmann and London¹ for instance, treat the excitation of electrons on collision when the motion of the nucleus is fixed. This is, however, contradicted by known wave-mechanical treatment, and leads to a strange vanishing of the energy. In a paper by Morse and Stückelberg this error is avoided, but only by taking the opposite case and treating the interactions between the atom as a perturbation, which has still less bearing on reality. Only recently has attention been drawn to the fundamental role of the intersection of the energy curves². However, no quantitative theory has been developed. The basis of the difficulties seems to be that we have here a kind of "quasi-classical" process; the strong periodicity of the wave function makes a direct estimate of the perturbation integral impossible. G. Beck³ has attempted to solve this for the case of a Coulomb field, but he has not really answered the problem as he has substituted for the continuous field an infinite potential barrier. This allows no evaluation of the correct order of magnitude of the effect. The aim of this article is to estimate this value.

2. Let us consider a system which is comprised of a "quantised" part as well as a "quasi-classical" part. This is to say a part for which the wavelength is small compared to the dimensions of the system. Then to a first approximation the eigenfunction of the system is made up of the product of the eigenfunction of the quantised part in the fixed quasi-classical part and the eigenfunction of the classical part in the external field produced by the state of the quantised part. In this first approximation there is no energy exchange between the two parts of the system, and the quasi-classical part behaves "adiabatically". If we wish to estimate the probability of energy transfer

we must go to the next order of magnitude and evaluate the perturbation integral. This we know is given by the expression $\int \psi'^* D \psi d\tau$ where D is the Schrödinger operator for the entire system. (For an exact solution $D\psi = 0$.) The integration for the quantised variables is carried out in the normal manner, but for the quasi-classical variables this is not possible. The corresponding parts of ψ are strongly periodic functions, which does not allow an evaluation in the normal manner. However, in order to solve the integral we can make use of just these classical properties by applying a canonical transformation. We introduce as a variable the action variable of the motion of the quasi-classical part in one of the two states. It is well known that we can perform this canonical transformation by subtracting from the differential of the action another total differential action of the kinetic potential function of the co-ordinates and of the action variables.

$$dS = \sum p dq - dS_1(I_1, q). \quad (1)$$

If we calculate the change of this action function with regard to the action variable of the first state for the motion of the second state, that is for given I_2 , we have by integration:

$$S = S_2(I_2, q) - S_1(I_1, q). \quad (2)$$

Here I_1 is considered as an independent variable and I_2 as a known quantity and q as functions of I_1 and I_2 . The corresponding ψ function for I_1 has the form $\psi_2(I_1) = \alpha \exp(iS/\hbar)$. The integral $\int \psi_2 \eta \psi_1 d\tau$ (η is the perturbation energy) is equal to $\psi_2(I_1) \eta(I_1)$ since ψ_1 is a δ -function of I_1 . Here I_1 is not a variable, but denotes the value of this variable in the first state. The essential part of this expression is of the form:

$$\exp \left\{ \frac{i}{\hbar} [S_2(I_2, q) - S_1(I_1, q)] \right\}. \quad (3)$$

As we should expect, we have an equation which is symmetrical in the two systems. The co-ordinates q can be obtained as functions of I_1 and I_2 by elimination of the momentum in the expressions $I_1(q, p)$ and $I_2(q, p)$ or, equivalently, from the equations:

$$p = f_1(I_1, q) = f_2(I_2, q). \quad (4)$$

The physical meaning of this is that the important points for the transition are those at which both q and p remain unaltered. This implies more here than merely the derivation of the so-called Franck-Condon rule, as it is valid not only for the real, but also for the imaginary part.

If the condition (4) is not applicable to the real part then the expression (3) defines the probability when we take the imaginary solution for which the expression $i(S_2 - S_1)/\hbar$ corresponds to the smallest absolute value of the real part which must naturally always be taken to be negative. If only the energies of the two states are specified, then the condition

$$E_1 - U_1(q) = E_2 - U_2(q) \quad (5)$$

must be fulfilled. As we are dealing with radiationless transitions, we set $E_1 = E_2$, giving:

$$U_1(q) = U_2(q) \quad (6)$$

The intersections of the potential energy curves or (6) (more generally, of the multi-dimensional surfaces) are important.

As far as the order of magnitude of the perturbation integral is concerned we can thus say that it is relatively large if the curves $U_1 = U_2$ cut one another in the real region and thus in the region of real kinetic energy. On the other hand it becomes exponentially small if this condition is not fulfilled. It is also easy to see what a "near miss" for the intersecting curves means: nothing more than that the curves just cut in the imaginary region. In certain circumstances it can happen that the intersection lies so close to the real axis that the real part of S becomes very small, and that we no longer have an exponentially vanishing expression.

3. Now let us consider more exactly the collision of two atoms. We write the Hamiltonian for the whole system in the well-known form:

$$H = H_e + \frac{p^2}{2m}, \quad (7)$$

where H_e is the Hamiltonian of the electrons, m the reduced mass ($1/m = 1/m_1 + 1/m_2$) of the nuclei, and p is the nuclear momentum. We can split the kinetic energy of the nuclei into its radial and transverse components. We can now write H in the form:

$$H = H_{e,r} + \frac{m^2}{2m r^2}, \quad (8)$$

where $H_{e,r}$ now contains the radial kinetic energy as well as H_e and m is the nuclear angular momentum. Finally, we introduce the total angular momentum, M , of the system, and the electronic angular momentum \mathfrak{A} instead of the nuclear angular momentum:

$$H = H_{e,r} + \frac{1}{2m r^2} (M - \mathfrak{A})^2. \quad (9)$$

The operator $H_{e,r}$ contains only radial operators, and so clearly only those matrix elements correspond to it which give rise to terms with the same electronic angular momentum. But by a well-known rule⁴ such terms may only cross to terms with different total spin. But owing to the small transition probability these terms are unimportant for us, and the corresponding perturbation elements can be neglected. The operator M is, of course, a diagonal matrix. There exist transitions for \mathfrak{A} where the electronic angular momentum quantum number changes by 0, ± 1 . Thus \mathfrak{A}^2 is also diagonal with respect to k . The remaining perturbation energy is thus only

$$\eta = - \frac{1}{m r^2} (\mathfrak{A} \cdot M), \quad (10)$$

and we can thus formulate the law that for atomic collisions the transitions with $\Delta k = \pm 1$ (thus from Σ to Π , Π to Σ and Δ , and so on) are by far the most probable. Since, for example, from an S and a P term both a Σ and a Π term can arise, this selection rule, in fact, says little. However, a transition from two S states to two S states is expressly forbidden since two S states always give rise only to a Σ term. It must be emphasised here that this condition is indeed necessary, but by no means sufficient, since it only starts with terms, but not those which must cross. Thus not all $\Delta k = \pm 1$ correspond to a transition on collision.

We next evaluate the matrix element for η with respect to all co-ordinates except r . We can then consider the angular momentum M as a normal vector in our quasi-classical treatment. This gives:

$$\eta(r) = - \frac{M}{m r^2} D(r), \quad (11)$$

where M is the absolute value of the angular momentum of the system, and D the matrix element of the electronic angular momentum corresponding to the transition $\Delta k = \pm 1$. This is, of course, also a function of the co-ordinates r .

The general formula of perturbation theory gives for the probability of a collision transition:

$$\omega = \left(\frac{|\eta|}{\hbar} \right)^2, \quad (12)$$

where $|\eta|$ is the matrix element of η in which the ψ functions are normalised for a flux of 4π atoms per unit time. In the quasi-classical approximation we have:

$$\psi = \frac{1}{\sqrt{\pi v(r)}} \frac{\cos[S(r)/\hbar]}{r}. \quad (13)$$

Here v is the speed expressed as function of the co-ordinates. Substitution of this expression gives:

$$\begin{aligned} \sqrt{\omega} &= \int \frac{\eta}{\hbar} \psi_1 \psi_2^* 4\pi r^2 dr = - \int \frac{M D(r) \cos[S_1(r)/\hbar] \cos[S_2(r)/\hbar] 4\pi dr}{m r^2 \hbar \sqrt{\pi v_1(r)} \sqrt{\pi v_2(r)}} \\ &= - \int \frac{4 M D(r) \cos[S_1(r)/\hbar] \cos[S_2(r)/\hbar]}{\hbar m r^2 \sqrt{v_1(r) v_2(r)}} dr. \end{aligned} \quad (14)$$

In this integral, as we have already said, we are considering the element with the difference of the action functions, or:

$$\sqrt{\omega} = - \int \frac{2 M D(r) \cos\{[S_1(r) - S_2(r)]/\hbar\}}{\hbar m r^2 \sqrt{v_1(r) v_2(r)}} dr. \quad (15)$$

We must take only that part of this integral coming from the neighbourhood of the intersection of the two curves. Here we take all quantities except the

rapidly changing cosine to be constant. We also expand the action function in powers of the distance from the intersection. We take into account that $dS/dr = p$ and we determine $d^2S/dr^2 = dp/dr$ from $p^2/2M + V = E$ (V , of course also, contains the centrifugal energy). From this, we have

$$v \frac{dp}{dr} = F, \quad (16)$$

where $F = -dv/dr$ is the force (including the centrifugal force) at the point r . Since at the intersection $p_1 = p_2$ and $V_1 = V_2$, and we have

$$S_1 - S_2 = A + \frac{F_1 - F_2}{2v} (r - r_0)^2, \quad (17)$$

where A denotes the values of $S_1 - S_2$ at the intersection. The integral is now of the form:

$$\sqrt{\omega} = - \int_{-\infty}^{+\infty} \frac{2MD}{\hbar m r^2 v} \cos \left[\frac{A}{\hbar} + \frac{F_1 - F_2}{2\hbar v} \xi^2 \right] d\xi,$$

where $\xi = r - r_0$ is the new integration variable. All variables take their values at the point of intersection. We have chosen $\pm\infty$ as integration limits since the region important for the integration lies anyway near the intersection. We have

$$\int_{-\infty}^{+\infty} \cos(\alpha + \beta \xi^2) d\xi = \sqrt{\frac{\pi}{\beta}} \cos\left(\alpha + \frac{\pi}{4}\right), \quad (19)$$

and thus

$$\sqrt{\omega} = - \frac{2MD}{\hbar m v r^2} \sqrt{\frac{2\pi\hbar v}{F_1 - F_2}} \cos\left(\frac{A}{\hbar} + \frac{\pi}{4}\right), \quad (20)$$

or

$$\omega = \frac{8\pi M^2 D^2}{\hbar m^3 v r^4 (F_1 - F_2)} \cos^2\left(\frac{A}{\hbar} + \frac{\pi}{4}\right). \quad (21)$$

The phase A/\hbar changes very quickly with different M , corresponding to the quasi-classical character of the system. Thus we can replace the expression, $\cos^2(A/\hbar + \pi/4)$ by its mean value $\frac{1}{2}$ even over a small interval. This gives finally:

$$\omega = \frac{4\pi M^2 D^2}{\hbar v (F_1 - F_2) m^2 r^4}.$$

We now insert the value of the speed v . We have

$$\frac{1}{2} m v^2 + V = E, \quad (22)$$

or, when we write out the centrifugal energy explicitly,

$$\frac{1}{2} m v^2 + U + \frac{M^2}{2m r^2} = E. \quad (23)$$

For the force we have $F = -dV/dr = -d/dr(U + M^2/2mr^2)$, or, since the two states are only distinguishable by the form of U , $F_1 - F_2 = d/dr(U_2 - U_1)$. We thus have for ω :

$$\omega = \frac{4\pi D^2 M^2}{\hbar m r^4 (F_1 - F_2) \sqrt{2m(E - U) - \frac{M^2}{r^2}}} \quad (24)$$

To obtain the total reaction cross-section we need only to take into account that $M = m v_0 \varrho$ where v_0 is the speed at infinity, and ϱ is related to the largest possible reaction cross-section by $d\lambda = 2\pi \varrho d\varrho$. We then have:

$$d\lambda = \frac{2\pi M dM}{m^2 v_0^2} = \frac{\pi M dM}{m E} \quad (25)$$

The total reaction cross-section is then clearly:

$$\sigma = \int \omega d\lambda = \int \frac{4\pi^2 D^2 M^3 dM}{\hbar m^2 E r^4 (F_1 - F_2) \sqrt{2m(E - U) - \frac{M^2}{r^2}}}. \quad (26)$$

The integration is to be taken from $M=0$ to that M for which the speed at the intersection is still real, i.e. up to $M^2/r^2 = 2m(E - U)$. If we choose $M/r\sqrt{2m(E - U)}$ as variable, we obtain:

$$\begin{aligned} \sigma &= \frac{4\pi D^2 [2m(E - U)]^{3/2}}{\hbar m^2 E (F_1 - F_2)} \int_0^1 \frac{x^3 dx}{\sqrt{1 - x^2}} = \frac{4\pi D^2 [2m(E - U)]^{3/2}}{\hbar m^2 E (F_1 - F_2)} \cdot \frac{2}{3} \\ &= \frac{2^{9/2} \pi D^2}{3 \hbar \sqrt{m} \frac{d}{dr} (U_2 - U_1)} \frac{(E - U)^{3/2}}{E}. \end{aligned} \quad (27)$$

This final formula describes the reaction cross-section as a function of the collision velocity. We can estimate its order of magnitude from the fact that the denominator contains \sqrt{m} . We should thus expect that when E is equal to a few volts, the reaction cross-section is about 100 times smaller than the so-called kinetic cross-section of the atom, in other words that an effective collision only occurs about once every few hundred times. If U is negative effective collisions can also result at smaller speed with relatively great prob-

ability. For small E , σ is inversely proportional to E and σ can then become appreciable for small E . The reaction cross-section reaches a minimum at $E = -2U$ and then increased as \sqrt{E} for large E , i.e. proportional to the speed. For positive U , a noticeable effect begins at $E = U$ and increased steadily with increasing E .

To calculate the angular distribution we must know not only the value of U at the intersection, but also the precise nature of the function. If we know this, we can obtain the deflection, Θ , as a function of M from the classical equations of motion. Here we would assume that the transition always takes place from the surface of the "sphere of intersection" so that the co-ordinates and momentum remain unaltered.

4. The relationships are much more complicated for polyatomic molecules. In any case, there is in general no crossing of the terms for complicated molecules except when the transition corresponds to different spin values. This case can thus be significant, although the probability for it is relatively small (reduced by the order of the relativistic correction). In the case of complicated molecules an approximately intersecting may occur. The available experimental material is too limited to enable a thorough investigation of the various cases which occur. For collisions between atoms and diatomic molecules there are generally terms having various mirror symmetry with respect to the nuclear plane which can cross. Then we should have conditions analogous to those for atomic collisions. On the other hand, for two diatomic molecules there are no crossing points in the general case, and an intersection is only possible for certain symmetrical conditions. For this reason two diatomic molecules do not in general pass through an intersection during their motion. There are, however, certain directions of motion (in multi-dimensional co-ordinate space) which do possess this property. Thus, the probability of a transition here depends on the product of the expression considered above with the probability that the collision occurs in a particular direction, and is thus distinctly smaller.

5. A quite different application of the theory developed above arises in the case of nuclear excitations or reactions due to a passing α -particle. In this case we always have a Coulomb field at large distances from the nucleus, both for the incoming and for the outgoing α -particle. In other words, at large distances $U_2 - U_1$ is constant and equal to the energy given up by the α -particle. An intersection of the energy curves is not possible at large distances. This must always happen somewhere in the region of the nuclear radius (whether the intersection is real or imaginary is here of no interest to us). The important part of the probability is thus determined by the value of the exponential function $\exp[2i(S_2 - S_1)/\hbar]$ in the region of the nucleus. But for small r_0 :

$$2iS = -2 \int_{r_0}^{\infty} \sqrt{\frac{2m}{\hbar^2} \left(\frac{2Ze^2}{r} - E \right)} dr \approx -\frac{4\pi Ze^2}{\hbar v} + \frac{8e\sqrt{mZr_0}}{\hbar}, \quad (28)$$

so that we have for the difference

$$2i(S_2 - S_1) = -\frac{4\pi Ze^2}{\hbar} \left(\frac{1}{v_2} - \frac{1}{v_1} \right), \quad (29)$$

which is independent of r_0 . We obtain an expression for the effective nuclear radius which is entirely independent of the nature of the potential energy near the nucleus

$$\sigma = A \exp \left[- \frac{4\pi Z e^2}{\hbar} \left(\frac{1}{v_2} - \frac{1}{v_1} \right) \right], \quad (30)$$

where A is a factor which only varies slowly with the speed. We obtain an entirely analogous expression for the case of the emission of a proton from the nucleus. The only difference is that in the case where there is no resonance, the proton has another potential barrier to pass through, which leads to an expression:

$$\sigma = B \exp \left\{ - \left[\frac{4\pi Z e^2}{\hbar} \left(\frac{1}{v_2} - \frac{1}{v_1} \right) - \frac{2\pi(Z-1)e^2}{\hbar v_p} \right] \right\} \quad (31)$$

(v_p is the speed of the proton). In the case of a resonance reaction (with regard to the proton) the previous expression is valid. Resonance as far as the α -particle is concerned should be considered as a normal α -particle capture. The formula (30) was applied by G. Gamow at my suggestion to the experimental results of reactions. It was, however, established that in all cases known so far we are dealing with the capture of an α -particle. The same seems to be the case for the observed artificial gamma radiation.

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8. ON THE THEORY OF STARS

From the theoretical point of view the physical nature of stellar equilibrium is considered.

THE astrophysical methods usually applied in attacking the problems of stellar structure are characterised by making physical assumptions chosen only for the sake of mathematical convenience. By this is characterised, for instance, Milne's proof of the impossibility of a star consisting throughout of classical ideal gas; this proof rests on the assertion that, for arbitrary L and M , the fundamental equations of a star consisting of classical ideal gas admit, in general, no regular solution. Milne seems to have overlooked the fact, that this assertion results only from the assumption of the opacity being constant throughout the star, which assumption is made only for mathematical purposes and has nothing to do with reality. Only in the case of this assumption the radius R disappears from the relation between L , M and R necessary for regularity of the solution. Any reasonable assumptions about the opacity would lead to a relation between L , M and R , which relation would be quite exempt from the physical criticisms put forward against Eddington's mass-luminosity relation.

It seems reasonable to try to attack the problem of stellar structure by methods of theoretical physics, i.e. to investigate the physical nature of stellar equilibrium. For that purpose we must at first investigate the statistical equilibrium of a given mass without generation of energy, the condition for which equilibrium being the minimum of free energy F (for given temperature). The part of free energy due to gravitation is negative and inversely proportional to some average linear dimensions of the system in question, or, in other words, it is proportional to $-\varrho^{1/3}$ (ϱ some average density).

The remaining inner part of free energy depends on the equation of state; for the classical ideal gas it is proportional to $\log \varrho$. In view of the fact that $\log \varrho$ tends to infinity for $\varrho \rightarrow \infty$ more slowly than $\varrho^{1/3}$ we will always have a minimum of free energy at $\varrho = \infty$. That means that, in the case of classical ideal gas, we obtain no equilibrium at all. Every part of the system would tend to a point. The state of affairs becomes quite different when we consider the quantum effects. For the non-relativistic Fermi-gas the inner free energy is changing with ϱ as $\varrho^{2/3}$, that means, more rapidly than the gravitational. It would lead to the existence of a stable equilibrium. The presence of sources would produce only an additional expansion due to the radiation pressure. Milne tries to escape from this conclusion by introducing a condensed inner part of the system, but he does not tell the reasons why such condensations could appear at all. The connection of the condensed state with the normal

state remains rather mysterious. It is easy to see that any equation of state leading to no discontinuous transitions (as admitted in Milne's calculations) would never make such condensations possible.

As the velocities of electrons in the Fermi-distribution rise with the density we have to apply, for sufficiently great densities, the relativistic theory. In the extreme-relativistic case the inner free energy per unit volume varies as $\rho^{1/3}$, i.e. the same power of density as the gravitational energy. The free energy F is therefore of the form $F = a \rho^{1/3}$. If a is positive the system will expand in order to have F minimum, until the density becomes too small for the extreme-relativistic relation $F = a \rho^{1/3}$ to be valid. If a is negative the system will have a tendency to collapse to a point. In order to find the criterion separating these two cases we have to investigate the solution of the general equation

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\mu}{dr} \right] = -4\pi G \rho. \quad (1)$$

The chemical potential μ is in the extreme-relativistic case equal to $\hbar c [3\pi^2 \rho / m^4]^{1/3}$ where m is the mass per electron, that means for most elements two protonic masses. The equation (1) is the $n = 3$ polytropic equation of Emden. With known substitutions it can be brought to the form

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dx}{dr} \right] = -x^3$$

with the boundary condition $-r^2(dx/dr) = M[G/\hbar c]^{3/2} m^2 [4/2\pi]^{1/2}$ on the outer boundary. As Emden has shown, this equation has a regular solution only in the case $-r^2(dx/dr) = 2.015$, in which case it admits arbitrary radii; that means, we will have an indifferent equilibrium corresponding to $a = 0$ in the former rough treatment. Thus we get an equilibrium state only for masses greater than a critical mass $M_0 = (3 \cdot 1/m^2) [\hbar c/G]^{3/2} = 2.8 \times 10^{33}$ g or about $1.5 \odot$ (for $m = 2$ protonic masses). For $M > M_0$ there exists in the whole quantum theory no cause preventing the system from collapsing to a point (the electrostatic forces are by great densities relatively very small). As in reality such masses exist quietly as stars and do not show any such ridiculous tendencies we must conclude that *all stars heavier than $1.5 \odot$ certainly possess regions in which the laws of quantum mechanics (and therefore of quantum statistics) are violated*. As we have no reason to believe that stars can be divided into two physically different classes according to the condition $M >$ or $< M_0$, we may with great probability suppose that all stars possess such pathological regions. It does not contradict the above arguments, which prove only that the condition $M > M_0$ is sufficient (but not necessary) for the existence of such regions. It is very natural to think that just the presence of these regions makes stars stars. But if it is so, we have no need to suppose that the radiation of stars is due to some mysterious process of mutual annihilation of protons and electrons, which was never observed and has no special reason to occur in stars. Indeed we have always protons and electrons in atomic nuclei very close together, and they do not annihilate themselves; and it would be very strange if the high temperature

did help, only because it does something in chemistry (chain reactions!). Following a beautiful idea of Professor Niel Bohr's we are able to believe that the stellar radiation is due simply to a violation of the law of energy, which law, as Bohr has first pointed out, is no longer valid in the relativistic quantum theory, when the laws of ordinary quantum mechanics break down (as it is experimentally proved by continuous-rays spectra and also made probable by theoretical considerations)¹. We expect that this must occur when the density of matter becomes so great that atomic nuclei come in close contact, forming one gigantic nucleus.

On these general lines we can try to develop a theory of stellar structure. The central region of the star must consist of a core of highly condensed matter, surrounded by matter in ordinary state. If the transition between these two states were a continuous one, a mass $M < M_0$ would never form a star, because the normal equilibrium state (i.e. without pathological regions) would be quite stable. Because, as far as we know it is not the fact, we must conclude that the condensed and non-condensed states are separated by some unstable states in the same manner as a liquid and its vapour are, a property which could be easily explained by some kind of nuclear attraction. This would lead to the existence of a nearly discontinuous boundary between the two states.

The theory of stellar structure founded on the above considerations is yet to be constructed, and only such a theory can show how far they are true.

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9. A THEORY OF ENERGY TRANSFER. II

The results of I are applied to: 1, predissociation; 2, excitation of oscillations during an optical transition; 3, the case of closely approaching potential curves. The question of the cross-section is discussed.

The results obtained in I¹ can be applied to a number of effects. It must be added, however, that during the collision of similar atoms it is necessary to add to the selection rules developed in I the postulate of the change in parity which is evident from the fact that the electron-spin components which are normal to the line connecting the nuclei change their sign when mirrored.

1. The most direct generalisation is the predissociation. For the dissociation probability during an oscillation we can use here expression (21) directly. This can also be written in the following form:

$$\omega = \frac{4\pi \hbar D^2}{(F_1 - F_2) m^2 r^4} \cdot \frac{j^2}{v}. \quad (1)$$

The proportionality to j^2/v can be found in a paper by Kronig² where, however, v designated the velocity at infinity instead of the actual radial velocity at the intersection, the reason lying in the estimates used by Kronig.

The treatment given in I can also be applied to the case of an intersection made possible by the difference of the total spin. Elementary considerations lead to the following selection rules: variation of the total spin by ± 1 , in addition to the optical selection rules for the azimuthal quantum number and possibly the parity. Now, since the matrix element no longer depends on the nuclear velocity one obtains instead of (1) a result simply proportional to $1/v$ as might be deduced from the time spent at the intersection.

2. A very special case of the application of the theory is the excitation of oscillations during an optical transition when the temperature corresponds to a non-oscillating molecule. In this case, the energy of the most highly excited state is determined by the Franck-Condon rule and the co-ordinates of the points of lowest energy of the normal state. When calculating the drop in intensity when moving away from this most highly excited state we have, according to the theory developed here, to determine the intersections of the following curves:

$$E_1 - U_1 = E_2 - U_2. \quad (2)$$

When denoting the distance of the intersection from the normal state by ξ and considering that the parabola of the lower state can for this purpose be

L. Landau, Zur Theorie der Energieübertragung II, *Physik. Z. Sowjet.* 2, 46 (1932).

replaced by a constant value of U and the upper curve by a straight line, one obtains approximately:

$$\varepsilon = F \xi, \quad \text{or,} \quad \xi = \frac{\varepsilon}{F}, \quad (3)$$

where F is the force acting on the upper curve, and ε the energy difference from the most highly excited state. Now, we only have to determine the action integrals up to the point ξ . The integral along the parabola is obviously the decisive one. One hence obtains for the exponent:

$$2 \int \frac{m \omega \xi d\xi}{\hbar} = \frac{m \omega}{\hbar} \xi^2 = \frac{m \omega}{\hbar} \left(\frac{\varepsilon}{F} \right)^2 \quad (4)$$

($\omega = 2\pi \cdot \text{frequency of normal oscillation}$). It is seen that the intensity distribution is Gaussian. The width of the intensity distribution is given by

$$\Delta \varepsilon = \sqrt{\frac{\hbar}{2m\omega}} F. \quad (5)$$

It thus usually includes many oscillation levels.

3. It is interesting to determine the probability when the energy curves approach closely to one another. This is invariably the case when two ions have at infinity an energy which is not very different from the energy of two atoms. Then the attractive Coulomb force gives an intersection in a region where all the other interactions, which might destroy the intersection, are still very small.

When the curves intersect in the near complex plane, we must extend our action integral into a region where the two potential energies approach one another very closely. we can write accordingly

$$p_1 - p_2 = \frac{\partial p}{\partial U} (U_1 - U_2) = - \frac{\partial p}{\partial E} (U_1 - U_2) = - \frac{U_1 - U_2}{v}, \quad (6)$$

where v is the velocity in the given region. The value of the probability is consequently represented by the quantity

$$\exp \left[- \frac{2i}{\hbar v} \int (U_1 - U_2) dx \right]. \quad (7)$$

According to a well-known formula we have, however:

$$U_1 - U_2 = \sqrt{(U_{10} - U_{20})^2 + 4a^2},$$

where U_{10} and U_{20} are the eigenvalues in zeroth approximation and a the "exchange energy" of the two terms. The width of the smallest distance is obviously equal to $2a$.

In the vicinity of the intersection we can write, except for an unimportant constant,

$$U_{10} = F_1 x, \quad U_{20} = F_2 x. \quad (8)$$

In the next approximation, the position of the intersection is given by $(F_1 - F_2)^2 x^2 + 4a^2 = 0$ whence follows

$$x = \frac{2ia}{F_1 - F_2}. \quad (9)$$

The exponent of (7) is consequently equal to

$$-\frac{2i}{\hbar v} \int_0^{\frac{2ia(F_1-F_2)}{(F_1-F_2)^2 x^2 + 4a^2}} \sqrt{(F_1 - F_2)^2 x^2 + 4a^2} dx = -\frac{2}{\hbar v} \int_0^{\frac{2a(F_1-F_2)}{\sqrt{4a^2 - (F_1-F_2)^2 \xi^2}}} \sqrt{4a^2 - (F_1 - F_2)^2 \xi^2} d\xi \quad (10)$$

The probability is therefore proportional to

$$\exp \left[\frac{-2\pi a^2}{\hbar v (F_1 - F_2)} \right]. \quad (11)$$

To estimate the exponent in the above Coulomb case, we put

$$F = \frac{e^2}{r^2} = \frac{\Delta^2}{e^2}, \quad (12)$$

where Δ is the energy difference at infinity. In the case of an ion collision v is simply determined by Δ . If $\Delta = 1$ eV, the condition:

$$\frac{4\pi a^2}{\hbar v} \frac{1}{F_1 - F_2} = 1$$

leads to $2a = 0.06$ eV.

4. When the exponent in (11) is small, we can neglect to a first approximation the splitting up of the curves.

The probability of the electron transition is therefore given by the probability of the transition in the case of two intersecting curves. This case can be treated by the method used in part I with the sole difference that the place of the matrix elements of $\partial M/mr^2$ is taken by a which hence gives for the probability

$$\frac{4\pi a^2}{\hbar v (F_1 - F_2)}. \quad (13)$$

In the intermediate case the probability will be of the order of magnitude 1. To express this probability in terms of the effective cross section, we introduce the following relations. The effective cross section in the case $\omega = 1$ is $2\pi \rho d\rho$, where ρ is the minimum distance without interaction. Because of the conservation of angular momentum $\rho v_0 = r v_\phi$ (r being the co-ordinate of the intersection, and v_ϕ the angular velocity at the intersection) we have in consequence:

$$d\sigma = \frac{2\pi r^2}{v_0^2} v_\phi dv_\phi, \quad (14)$$

or, since $v_r^2 + v_0^2 = v_0'^2$ equals a fixed value for a given energy,

$$d\sigma = \frac{2\pi r^2}{v_0^2} v_r dv_r. \quad (15)$$

Now, when multiplying by ω and integrating (v in (13) means of course v_r) with respect to all values of v_r from zero to v_0' , we obtain

$$\sigma = \frac{8\pi^2 a^2 r^2 v_0'}{\hbar F v_0^2}, \quad (16)$$

or

$$\sigma = \frac{4\pi^2 a^2 r^2 \sqrt{2m(E-U)}}{\hbar F E}. \quad (17)$$

5. The statements made in I require, on the other hand, considerable modification when the two energy states are situated close together at infinity (resonance case). This may be brought about either by transitions of fine structure levels into one another or by an accidental approach of the levels of the two atoms. To determine the range of validity of the previous statement, we calculate the action difference for the two curves which do not intersect in the real region. When denoting the order of magnitude of the complex value of r at the intersection by R we can write:

$$S_1 - S_2 = \frac{R \Delta}{v}, \quad (18)$$

where Δ is the energy difference of the two terms at infinity (R corresponds to the order of magnitude of r at which the interaction energy is of the order of the energy difference).

For the exponent we have:

$$\frac{R \Delta}{\hbar v}. \quad (19)$$

Here, too, we can therefore differentiate between two cases depending on whether the exponent is small or large compared with 1. In the first case the energy difference between the two states can be neglected. In this case we may apply for the calculation the method of Kallmann and London³. When estimating the exponent it will be convenient to put $R = 10^{-7}$ cm, since the interaction decreases rapidly for larger distances. When assuming for v the thermal velocity of about 3×10^4 cm/sec, one obtains Δ of the order of 0.1 meV. On the other hand, if the exponent is large, the intersection of the curves again plays an important part and we must use (I.27).

The author would like to thank L. Rosenkevich for valuable discussions.

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10. ELECTRON MOTION IN CRYSTAL LATTICES

It is well known that in a periodic field an electron can move without resistance. When the lattice is slightly distorted at a point, this only leads to scattering of the electrons at this point. This, however, does not mean the electron is trapped at this point. According to a familiar theorem in wave mechanics this will only be possible if, in addition to continuous eigenvalues, the distorted lattice would also have discrete eigenvalues. But this is not the case for slight distortions.

Let us consider a free electron, subjected in a certain region to a weak field. we can then demonstrate in accordance with Peierls¹ that the solution of the Schrödinger equation at $E = 0$ has no nodes at weak fields, that is it corresponds to the lowest possible eigenvalue. For, when determining the solution of the Schrödinger equation

$$\nabla^2 \psi = \frac{2mU}{\hbar^2} \psi \quad (1)$$

for small U in the form

$$\psi = 1 + \chi, \quad (2)$$

where χ is also small, one obtains:

$$\nabla^2 \chi = \frac{2mU}{\hbar^2}. \quad (3)$$

If U decreases at infinity more rapidly than $1/r^2$, then this equation has a solution finite throughout, and whose values are proportional to those of U . For a sufficiently small U one therefore has $|\chi| < 1$ hence $1 + \chi$ vanishes nowhere. (When denoting the dimension of the region where U is different from zero by a , we find that a discrete eigenvalue can only exist when mUa^2/\hbar^2 is of the order of unity.)

An analogous proof is possible for a periodic lattice by taking as starting point the solution corresponding to the lowest eigenvalue which is consequently nodeless for a strictly periodic field, and by writing the "distorted" ψ in the form $\psi = \psi_0 + \chi$.

Hence a small distortion does not yet lead to the possible trapping of the electron. This possibility only exists for large distortions. We can now differentiate between two essentially different cases. For, the energetically most favourable state of the total system may correspond, firstly, to the undistorted lattice and the electron moving about "freely" and, secondly, the electron

L. Landau, Über die Bewegung der Elektronen im Kristallgitter, *Phys. Z. Sowjet.* 3, 664 (1933).

trapped at a strongly distorted region. In the first case, the electron cannot be trapped at all by the lattice. This situation seems to be realised in the case of diamond. In the second case, the electron can only be trapped when passing over an energy barrier. For, as already stated, in the case of a small distortion, the eigenvalues of the electron are not changed. Hence the energy variation of the total system consists solely in the distortion energy and thus is essentially positive. We must therefore expect that the trapping of the electron is associated with activation effects. This corresponds to the situation in the case of NaCl which cannot be discoloured by X-rays at low temperatures. It would be interesting to verify in this effect the $\exp(-A/kT)$ law and to determine the value of the activation energy A .

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11. ON THE SECOND LAW OF THERMODYNAMICS AND THE UNIVERSE

The irreversibility of thermodynamic phenomena is discussed. It is seen that this irreversibility, as also in general the difference between the two time directions (past and future) can only be explained by the fact that there exist regions in the universe which are not covered by the theories (classical and wave mechanical) leading to thermodynamics.

It seems to have been accepted since Boltzmann that the irreversibility of the second law does not contradict in any way the symmetry of classical mechanics as regards the interchangeability of the sign of the time. The justification for this argument can be seen from fact that the behaviour of a macrosystem isolated

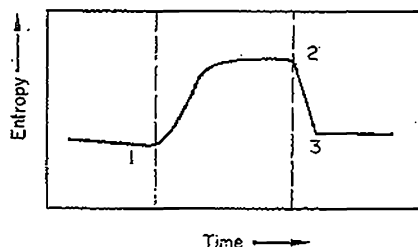


FIG. 1

for a period large in comparison to the relaxation time is completely symmetrical in both time directions. If, however, we have a system in thermodynamic equilibrium with its surroundings, and we isolate this system for a period which is not so large that the important fluctuations can be ignored, then the entropy curve behaves as shown in Fig. 1 (the beginning 1 and end 2 of the isolation are denoted by the dotted lines). When the system is isolated (point 1) the entropy begins to rise, reaches a maximum, and remains at this level until point 2, after which the thermodynamic equilibrium starts to reassert itself. If we change the direction of the time axis, the picture is completely different. In fact the reversed process means that at point 3 the entropy begins to increase, as though the system knew in advance that after a certain time it would be isolated. We are inclined to dismiss this behaviour as impossible, "since the

M. Bronstein und L. Landau, Über den zweiten Wärmesatz und die Zusammenhängeverhältnisse der Welt im Großen, *Phys. Z. Sowjet.* 4, 114 (1933).

future can have no influence on the behaviour of physical bodies." It is, however, a philological point, for it is far from clear how we can conclude from classical mechanics that one time direction possesses different properties from the other. The problem is namely to determine the origin of this asymmetry.

To do this we must consider the exact meaning of the second law somewhat more carefully. Let us consider a body which is isolated for a very long time (compared to the relaxation time). If we think of the volume of this body as partitioned into a number of relatively small parts 1, 2, 3, . . . (each, however, containing many atoms), then every part can be in statistical equilibrium, even though the body as a whole is far from being in equilibrium. The macroscopic description of this state is given apportioning a statistical distribution function, ϱ_i , to the i th part of the total system. These ϱ_i are functions of the number of atoms and of the point in phase space of the system comprised of this number of atoms. Thus we have $\int \varrho_i d\tau_i = 1$, where we understand the integral to include not only all physically different points of the phase space, but also the subsequent summation over all possible numbers of atoms. The parts, 1, 2, 3, . . . are quasi-independent of one another; that is to say a change of the initial state of such a relatively small part of the total body has virtually no effect on the state of the entire body. The quasi-independence of the parts is expressed mathematically in that when we evaluate the function $\varrho = \prod \varrho_i$ at t and transform it according to the laws of mechanics to refer to t' , so that ϱ changes to ϱ' , then we obtain the statistical partition function of the i th part at t' through the integration:

$$\varrho'_i = \int \dots \int \varrho' d\tau_1 d\tau_2 \dots d\tau_{i-1} d\tau_{i+1} \dots$$

The change of the macrostate between t and t' is thereby specified. For the entropy we obtain

$$\sigma = \sum_i \overline{\ln \frac{1}{\varrho_i}} = \overline{\ln \frac{1}{\varrho}},$$

and

$$\sigma' = \sum_i \overline{\ln \frac{1}{\varrho'_i}} = \overline{\ln \frac{1}{\prod_i \varrho_i}} = \overline{\ln \frac{1}{\varrho'}} - \overline{\ln \frac{\prod_i \varrho'_i}{\varrho'}}.$$

With the help of the inequality

$$\ln x \leq x - 1$$

we have

$$\overline{\ln \frac{\prod_i \varrho'_i}{\varrho'}} \leq \overline{\ln \frac{\prod_i \varrho'_i}{\varrho'}} - 1 = 0.$$

This gives

$$\sigma' \geq \overline{\ln \frac{1}{\varrho'}},$$

but from Liouville's theorem we have

$$\overline{\ln \frac{1}{\varrho'}} = \overline{\ln \frac{1}{\varrho}},$$

whence

$$\sigma' \geq \sigma.$$

This inequality has a two-fold meaning: (1) we consider a body isolated for an interval large in comparison with the relaxation time. If we seek out all the microstates which have been attained in the interval and which describe one and the same macroscopic state, then in the next instant from the vast majority, of these microstates will arise states of greater or at least equal entropy; (2) the great majority of these microstates come from the immediately preceding states of greater entropy.

The necessity of these two assertions is seen from the fact that in the proof introduced above, no use was made of the inequality $t < t'$. Naturally this is connected with the symmetry of mechanics regarding the changeability of the sign of the time. The two statements are independent of each other; that is, if we know nothing about a microstate except that it does not fulfil the conditions connected with the first statement, then there is a high probability that it will fulfil those of the second statement and vice versa. (Naturally, in the majority of cases considered which specify one and the same macroscopic state, both sets of conditions are satisfied. This leads to a characteristic, completely symmetrical appearance of the fluctuation curve, which has been much discussed in the literature.) Of interest to us is the following sequence: let us take on the fluctuation curve (i.e. on the curve of the entropy change over a period of time large in comparison with the relaxation time) an interval comparable with the relaxation time, about which nothing is known except that the end t_0 (which end is unimportant) corresponds to the entropy σ_0 , which is less than or equal to σ . Then we will obtain a series of inequalities $\sigma_0 \leq \sigma_1 \leq \sigma_2 \dots$ corresponding to the monotonic series of sequential moments, $t_0, t_1, t_2 \dots$ belonging to the time interval under consideration. This law, which we can call the law of *monotonic entropy change*, can also be expressed as follows: when a body is isolated over a period which is not so big that the fluctuations may be ignored, and when, moreover, nothing else is known of this time interval than possibly the macrostate at one of its two ends, then we may postulate that the entropy changes monotonically during this time interval.

These "thermodynamic" inferences of the classical statistics are more general than the classical statistics itself: as is known they may also be deduced from the quantum statistics based on wave mechanics. Only in relativistic conditions are the thermodynamic considerations no longer valid; for example the diverging integral density of the radiation energy in the Rayleigh-Jeans law means simply that statistical equilibrium in this region does not exist.

Nor do we expect thermodynamics to be upheld in the case of relativistic quantum theory. In the following we will, for brevity, denote the systems which obey the classical and wave mechanics as "statistical bodies". If a statistical body is isolated from its surroundings for a period which is not too large compared with the relaxation time, its entropy changes monotonically. If two such bodies are isolated, then the monotonic changes cannot occur in mutually opposed directions. This is immediately clear, since these two systems can be connected by an arbitrarily weak coupling into one single system, at any rate when their periods of isolation are simultaneous or partly so. If we apply the proof introduced above of the monotonic entropy change in this case of two weakly linked parts of one system, then we can see immediately that the directions of the entropy changes of the parts cannot be opposed to each other. If different systems are not simultaneously isolated from their surroundings, then we can introduce to the argument any necessary intermediary steps. We can now see that either the entropy of every statistical body which is not isolated from the surrounding universe for too long is constant, or there is a particular universal time direction for entropy increase.

Which of these two possibilities occurs in nature? If we consider the world as an individual isolated macro-system, obeying classical or wave mechanics, then we are led to the conclusion, as was Boltzmann, that the world as a whole is in statistical equilibrium. Boltzmann attempted to reconcile this conclusion with the apparent lack of statistical equilibrium in the observable parts of the world through his well-known fluctuations hypothesis. The fact that we are able to observe such a terrific fluctuation, he attempted to explain by saying that precisely the occurrence of such a variation is a necessary condition for the existence of the observer (the favourable conditions for the biological evolution of organism and so on). The argument is, however, entirely false. For in such a world of fluctuations, the existence of an individual observer without the heavens containing myriads of stars corresponds to a far greater entropy, and therefore to a much higher probability. We conclude from this that the world obeying statistics would display no important fluctuations; in such a world all inequalities in the law of monotonic entropy change would be replaced by equations, and both directions of the time axis would be entirely equivalent.

Everyday experience shows, however, that this is not the case. This alone is quite sufficient (even without Bohr's hypothesis that the inner layers of the stars may only be described in terms of relativistic quantum theory) to conclude that there are at least regions in the world which do not obey the statistics. This makes the two directions of the time axis non-equivalent; the direction which is associated with the increase of the entropy of a system isolated over not too long a period from its surroundings can be defined as the future. For classical mechanics the conceptions of past and future are entirely without meaning. Only then do we obtain the second law of thermodynamics as the *law of increase of entropy*. This law, as well as the existence of past and future observed in everyday experience, is only possible because the world as a whole does not obey the laws of thermodynamics.

12. A POSSIBLE EXPLANATION OF THE FIELD DEPENDENCE OF THE SUSCEPTIBILITY AT LOW TEMPERATURES

THE chlorides of chromium, bivalent iron, cobalt and nickel have the property that at low temperatures their magnetic susceptibility increases more rapidly as the temperature decreases than predicted by Curie's law, which according to the usual concepts ought to indicate the presence of ferromagnetism.

At low temperatures, however, these materials do not show ferromagnetism but only a field dependence of the susceptibility.

I should like to draw attention to an explanation of this anomaly which follows directly from the theoretical analysis and is in agreement with the known facts. Since the above elements may form both ferro-alums and paramagnetic compounds, we may draw the conclusion that the mutual orientational forces of the spins may be either positive or negative, probably depending chiefly on the interatomic spacing.

All these materials form crystal lattices in which the paramagnetic atoms are arranged in layers, the spacing between the atomic layers being considerably larger than the interatomic spacings in the layer. Now, if we assume that the orientational forces within a layer are positive, but those between different layers are negative, and also much smaller, we obtain the following picture. At low temperatures we have spontaneously magnetised layers whose magnetic moments are, however, oriented in opposite directions so that we have no spontaneous magnetisation of microscopic regions and hence no ferromagnetism. Since the mutual interaction between different layers is relatively small, a relatively small field is sufficient to modify the mutual orientation of the moments considerably. One thus obtains deviations from the linear law of the dependence of the total moment on the field, and finally one obtains even saturation effects, because the magnetic field orientates the magnetic moments of all layers parallel to its direction.

To arrive at a quantitative theory of this phenomenon we must consider three different effects: 1. The saturation magnetisation for each separate layer the temperature dependence of which is given approximately by a Weiss curve. 2. The exchange effect between different layers which as a first approximation may be assumed proportional to the scalar product of the magnetic moments of these layers. 3. The (relativistic) interaction of the magnetic moments of each separate layer with the lattice which through symmetry, is

L. Landau, Eine mögliche Erklärung der Feldabhängigkeit der Suszeptibilität bei niedrigen Temperaturen, *Phys. Z. Sowjet.* 4, 675 (1933).

to a first approximation proportional to the square of the component of the magnetic moment parallel to the symmetry axis.

Thus, in the light of the above hypotheses the materials should therefore, as with ferromagnetics, have a Curie point where the saturation magnetisation of the layers equals zero. By the same analogy, one can expect at this Curie point a discontinuity of the specific heat whose value can be calculated just as in the case of ferromagnetics.

To investigate the susceptibility in the region of the Curie point, we consider the free energy which may be made up of the three above components. In the field-free case, we may assume that the magnetic moments of the layers are situated in pairs in opposite directions so that the layers can be divided into two different kinds. This division is then also retained in the presence of a magnetic field. One thus arrives at the following free energies:

1. The free energy per atom for the exchange effect within a layer which we expand in powers of the mean moment per atom of the layer, retaining only the first terms:

$$F_1 = \frac{1}{2} a m_1^2 + \frac{1}{4} b m_1^4, \quad (1)$$

(odd powers naturally do not occur on symmetry grounds), and the analogous expression

$$F_2 = \frac{1}{2} a m_2^2 + \frac{1}{4} b m_2^4 \quad (2)$$

for the layers of the second kind.

2. The exchange energy of the two kinds layers of per pair of atoms:

$$F_3 = A (m_1 \cdot m_2). \quad (3)$$

3. The free energy for the interaction of the moment with the lattice

$$F_4 = \frac{1}{2} \alpha (m_{1x}^2 + m_{1y}^2), \quad F_5 = \frac{1}{2} \alpha (m_{2x}^2 + m_{2y}^2), \quad (4)$$

and, finally, the energy in the external field

$$F_6 = -(\mathbf{H} \cdot \mathbf{m}_1), \quad F_7 = -(\mathbf{H} \cdot \mathbf{m}_2). \quad (5)$$

The total free energy per atom is thus equal to

$$F = \frac{1}{2} (F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7) \quad (6)$$

we now put

$$\frac{1}{2} (m_1 + m_2) = m; \quad \frac{1}{2} (m_1 - m_2) = l. \quad (7)$$

Then, as is readily calculated, we have:

$$\begin{aligned} F = \frac{1}{2} (a - A) l^2 + \frac{1}{4} b l^4 + \frac{1}{2} \alpha (l_x^2 + l_y^2) + \frac{1}{2} (a + A) m^2 + \frac{1}{2} \alpha (m_x^2 + m_y^2) \\ + \frac{1}{2} b m^2 l^2 + b (\mathbf{m} \cdot \mathbf{l})^2 + \frac{1}{4} b m^4 - (\mathbf{H} \cdot \mathbf{m}). \end{aligned}$$

When the external field equals zero, the total moment per atom is also equal to zero. The value of l can then be calculated from the minimum postulate for the first three terms. We assume a to be positive, i.e., the Z -axis is assumed to be the most favourable direction of magnetisation. We then have a minimum or maximum of the free energy depending on whether $a - A$ is positive or negative.

At the Curie point we thus have $a - A = 0$, and we can write:

$$a = A + \beta(-\Theta + T),$$

where Θ is the Curie temperature, and β is positive. The other quantities are assumed to be approximately independent of temperature. Above the Curie point $l = 0$, and the fraction of the free energy dependent on m , thus assumes the following form:

$$[2A + \beta(T - \Theta)] m_z^2 + [2A + \alpha + \beta(T - \Theta)] (m_x^2 + m_y^2) - H_x m_x - H_y m_y - H_z m_z,$$

where, in addition, the term with m^4 has been neglected. From the minimum condition it follows that:

$$m_x = \frac{H_x}{\beta(T - \Theta) + \alpha + 2A}, \quad m_y = \frac{H_y}{\beta(T - \Theta) + \alpha + 2A},$$

$$m_z = \frac{H_z}{\beta(T - \Theta) + 2A}.$$

We thus have for the susceptibility per atom:

$$\chi_x = \chi_y = \frac{1/\beta}{T - \Theta + (\alpha + 2A)/\beta}, \quad \chi_z = \frac{1/\beta}{T - \Theta + (2A/\beta)}.$$

Below the Curie point the minimum condition for l gives:

$$\beta(T - \Theta) + b l^2 = 0, \quad l^2 = \frac{\beta(\Theta - T)}{b}. \quad (8)$$

Now, when substituting this expression into the term dependent on m , one obtains:

$$\left[A + \frac{1}{2} \beta(T - \Theta) + \frac{1}{2} b l^2 \right] m^2 + \frac{1}{2} \alpha (m_x^2 + m_y^2) + b l^2 m_z^2$$

or, by substituting in equation (8):

$$[A + \beta(\Theta - T)] m_z^2 + \left(A + \frac{1}{2} \alpha \right) (m_x^2 + m_y^2) - H_x m_x - H_y m_y - H_z m_z.$$

From this we now obtain in analogous fashion:

$$\chi_x = \chi_y = \frac{1/\beta}{\alpha + (2A/\beta)}, \quad \chi_z = \frac{1/\beta}{2(\Theta - T) + (2A/\beta)}. \quad (9)$$

Thus, at high temperatures, one really has a Weiss law with a negative Θ . But the Curie temperature is higher than Θ so that at the Curie point χ attains a finite value. The expressions derived here show that when the temperature is lowered further χ must continue to decrease further (which, it is true, is only the case for some of the above-mentioned substances. The question of the reason for this discrepancy is left open). At low temperatures one observes, in accordance with the expressions (9), a specially marked anisotropy in that χ_x and χ_y become large compared with χ_z . $A/\beta \Theta$ gives the ratio between the exchange effect of two layers and the exchange within a layer, and must therefore always be small. The same also holds for $\alpha/\beta \Theta$, which represents the ratio of the relativistic effects to the exchange effect.

This statement is, of course, related to the assumption $\alpha > 0$. A comparison with experiment is not possible at present since no measurements on single crystals are available.

13. INTERNAL TEMPERATURE OF STARS

It may be of interest to notice that the investigation of the process of thermal transformation of light elements in stars¹ enables us to check the upper limit for the temperature of internal regions. In fact, so far as lithium is present, for example, on the star surface, it is natural to accept that it is in equilibrium with the lithium content in the internal regions of the star near the stellar nucleus, where the production of different elements takes place. On its way from the stellar nucleus through the hot regions of the star, lithium atoms will be partly destroyed by thermal collisions with hydrogen atoms (${}^7\text{Li} + {}^1\text{H} \rightarrow 2{}^4\text{He}$) and will not reach the surface at all if the temperature of the internal regions is too high.

For the rate of the reaction in question we have:

$$\omega \sim \int \pi \left(\frac{\hbar}{m v} \right)^2 \cdot e^{-2\pi Z e^2 / \hbar v} \cdot v \cdot N \cdot 4\pi v^2 \left(\frac{m}{2\pi k T} \right)^{3/2} \cdot e^{-m v^2 / 2 k T} dv, \quad (1)$$

where Z is the atomic number of the element, v and N the velocity and density of protons, and T the absolute temperature.

Calculating the integral we obtain:

$$\omega \sim N \hbar^{5/3} \frac{(4\pi Z e^2)^{1/3}}{m^{4/3} (k T)^{2/3}} e^{-3/2 (m/k T)^{1/2} (2\pi Z e^2 / \hbar)^{2/3}}. \quad (1a)$$

On the other hand, in the time $1/\omega$ a lithium atom will travel through the distance

$$l \sim \sqrt{\frac{D}{\omega}} \quad (2)$$

where the diffusion coefficient D is given by the expression:

$$D \sim \frac{(k T)^{1/2}}{N^1} \sigma M^{1/2} m^{1/2}. \quad (2a)$$

Here N^1 is the total number of atoms in a cubic centimetre, σ the cross section of collision, and M the atomic weight of the atoms in question.

Using (1a) and (2a), we obtain from (2):

$$l \sim \frac{m^{5/12} (k T)^{7/12}}{\sqrt{N N^1} \sigma^{1/2} \hbar^{5/6} M^{1/4} (4\pi Z e^2)^{1/6}} e^{3/4 (m/k T)^{1/2} (2\pi Z e^2 / \hbar)^{2/3}} \quad (3)$$

G. Gamow and L. Landau, Internal temperature of stars, *Nature*, **132**, 567 (1933).

Accepting $N \sim N^1 \sim 10^{24} \text{ cm}^{-3}$ and $\sigma \sim 10^{-18} \text{ cm}^2$, we obtain for lithium ($Z = 3$) the following numbers:

$T (C^\circ)$	10^6	5×10^6	10^7	5×10^7	10^8
$l (\text{cm})$	10^{12}	10^6	10^4	10	1

From this table the conclusion is reached that either lithium is present on the star surface only occasionally or that no regions with temperatures of more than several millions of degrees can exist in the interior of a star.

REFERENCE

1. ATKINSON and HOUTERMANS, *Z. Phys.* 54, 656 (1929).

14. STRUCTURE OF THE UNDISPLACED SCATTERING LINE

THE fine structure of the undisplaced line produced by light scattering has been investigated a number of times¹ in recent years, yet the problem does not seem to us to have been completely solved either from the experimental or theoretical point of view. We propose to here give a few results from a theoretical study in this connection, the complete publication of which is delayed for reasons beyond our control.

If we start by confining ourselves to the scattered radiation connected with density oscillations, we find from the theory of fluctuations that the undisplaced line splits up into a triplet in the case of liquids and gases above a certain degree of dilution (see below for limits of validity). The two outer components represent the familiar Brillouin-Mandelstam doublet with angle-dependent separation $\Delta \nu = \pm \nu(v/c) 2 \sin \frac{1}{2} \theta$ (ν is the velocity of sound); but there is also an undisplaced component, and the contribution of the two outer components to the intensity of the triplet is in fact given to a good approximation by the ratio c_p/c_v of the specific heats. For a monatomic ideal gas, for instance, the relative intensities of the three components are 3: 4: 3; and the intensity of the central component is also for most liquids ($c_p/c_v \sim 1.4$) of the same order as that of the two outer ones. On approaching the critical point, the intensity of the two outer components remains constant, whilst that of the central component increases sharply.

The width of the three components can be found quantitatively; it is determined by the viscosity and the thermal conductivity.

These results only hold for gases as long as $l \ll \lambda/2 \sin \frac{1}{2} \theta$ (l is the mean free path). The three components gradually merge together for longer mean free path, and when $l \gg \lambda/2 \sin \frac{1}{2} \theta$ the line structure reaches the Gaussian form with an angle-dependent width such as is produced by the Doppler effect.

The situation is more complicated for that part of the Rayleigh line that has no connection with density fluctuations. In liquids, the decisive factor for the structure of this part is the Debye relaxation time. A more detailed discussion of this subject and of the situation in crystals, as also a comparison with experimental results, will be found in a more detailed paper.[†]

REFERENCE

1. Cf. in particular, GROSS, *Nature* 126 201, 400, 603 (1930); 129, 722 (1932). For further experimental literature see, e.g. MEYER and RAMM, *Phys. Z.* 33, 270 (1932), and for theoretical literature see, LEONOWITSON, *Z. Phys.* 72, 247 (1931).

L. Landau und G. Placzek, Struktur der unverschobenen Streulinie, *Phys. Z. Sowjet.* 5, 172 (1934).

[†] This paper does not seem to have been published (Editor's note).

15. ON THE THEORY OF THE SLOWING DOWN OF FAST ELECTRONS BY RADIATION

A simple method is pointed out for calculating the radiation from electrons which are slowing down. This calculation establishes the limits of the validity of the Heitler formula.

IN the work of Heitler¹ the effective cross-section for the bremsstrahlung effect for fast electrons was calculated. It was established that this cross-section does not depend on energy, so that the range of fast electrons seems to approach a limit at high energies, which appears to be improbable. I would like to point out here a simple method for calculating the radiation, which establishes the limits of validity of the Heitler formula, and at the same time gives more general results.

Instead of considering a system of co-ordinates in which the nucleus is at rest and the electron moving nearly with the velocity of light, we consider a co-ordinate system in which the electron is approximately at rest, so that the uncertainty in its velocity is small compared with c . Denote the minimum distance between the electron and nucleus by r (it follows from the Lorentz transformation that this distance is the same in both co-ordinate systems). This quantity is also not exactly known in our case, its uncertainty being connected with the uncertainty in the velocity by the well known relation

$$\Delta r \Delta v \sim \frac{\hbar}{m}$$

Since $\Delta v \ll c$ and Δr is small in comparison with c ,

$$r \gg \frac{\hbar}{mc}. \quad (1)$$

It is possible to show, since $Ze^2/\hbar c$ is always less than 1, that in this case the motion of the electron may also be considered to be approximately free.

The time of the collision τ in the rest co-ordinate system is of the order of magnitude

$$\tau \sim \frac{r}{c}.$$

¹ L. Landau, Zur Theorie der Bremsung von schnellen Elektronen durch Ausstrahlung, *Phys. Z. Sowjet.* 5, 761 (1934).

Л. Д. Ландау, К теории торможения быстрых электронов излучением, *Журнал Экспериментальной и Теоретической Физики* 5, 255 (1935).

In the moving co-ordinate system this decreases because of the Lorentz contraction. If one introduces the notation

$$\varepsilon = \frac{E}{m c^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (2)$$

then the time of the collision τ' is:

$$\tau' = \frac{r}{c \varepsilon}. \quad (3)$$

There corresponds to this the frequency

$$\omega = \frac{c \varepsilon}{r}. \quad (4)$$

It is well known that the force in the moving co-ordinate system is essentially parallel to the direction of r and is multiplied by ε in comparison with the rest system. Thus

$$E' = \frac{Z e}{r^2} \varepsilon. \quad (5)$$

The electron in the moving co-ordinate system is under the influence of a plane wave, the frequency of which is given by equation (4), the time of action by (3), and the force by (5).

The radiation can be interpreted as a scattering of this wave. It is well known that the way this scattering takes place is essentially different depending on the ratio of $\hbar \omega$ to $m c^2$.

Let us consider first of all the case where $\hbar \omega \ll m c^2$, i.e. $r \gg (\hbar/mc) \varepsilon$. In this case the scattering behaves classically. The total radiated energy is given in the case by the well known formula

$$\Delta E' \cong \frac{e^2}{c^3} w'^2 \tau' \quad (6)$$

where w' is the acceleration.

If one puts

$$w' = \frac{e E'}{m} \quad (7)$$

and substitutes equations (5) and (3) then one obtains

$$\Delta E' \cong \frac{Z^2 e^6 \varepsilon}{m^2 c^4 r^3}. \quad (8)$$

It is well known that the radiated momentum for the rest electron is, according to the classical theory,

$$\Delta p' = 0. \quad (9)$$

Therefore, in the rest co-ordinate system

$$\Delta E = \frac{\Delta E' + w \Delta p'}{\sqrt{1 - \frac{v^2}{c^2}}} \sim \Delta E' \varepsilon \sim \frac{z^2 e^6 \varepsilon^2}{m^2 c^4 r^3}. \quad (10)$$

In the moving co-ordinate system the scattering is approximately uniformly distributed over the different directions. Hence in the rest co-ordinate system we obtain forward scattering into an angular range of the order of magnitude

$$\theta \sim \frac{1}{\varepsilon}. \quad (11)$$

Consider now the second limiting case where

$$\hbar \omega \gg m c^2, \quad (12)$$

$$r \ll \frac{\hbar}{m c} \varepsilon. \quad (13)$$

In this case we can no longer calculate classically but must apply to the scattering the theory of the Compton effect. The number of scattered light quanta is, according to the Klein-Nishina formula, determined by the effective cross-section

$$\sigma = \left(\frac{e^2}{m c^2} \right)^2 \cdot \frac{\ln \frac{\hbar \omega}{m c^2}}{\frac{\hbar \omega}{m c^2}}. \quad (14)$$

The total numbers of scattered quanta is

$$\Delta N \sim \frac{c E'^2}{\hbar \omega} \sigma \sim \frac{Z^2 e^6}{\hbar^2 m^2 c^4 \varepsilon r} \ln \frac{\hbar \varepsilon}{m c r}. \quad (15)$$

In the moving co-ordinate system the direction of the scattered quanta is arbitrary, but their energy on the other hand is of the order of magnitude of $m c^2$. Hence in the rest co-ordinate system the energy of the quanta is of the order of magnitude of $m c^2 \varepsilon$, i.e. of the order of magnitude of the kinetic energy. The mean radiated energy is

$$\Delta E \sim \Delta N m c^2 \varepsilon \sim \frac{Z^2 e^6}{\hbar^2 c^2 r} \ln \frac{\hbar \varepsilon}{m c r}. \quad (16)$$

If one puts $\hbar \omega \cong m c^2$, $r_0 \cong (\hbar/mc) \varepsilon$, then the formulae (16) and (10) of course coincide, giving

$$\Delta E \sim \frac{m Z^2 e^6}{\hbar^3 c \varepsilon}. \quad (17)$$

The region of validity of formula (16) is determined, apart from by (13) and (1), by the limits of validity of the Klein-Nishina formula. We could expect that these latter limits are given by the condition that the wavelength should be large in comparison with the "electronic radius". In our case this gives

$$\frac{r}{\varepsilon} \gg \frac{e^2}{m c^2}, \quad (18)$$

$$r \gg \frac{e^2}{m c^2} \varepsilon. \quad (19)$$

This limit can have a meaning only if

$$\varepsilon \sim \frac{\hbar c}{e^2}.$$

The effective cross-section for the bremsstrahlung effect is given by the formula

$$\Phi = \frac{S \cdot \Delta E \cdot 2\pi r d r}{E}. \quad (20)$$

If the expression (10) is substituted into (20) we see that the integral is convergent for large r . If (16) is substituted into (20) we see that the same happens for small r . Therefore, the intermediate case

$$\hbar \omega \sim m c^2, \quad r \sim \frac{\hbar}{m c} \varepsilon, \quad \Delta E \sim \frac{m Z^2 e^6}{\hbar^3 c \varepsilon}$$

is the most important one. The effective cross-section is thus obtained in the form

$$\Phi \sim \frac{\Delta E r^2}{E} \sim \frac{Z^2 e^6}{\hbar m^2 c^5}. \quad (21)$$

This is nothing but the Heitler formula.

REFERENCE

1. W. HEITLER, *Z. Phys.* 84, 145 (1933).

16. ON THE PRODUCTION OF ELECTRONS AND POSITRONS BY A COLLISION OF TWO PARTICLES

The production of an electron and positron by a collision of two particles, moving with a velocity near to the velocity of light, is investigated. The cross-section of this effect is obtained; it increases with the cube of the logarithm of the energy of the colliding nuclei.

1. Dirac's theory of electrons and positrons gives us the possibility to investigate the effects of the production of electronic pairs by collisions of various kinds. The whole effect may then be described as a transition of a negative-energy electron into a positive-energy state under the influence of colliding particles; thus, an electron and a "hole", i.e. a positron, are produced.

If we regard the process in first order perturbation theory (that is, the unperturbed wave-functions for the electron and the interaction of the two particles), the effect is due to the interaction of both particles, and it can be shown without difficulty that for velocities of colliding particles near to the velocity of light the probability of the production of an electronic pair is dependent on the relative acceleration of the colliding particles, i.e. inversely proportional to the square of their mass. Therefore in this case (we shall here regard only the case $1 - v/c \sim 0$, that is, the energy is great comparing with the mass multiplied with c^2) the first order effect is small in comparison with the second-order effect. In the latter approximation we may regard the nuclei as non-interacting, i.e. as moving rectilinearly, because for large velocities the interaction can be neglected. The effect is then due merely to the superposition of the fields of both nuclei. The first approximation of the perturbation theory would give zero under such conditions.

The method adopted is as follows: the wave function of the electron in a negative-energy state (before the collision), perturbed by the field of the particles, is calculated. With the aid of this function and the wave function of a positive-energy electron state (after the collision) the matrix element of the perturbation is then constructed and the square of its modulus gives the probability of the transition, i.e. the production of an electron and a positron. The interaction of both nuclei is neglected.

The method is of course symmetrical as regards the electron and positron and also in the two nuclei. Therefore it is just the same which of both is regarded as fixed and the result must be invariant against the Lorentz-transformation (v is the relative velocity of the nuclei, E_2 the energy of the positive-state

L. Landau and E. Lifshitz, On the production of electrons and positrons by a collision of two particles, *Phys. Z. Sowjet.* 6, 244 (1934).

electron; $E_1 = -|E_1|$ that of a negative-state, i.e., positron; the x -axis is in the direction of v):

$$E_1 = \frac{E'_1 - v p'_{1x}}{\sqrt{1 - v^2}}, \quad E_2 = \frac{E'_2 - v p'_{2x}}{\sqrt{1 - v^2}}$$

(in the following we put $c = 1$, i.e. the dimension of velocity equal to 1; the necessary power of c can be inserted in the final result so as to secure the right dimensions). If the energy is large and the momentum makes a small angle with the x -axis (as is really the case, as will be shown farther on), we have:

$$p_x = E - \frac{m^2 + p^2}{2E} - \frac{(m^2 + p^2)^2}{8E^3} - \dots \quad (1)$$

($p^2 = p_y^2 + p_z^2$) and we get the Lorentz-transformation up to terms of higher order in the form:

$$E_1 = \frac{m^2 + p_1^2}{2E'_1 \sqrt{1 - v^2}}, \quad E_2 = \frac{m^2 + p_2^2}{2E'_2 \sqrt{1 - v^2}}. \quad (2)$$

If the particles lose but a small part of their energy, which makes it possible to neglect their interaction, the result is independent of the nature of the particles and is the same for the case of two colliding nuclei (we shall in the following refer only to this case) or a nucleus and an electron. The latter case was investigated by Carlson and Furry¹ but we shall see that the result obtained here is different from that of Carlson and Furry. It may be noted that their result is not invariant against the transformation (2) notwithstanding that E_1 and E_2 are there large enough.

2. We proceed now to the calculation of the cross section for the production of pairs. Before the collision the electron occupies a negative energy state. Its four-dimensional momentum-energy vector† is then \mathbf{p}_1 with components p_{1k} ($p_{11}, p_{12}, p_{13} = p_{1x}, p_{1y}, p_{1z} = -|p_{1x}, p_{1y}, p_{1z}|$, $p_{14} = iE_1$, $E_1 = -|E_1|$). The first nucleus (with the charge $Z_1 e$) is fixed and the second ($Z_2 e$) has a velocity v along the x -axis; their four-dimensional velocities are

$$u^{(1)} = u_0^{(1)} = (0, 0, 0, i), \quad u^{(2)} = \frac{u_0^{(2)}}{\sqrt{1 - v^2}} = \left(\frac{v}{\sqrt{1 - v^2}}, 0, 0, \frac{i}{\sqrt{1 - v^2}} \right)$$

respectively (in the following we always assume $1 - v/c \sim 0$). The field we assume for the sake of convenience as consisting of a strong discrete plane wave

$$\varphi_k^{(1)} e^{i\omega_k^{(1)} x_k}, \quad \omega_k^{(1)} = \frac{p'_k - p_{1k}}{\hbar} \quad (k = 1, 2, 3, 4; \quad x_{1,2,3,4} = x, y, z, it)$$

and a continuous spectrum $\varphi_k^{(2)}$; $\varphi_{1,2,3}$ are the components of the vector potential \mathfrak{A} and $\varphi_4 = i\varphi$ is the scalar potential, multiplied by i . After the

† All four-dimensional vectors are printed bold-face in this paper.

collision the electron is in a positive-energy state and its momentum-energy vector is p_2 . We also introduce p for the two-dimensional vector p_y, p_z .

The initial, i.e. the negative-energy states, form a continuous range, which is before the collision (the space is empty) wholly occupied. Therefore the probability of the transition must be referred to a differential interval of the momentum.

The unperturbed wave functions of the electron before and after the collision must then be normalised on the δ -function in momentum-space, that is

$$\Psi_1^{(0)} = \frac{1}{(2\pi \hbar)^{3/2}} \psi^{(0)} e^{\frac{i}{\hbar} p_{1k} x_k}, \quad \Psi_2^{(0)} = \frac{1}{(2\pi \hbar)^{3/2}} \psi_2^{(0)} e^{\frac{i}{\hbar} p_{2k} x_k}, \quad (4)$$

$$|\Psi_1^{(0)}|^2 = |\Psi_2^{(0)}|^2 = 1$$

respectively and satisfy the Dirac equation

$$[m + \gamma_k p_k] \Psi^{(0)} = 0 \quad (5)$$

where

$$\gamma_i \gamma_k + \gamma_k \gamma_i = -2\delta_{ik} \quad (6)$$

and

$$p_k = \frac{\hbar}{i} \frac{\partial}{\partial x_k}.$$

Substituting (4) in (5) we get a set of algebraic equations to determine $\psi_1^{(0)}$ and $\psi_2^{(0)}$:

$$[m + \gamma_k p_{1k}] \psi_1^{(0)} = 0, \quad [m + \gamma_k p_{2k}] \psi_2^{(0)} = 0.$$

The wave function of the electron in a negative-energy state, perturbed by the field of the first nucleus, must then satisfy the equation

$$[m + \gamma_k (p_k + e \varphi_k^{(1)} e^{i\omega_i^{(1)} x_i})] \Psi_1 = 0$$

($\varphi_k^{(2)}$ is neglected compared with $\varphi_k^{(1)}$). If we seek Ψ_1 in the form

$$\Psi_1 = \Psi_1^{(0)} + \Psi', \quad \Psi' = \frac{1}{(2\pi \hbar)^{3/2}} \psi' e^{\frac{i}{\hbar} p'_k x_k},$$

neglect $e \gamma_k \varphi_k^{(1)} e^{i\omega_i^{(1)} x_i}$ in Ψ' and multiply with $(m - \gamma_k p'_k)$, we get

$$\psi' = \frac{(-m + \gamma_k p'_k) e \gamma_e \varphi_e^{(1)}}{(m^2 + p'^2)} \psi_1^{(0)} \quad (7)$$

p'_k may be called the momentum-energy vector of an "intermediate state", as it is often done in the perturbation theory in the second approximation. The matrix element of the perturbation referring to the transition from the "intermediate" (Ψ') to the positive-energy state ($\Psi_2^{(0)}$) (we take the perturbation in the form $e \gamma_k \gamma_e \varphi_k^{(2)}$ and neglect here $\varphi_k^{(1)} e^{i\omega_i^{(1)} x_i}$ because two discrete plane waves can give transitions only if the sum of their frequencies is equal

to $(p_{2k} - p_{1k})/\hbar$ and this case is negligible) is (using (7)):

$$\begin{aligned} & \frac{e}{(2\pi\hbar)^3} \int \psi_2^{(0)*} \gamma_4 \gamma_k \varphi_k^{(2)} \psi' d\tau = \\ & = \frac{e^2}{(2\pi\hbar)^3} \frac{\psi_2^{(0)*} \gamma_4 (\gamma \chi^{(2)}) [-m + (\gamma \mathbf{p}')] (\gamma \varphi^{(1)}) \psi_1^{(0)}}{(m^2 + \mathbf{p}'^2)} \\ & d\tau = dx_1 dx_2 dx_3 dx_4 \end{aligned} \quad (8)$$

where γ is a vector with the components $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and

$$\begin{aligned} \chi_k^{(2)} &= \int \varphi_k^{(2)} e^{-\frac{i}{\hbar}(p_{2k} - p'_{1k})x_k} d\tau = \int \varphi_k^{(2)} e^{-i\omega_k^{(2)}x_k} d\tau, \\ \omega_k^{(2)} &= \frac{p_{2k} - p'_{1k}}{\hbar} \end{aligned} \quad (9)$$

is a Fourier component of the continuous spectrum with the frequency $\omega_k^{(2)}$.

We now proceed from the discrete plane wave $\varphi_k^{(1)}$ to a Fourier component $\chi_k^{(1)}$ with a frequency:

$$\omega_k^{(1)} = \frac{p'_k - p_{1k}}{\hbar} \quad (9a)$$

for an interval $dp'_1 dp'_2 dp'_3 dp'_4$ of the continuous spectrum. From the Fourier integral it is then easily seen that instead of $\varphi_k^{(1)}$ we have to put

$$\frac{\chi_k^{(1)} dp'_1 dp'_2 dp'_3 dp'_4}{(2\pi\hbar)^4} = \frac{\chi_k^{(1)} d\pi'}{(2\pi\hbar)^4}, \quad d\pi' = dp'_1 dp'_2 dp'_3 dp'_4.$$

Integrating over the whole spectrum to obtain the whole effect and dividing by \hbar , we get the probability amplitude

$$P = \frac{e^2}{(2\pi\hbar)^7 \hbar} \int \frac{\psi_2^{(0)*} \gamma_4 (\gamma \chi^{(2)}) [-m + (\gamma \mathbf{p}')] (\gamma \chi^{(1)}) \psi_1^{(0)}}{(m^2 + \mathbf{p}'^2)} d\pi'. \quad (10)$$

We must now find the potentials $\chi_k^{(1)}$ and $\chi_k^{(2)}$ of the field of the nuclei, moving rectilinearly without interaction. This is better done in a three-dimensionals form, and so we have for the scalar and vector potentials of the first or the second nucleus:

$$\square \varphi = -4\pi e Z \delta(\mathbf{r} - \mathbf{v}t - \mathbf{r}_0), \quad \square \mathcal{A} = -4\pi e Z \mathbf{v} \delta(\mathbf{r} - \mathbf{v}t - \mathbf{r}_0) \quad (11)$$

where \mathbf{r} and \mathbf{v} are the three-dimensional radius-vector and the velocity of the nucleus; \mathbf{r}_0 is the radius-vector of either of the two nuclei when both are in a closest distance one from another. We assume the first nucleus to be at rest in the co-ordinate origin and the second moving parallel to the x -axis; $\mathbf{r}_0^{(1)} = 0$ and $\mathbf{r}_0^{(2)}$ has only the Y and Z components. Taking from both sides of (11) a

Fourier component with the frequency ω_k we obtain

$$\begin{aligned}
 & (\mathfrak{k}^2 - \omega^2) \int \int \varphi e^{-i(\mathfrak{k}v - \omega t)} dV dt = \\
 & = 4\pi Z e \int \int \delta(r - vt - r_0) e^{-i(\mathfrak{k}v - \omega t)} dV dt = 4\pi Z e \int e^{-i(\mathfrak{k}vt + \mathfrak{k}r_0 - \omega t)} dt, \\
 & (\mathfrak{k}^2 - \omega^2) \int \int \mathfrak{A} e^{-i(\mathfrak{k}v - \omega t)} dV dt = 4\pi Z e v \int e^{-i(\mathfrak{k}vt + \mathfrak{k}r_0 - \omega t)} dt \quad (12) \\
 & i\omega = \omega_A, \quad \mathfrak{k}_{x,y,z} = \omega_{1,2,3}, \quad dV = dx dy dz.
 \end{aligned}$$

With a well-known formula

$$\int_{-\infty}^{+\infty} e^{ixt} dt = 2\pi \delta(x) \quad (13)$$

(12) gives us:

$$\begin{aligned}
 & (\mathfrak{k}^2 - \omega^2) \int \int \varphi e^{-i(\mathfrak{k}v - \omega t)} dV dt = 8Ze\pi^2 e^{-i\mathfrak{k}r_0} \delta(\omega - \mathfrak{k}v), \\
 & (\mathfrak{k}^2 - \omega^2) \int \int \mathfrak{A} e^{-i(\mathfrak{k}v - \omega t)} dV dt = 8Ze\pi^2 v e^{-i\mathfrak{k}r_0} \delta(\omega - \mathfrak{k}v). \quad (14)
 \end{aligned}$$

This can be written in a four-dimensional form if we use the notations (9) and we get for the field produced by both nuclei

$$\begin{aligned}
 \omega_i^{(1)2} \chi_k^{(1)} &= 8\pi^2 Z_1 e u_k^{(1)} e^{-i\omega_i^{(1)} x_0 i^{(1)}} \delta(\omega_e^{(1)} u_e^{(1)}) + \\
 &+ 8\pi^2 Z_2 e u_k^{(2)} e^{-i\omega_i^{(1)} x_0 i^{(2)}} \delta(\omega_e^{(1)} u_e^{(2)}), \\
 \omega_i^{(2)2} \chi_k^{(2)} &= 8\pi^2 Z_1 e u_k^{(1)} e^{-i\omega_i^{(2)} x_0 i^{(1)}} \delta(\omega_e^{(1)} u_e^{(2)}) + \\
 &+ 8\pi^2 Z_2 e u_k^{(2)} e^{-i\omega_i^{(2)} x_0 i^{(2)}} \delta(\omega_e^{(2)} u_e^{(2)}).
 \end{aligned}$$

These expressions are to be inserted in the probability amplitude (10). The square of its modulus summed over the two states with the opposite directions of spin, gives the probability of a collision with the production of a positron with the energy and momentum lying in an interval $d|p_{1y}| d|p_{1z}| d|E_1| = d\pi_1$ and an electron in $dp_{2y} dp_{2z} dE_2 = d\pi_2$

$$dW_{p_1 E_1 p_2 E_2} = \sum |P|^2 d\pi_1 d\pi_2$$

(p_x is determined by E and p). The cross-section $d\Phi_{p_1 E_1 p_2 E_2}$ is obtained by integration over dy and dz .

Before writing down the expression for the probability amplitude we note that only two nuclei can give an effect and that therefore, substituting (15) in (10), we must leave only the terms which contain both $u^{(1)}$ and $u^{(2)}$, i.e. are due to both nuclei.

We then get

$$P = \frac{e^4 Z_1 Z_3}{2\hbar^2 \pi^3} \psi_2^{(0)*} Q \psi_1^{(0)}, \quad (16a)$$

$$Q = \gamma_4 (\gamma u_0^{(2)}) [-I_0 + (\gamma I)] (\gamma u_0^{(1)}) + \gamma_4 (\gamma u_0^{(1)}) [-I'_0 + (\gamma I')] (\gamma u_0^{(2)}),$$

$$I_i = \int \frac{p'_i e^{-i(\omega_k^{(2)} x_0 k^{(2)})} \delta[(p' - p_1) u_0^{(1)}] \delta[(p_2 - p') u_0^{(2)}] d\pi'}{(m^2 + p_2^2) (p_1 - p')^2 (p_2 - p')^2}, \quad (16b)$$

$$I'_i = \int \frac{p'_i e^{-i(\omega_k^{(1)} x_0 k^{(2)})} \delta[(p' - p_1) u_0^{(2)}] \delta[(p_2 - p') u_0^{(1)}] d\pi'}{(m^2 + p'^2) (p_1 - p')^2 (p_2 - p')^2} \quad (16c)$$

$$i = 0, 1, 2, 3, 4 \quad p'_0 = m$$

$[x_0^{(1)}]$ is put, as was mentioned, equal to zero and instead of $\delta(\omega_k^{(1)} u_k^{(1)})$ and $\delta(\omega_k^{(2)} u_k^{(1)})$ we write here:

$$\delta(\omega_k^{(1)} u_k^{(1)}) = \hbar \delta[(p' - p_1) u_0^{(1)}],$$

$$\delta(\omega_k^{(2)} u_k^{(2)}) = \hbar \sqrt{1 - v^2} \delta[(p_2 - p') u_0^{(2)}];$$

I is a vector with the components I_1, I_2, I_3, I_4

The integration over dp'_1 and dp'_4 , i.e. dp'_x and dE' can be performed at once owing to the presence of δ -functions. It leads to:

$$(p'_k - p_k) u_{0k}^{(1)} = 0, \quad (p_{2k} - p'_k) u_{0k}^{(2)} = 0,$$

$$-ip'_4 = E' = E_1, \quad p'_1 = p'_x = (E_1 + v p_{2x} - E_2)/v,$$

$$I_i = \iint \frac{p'_i e^{-i(\omega_k^{(2)} x_0 k^{(2)})} dp'_y dp'_z}{[m^2 + E_1^2(1 - v^2) - 2E_1(E_2 - v p_{2x}) + (E_2 - v p_{2x})^2 + p'^2]}$$

1

$$\frac{1}{[(E_2 - v p_{2x} - E_1 + v p_{1x})^2 + (p' - p_1)^2][(E_2 - E_1)^2(1 - v^2) + (p' - p_2)^2]} \quad (17a)$$

and analogous:

$$(p'_k - p_{1k}) u_{0k}^{(2)} = 0, \quad (p_{2k} - p'_k) u_{0k}^{(1)} = 0,$$

$$-ip'_4 = E' = E_2, \quad p'_1 = p'_x = (E_2 + v p_{1x} - E_1)/v,$$

$$I'_i = \iint \frac{p'_i e^{-i(\omega_k^{(1)} x_0 k^{(2)})} dp'_y dp'_z}{[m^2 + E_2^2(1 - v^2) - 2E_2(E_1 - v p_{1x}) + (E_1 - v p_{1x})^2 + p'^2]}$$

1

$$\frac{1}{[(E_2 - v p_{2x} - E_1 + v p_{1x})^2 + (p' - p_2)^2][(E_2 - E_1)^2(1 - v^2) + (p' - p_1)^2]} \quad (17b)$$

We must calculate now the square of the modulus of (16a) which gives us the probability:

$$dW_{p_1 E_1 p_2 E_2} = \frac{e^8 Z_1^2 Z_2^2}{4\hbar^4 \pi^6} \sum |\psi_2^{(0)*} Q \psi_1^{(0)}|^2 d\pi_1 d\pi_2. \quad (18)$$

Using Greek letters for the matrix element indices, we can write

$$\sum |\psi_2^{(0)*} Q \psi_1^{(0)}|^2 = Q_{\alpha\beta} (\sum \psi_{1\alpha}^{(1)} \psi_{1\delta}^{(0)*}) \tilde{Q}_{\delta\gamma}^* (\psi_{2\gamma}^{(0)} \psi_{2\delta}^{(0)*}) = Q_{\alpha\beta} \lambda_{1\beta\delta} \tilde{Q}_{\delta\gamma}^* \lambda_{2\gamma\alpha} \quad (19)$$

(\sim denotes the transposed matrix) where

$$\lambda_{1\alpha\beta} = \sum \psi_{1\alpha}^{(0)} \psi_{2\beta}^{(0)*}, \quad \lambda_{2\alpha\beta} = \sum \psi_{2\alpha}^{(0)} \psi_{1\beta}^{(0)*}$$

(\sum denotes summation over the two states with opposite spins).

The summation is extended over all indices, among them also over the first and the last, i.e. we have a circular product and its factors can be circularly transposed. Therefore only such products are not equal to zero, which contain an even number of every γ_k .

To find \tilde{Q}^* we note, that the γ_k are anti-hermitean, i.e.

$$\tilde{\gamma}_k^* = -\gamma_k \quad (20)$$

and therefore we can write

$$(\gamma \overline{u_0^{(1)}})^* = -\gamma_4 (\gamma u_0^{(1)}) \gamma_4 \quad (21)$$

and the same for $(\gamma u_0^{(2)})$ and (γI) . Hence

$$Q^* = \gamma_4 Q' \gamma_4,$$

$$Q' = (\gamma u_0^{(1)}) [-I_0 + (\gamma I)] (\gamma u_0^{(2)}) \gamma_4 + (\gamma u_0^{(2)}) [-I'_0 + (\gamma I')] (\gamma u_0^{(1)}) \gamma_4.$$

Now, it can be easily verified by calculation of ψ -functions that

$$\lambda_1 = \frac{2}{p_{14}} (\gamma_k p_{1k} - m) \gamma_4, \quad \lambda_2 = \frac{2}{p_{24}} (\gamma_k p_{2k} - m) \gamma_4 \quad (23)$$

(18), (16b), (19) and (22) then give (remembering that $\gamma_4^2 = -1$):

$$\begin{aligned} dW_{p_1 E_1 p_2 E_2} &= \frac{e^8}{4\hbar^4 \pi^6} Q \lambda_1 \gamma_4 Q' \gamma_4 \lambda_2 d\pi_1 d\pi_2 \\ &= \frac{e^8 d\pi_1 d\pi_2 \dots}{\hbar^4 \pi^6 E_1 E_2} \{[(\gamma u_0^{(2)}) [I_0 - (\gamma I)] (\gamma u_0^{(1)}) \\ &\quad + (\gamma u_0^{(1)}) [I'_0 - (\gamma I')] (\gamma u_0^{(2)})] [(\gamma p_1) - m] [(\gamma u_0^{(1)}) [I_0 - (\gamma I)] (\gamma u_0^{(2)}) \\ &\quad + (\gamma u_0^{(2)}) [I'_0 - (\gamma I')] (\gamma u_0^{(1)})] [(\gamma p_2) - m]\}. \end{aligned} \quad (24)$$

In the following we assume that

$$m \ll |E_1| \sim E_2 \ll m/\sqrt{1-v^2}, \quad p_{1y}, p_{1z}, p_{2y}, p_{2z} \sim m, \quad (25)$$

i.e. that the energy of the electron and the positron produced by the collision is large compared with their self-energy and small in comparison with the

energy of the colliding particles, and that the tracks of the electrons and positrons produced lie in a small angle around the direction of the moving nucleus. That this is really the most important region, i.e. that most of the pairs have such an energy and momentum, is verified by the final result.

In the multiplication in (24) we neglect all the terms (in figure brackets) of the order $E_{1,2}^2(1-v^2)I_i^2$ or $(m^4/E_{1,2})I_i^2$ ($i = 0, 2, 3$) and higher. A somewhat lengthy calculation then gives if we use (1)

$$\begin{aligned} dW_{p_1 E_1 p_2 E_2} = & \frac{4e^8 Z_1^2 Z_2^2 d\pi_1 d\pi_2}{\hbar^4 \pi^6 E_1 E_2} \left\{ m^2 I_0' I_0 - m^2 \mathfrak{I} \mathfrak{I}' - (p_1 p_2) (\mathfrak{I} \mathfrak{I}') \right. \\ & - (p_1 p_2) I_0 I_0' + (p_1 + p_2) m (I_0' \mathfrak{I} + I_0 \mathfrak{I}') + (p_2 \mathfrak{I}') (p_1 \mathfrak{I}) + (p_2 \mathfrak{I}) (p_1 \mathfrak{I}') \\ & + \frac{(m^2 + p_2^2)}{2} \frac{E_1}{E_2} (I_0^2 + \mathfrak{I}^2) + \frac{(m^2 + p_1^2)}{2} \frac{E_2}{E_1} (I_0'^2 + \mathfrak{I}'^2) \left. \right\}, \quad (26) \\ \mathfrak{I} = (I_2, I_3), \quad \mathfrak{I}' = (I_2', I_3'). \end{aligned}$$

For $I_0, I_0', \mathfrak{I}, \mathfrak{I}'$ we get, applying (1) to (17a), (17b) and leaving in the denominator only the highest terms (it can be shown, that the neglected terms give an effect of an order higher than the final result):

$$\begin{aligned} I_0 = & \iint \frac{E_2 m e^{-i(p_{2y}-p'_{2y})y/\hbar} e^{-i(p_{2z}-p'_{2z})z/\hbar} dp'_y dp'_z}{[(m^2 + p'^2) E_2 - (m^2 + p_2^2) E_1] [A + (p' - p_1)^2] [B + (p' - p_2)^2]}, \\ I_0' = & \iint \frac{E_1 m e^{-i(p'_{1y}-p_{1y})y/\hbar} e^{-i(p'_{1z}-p_{1z})z/\hbar} dp_y dp_z}{[(m^2 + p'^2) E_1 - (m^2 + p_1^2) E_2] [A + (p' - p_2)^2] [B + (p' - p_1)^2]}, \quad (27) \\ A = & \left[\left(\frac{m^2 + p_1^2}{2E_1} \right) - \left(\frac{m^2 + p_2^2}{2E_2} \right) \right]^2, \quad B = (E_1 - E_2)^2 (1 - v^2). \quad (27a) \end{aligned}$$

In \mathfrak{I} and \mathfrak{I}' instead of m in the numerator stands p' . The second and third factors in the denominators of I_0 etc. are much smaller than the first; it is evident therefore that they play the main role in the integral.

If we wish to get from the probability $dW_{p_1 E_1 p_2 E_2}$ the cross section $d\Phi_{p_1 E_1 p_2 E_2}$ we must integrate (26) over dy and dz . It is, however, more convenient to integrate over dy and dz before performing the integration over $dp'_y dp'_z$. For the sake of this we write the products of two integrals, as $I_0 I_0'$ etc., in the form of a double integral of the product of both integrated expressions, changing in one of them p' to another variable, say p'' . The integration is then extended over $dp'_y dp'_z dp''_y dp''_z$ and the integration over dy and dz can be performed before the latter. The exponential factors by this integration give according to (13) δ -functions of two kinds:

$$\delta(p_{1y} + p_{2y} - p'_y - p''_y) \delta(p_{1z} + p_{2z} - p'_z - p''_z)$$

and

$$\delta(p'_y - p''_y) \delta(p'_z - p''_z)$$

in $I_0 I_0', \mathfrak{I} \mathfrak{I}'$ and $I_0^2, I_0'^2, \mathfrak{I}^2, \mathfrak{I}'^2$ respectively. The integration over $dp'_y dp'_z$ or $dp''_y dp''_z$ is then obvious and we obtain from (26) and (27):

$$\begin{aligned}
d\Phi_{p_1 E_1 p_2 E_2} = & \frac{16e^8 Z_1^2 Z_2^2}{\hbar^2 \pi^4} \left\{ \iint \{ m^4 + m^2 p' (p_1 + p_2 - p') \right. \\
& + (p_1 p_2) [p' (p_1 + p_2 - p')] + m^2 (p_1 p_2) - m^2 (p_1 + p_2)^2 \\
& - [p_1 (p_1 + p_2 - p')] (p_2 p') - [p_2 (p_1 + p_2 - p')] (p_1 p') \} \\
& \times dp'_y dp'_z [(m^2 + p'^2) E_2 - (m^2 + p_2^2) E_1] [(m^2 + p_1^2) E_2 \\
& - (m^2 + (p_1 + p_2 - p')^2) E_1] [A + (p' - p_1)^2] [B + (p' - p_2)^2]^2 \\
& + \frac{1}{2} \iint \frac{(m^2 + p_2^2) (m^2 + p'^2) dp'_y dp'_z}{[(m^2 + p'^2) E_2 - (m^2 + p_2^2) E_1]^2} \\
& \times \frac{1}{[A + (p' - p_1)^2]^2 [B + (p' - p_2)^2]^2} \\
& + \frac{1}{2} \iint \frac{(m^2 + p_1^2) [m^2 + (p_1 + p_2 - p')^2] dp'_y dp'_z}{[(m^2 + (p_1 + p_2 - p')^2) E_1 - (m^2 + p_1^2) E_2]^2} \\
& \times \frac{1}{[A + (p' - p_1)^2]^2 [B + (p' - p_2)^2]^2} d\pi_1 d\pi_2 \Big\}. \quad (28)
\end{aligned}$$

We can introduce new variables

$$p_1 + p_2 = 2p, \quad p' - p_1 = 2a, \quad p' - p_2 = 2b \quad (29)$$

and integrate over $da_y da_z db_y db_z$. (29) then gives (the Jacobian for this transformation is equal to 16):

$$\begin{aligned}
d\varphi_{p E_1 E_2} = & \frac{2^8 e^8 Z_1^2 Z_2^2 d|E_1| dE_2 dp_y dp_z}{\hbar^2 \pi^4} \left\{ \iiint \int [- (m^2 + p^2)^2 \right. \\
& - 2(m^2 + p^2)(a^2 + b^2) - (a^2 - b^2)^2 + 4a^2 b^2 - 4(ab)^2 \\
& + 4(a p)^2 + 4(b p)^2] da_y db_z da_y db_z [(m^2 + (p + a + b)^2) E_2 \\
& - ((p + a - b)^2 + m^2) E_1] [(m^2 + (p + b - a)^2) E_2 \\
& - (m^2 + (p - a - b)^2) E_1] (A + 4a^2)^2 (B + 4b^2)^2 \\
& + \frac{1}{2} \iiint \int \frac{[m^2 + (p + a - b)^2]}{[(m^2 + (p + a + b)^2) E_2 - (m^2 + (p + a - b)^2) E_1]^2} \\
& \times \frac{[m^2 + (p + a + b)^2] da_y da_z db_y db_z}{(A + 4a^2)^2 (B + 4b^2)^2} \\
& + \frac{1}{2} \iiint \int \frac{[m^2 + (p + a - a)^2]}{[(m^2 + (p - a - b)^2) E_2 - (m^2 + (p + b - a)^2) E_1]^2} \\
& \times \frac{[m^2 + (p - a - b)^2] da_y da_z db_y db_z}{(A + 4a^2)^2 (B + 4b^2)^2} \Big\}. \quad (30)
\end{aligned}$$

After the reduction to a common denominator in the numerator remain only the terms of the fourth power in a and b and higher. In the denominator we can then put $a = b = 0$ everywhere except $(A + 4a^2)^2(B + 4b^2)^2$ (these factors play the most important role in the integral). We obtain (leaving only the terms with the fourth degree of a and b):

$$d\Phi_{p, E_1, E_2} = \frac{2^{11} Z_1^2 Z_2^2 e^8 d|E_1| dE_2 dp_y dp_z}{\hbar^2 \pi^4 (E_1 - E_2)^4 (m^2 + p^2)^2} \times \left[E_1^2 + E_2^2 - \frac{4E_1 E_2 m^2 p^2}{(m^2 + p^2)^2} \right] \iiint \frac{a^2 b^2 da_y da_z db_y db_z}{(A + 4a^2)^2 (B + 4b^2)^2}. \quad (31)$$

Now, the integral diverges logarithmically and we conclude that the region is of importance where

$$A + B < a^2 < m^2 \quad A + B < b^2 < m^2 \quad (32a)$$

and also

$$A + B < (p_1 - p_2)^2 < m^2. \quad (32b)$$

Equation (32b) shows that the tracks of the electron and positron are on different sides of the x -axis. The smallness of a and b justifies our leaving only the terms of the lowest power in a and b .

Integrating (31) in the region from zero till m (leaving only the logarithmic terms; the coefficient in the logarithm must not be written in our approximation and we can also write there m^2 for p^2 according to (25)):

$$d\Phi_{p, E_1, E_2} = \frac{8e^8 Z_1^2 Z_2^2 d|E_1| dE_2 dp_y dp_z}{\pi^2 \hbar^2 (m^2 + p^2)^2 (E_2 - E_1)^4} \left\{ E_1^2 + E_2^2 - \frac{4E_1 E_2 p^2 m^2}{(m^2 + p^2)^2} \right\} \times \ln \frac{m}{(E_2 - E_1) \sqrt{1 - v^2}} \ln \frac{1}{m \left(\frac{1}{E_2} - \frac{1}{E_1} \right)}.$$

Finally, integrating over $dp_y dp_z$, we obtain the cross section referring to $d|E_1| dE_2$:

$$d\Phi_{E_1, E_2} = \frac{8e^8 Z_1^2 Z_2^2}{3\pi \hbar^2 m^2} \frac{(3E_1^2 + 3E_2^2 - 2E_1 E_2)}{(E_2 - E_1)^4} \ln \frac{m}{(E_2 - E_1) \sqrt{1 - v^2}} \times \ln \frac{1}{m \left(\frac{1}{E_2} - \frac{1}{E_1} \right)} d|E_1| dE_2. \quad (34)$$

From this expression the region (25) is easily verified; it may also be noted that this expression is invariant in respect to the transformation (2).

If we write E_1 for the absolute value of the energy of the positron ($E'_1 = -E_1$) and go over to ordinary units for the velocity, we get

$$d\Phi_{E_1 E_2} = \frac{8}{3\pi} (Z_1 Z_2 \alpha)^2 \left(\frac{e^2}{m c^2} \right)^2 \frac{3 E_1'^2 + 3 E_2^2 + 2 E_1' E_2}{(E_1' + E_2)^3} \\ \times \ln \frac{m c^2}{(E_1' + E_2) \sqrt{1 - \frac{v^2}{c^2}}} \ln \frac{1}{m^2 \left(\frac{1}{E_1'} + \frac{1}{E_2} \right)} dE_1' dE_2 \quad (35)$$

(α is the fine-structure constant).

The whole cross-section for the production of pairs is obtained by integration over $dE_1' dE_2$. To perform the first integration we note that for the integral the region $E_1' \sim E_2$ is of importance, for, assuming $E_2 \gg E_1'$, it has the form $\iint \frac{dE_1' dE_2}{E_2}$, that is the large values of E_1' ($E_1' \sim E_2$) are of importance. Therefore we may write in the logarithms E_2 instead of E_1' and integrate the rest. If we remember that the coefficient in the logarithm is not written, we obtain

$$d\Phi_{E_2} = \frac{56}{9\pi} (Z_1 Z_2 \alpha)^2 \left(\frac{e^2}{m c^2} \right)^2 \ln \frac{m c^2}{E_2 \sqrt{1 - \frac{v^2}{c^2}}} \ln \frac{E_2}{m c^2} \cdot \frac{dE_2}{E_2}. \quad (37)$$

The integral over dE_2 is logarithmically divergent, and according to (25) it must be taken in the limits between $m c^2$ and $m c^2 / \sqrt{1 - v^2/c^2}$. Introducing $x = \ln(E_2/m c^2)$ as a new variable, we have

$$\int_{m c^2}^{\frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}}} \ln \frac{m c^2}{E_2 \sqrt{1 - \frac{v^2}{c^2}}} \ln \frac{E_2}{m c^2} \cdot \frac{dE_2}{E_2} = \int_0^{\ln \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} x \left(\ln \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - x \right) dx \\ = \frac{1}{6} \left(\ln \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^3.$$

Inserting this in (32) we obtain finally

$$\Phi = \frac{28}{27\pi} (Z_1 Z_2 \alpha)^2 \left(\frac{e^2}{m c^2} \right)^2 \ln^3 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (38)$$

Hence, the cross-section increases with the cube of logarithm of the energy of the colliding particles.

Numerically (38) gives:

$$\Phi = 1.4 \times 10^{-30} Z_1^2 Z_2^2 \ln^3 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ cm}^2. \quad (39)$$

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17. ON THE THEORY OF SPECIFIC HEAT ANOMALIES

A quantitative theory is outlined for the occurrence of a critical point due to order phenomena. Below the transition point a $1/\sqrt{\Theta - T}$ law follows for the specific heat. The order of magnitude of the coefficients is determined for a particular case.

SPECIFIC heat anomalies have been observed in many solids. These take the form of a rise in the specific heat followed by a sharp fall. The specific heat sometimes takes very high values (up to 100 kcal/deg mole). These anomalies are often described as λ -points. Attempts have been made to explain them through molecular rotation. However, this suggestion offers no explanation of the anomalous relationships of the specific heats, whether or not the molecular rotation is connected with λ -points. Dehlinger has shown in a recently published work that in certain cases phenomena connected with order changes can lead to abnormally high values of the specific heat. The aim of this paper is to develop this idea quantitatively.

We follow Dehlinger in introducing two kinds of transition points: normal phase changes and Curie points (Dehlinger: phase changes of the second order). For the first type a state of partial order arises instantaneously from the complete disorder which is thermodynamically most probable at higher temperatures. For the Curie points, on the other hand, there is no jump in the entropy at the Curie point, but the assuming of an ordered state merely begins there. If we consider the temperature of the phase change as a function of the pressure, it can happen that at some pressures we have normal transition points, and at others Curie points. The coincidence of the two types gives rise to a critical point.

We describe the degree of order by means of a variable, ξ , which is zero for complete disorder, and can only take positive values. We develop the thermodynamic potential in powers of ξ . Then the conditions for equilibrium are either $\partial\Phi/\partial\xi = 0$ or also $\xi = 0$ since ξ cannot be negative. At the critical point we must have $\xi = \partial\Phi/\partial\xi = \partial^2\Phi/\partial\xi^2 = 0$; $\partial^3\Phi/\partial\xi^3$ must here be positive (otherwise no equilibrium state would correspond to the critical point). Thus writing Φ in the form

$$\Phi = \Phi_0 + \alpha\xi + \beta\xi^2 + \frac{1}{3}\gamma\xi^3, \quad (1)$$

where α , β , and γ are functions of temperature and pressure. (The higher order coefficients can be neglected.) At the critical point $\alpha = \beta = 0$.

L. Landau, Zur Theorie der Anomalien der spezifischen Wärme. *Phys. Z. Sowjet.* 8, 113 (1935).

If the pressure is different from the critical value, then at the temperature for which $\alpha = 0$, β is different from zero.

In the region near this value of the temperature we have $\alpha = a t$ where t is the temperature difference from the point defined above. On the other hand we can neglect the temperature dependence of β (the corresponding terms are multiplied by a higher power of ξ). The quantity a is taken to be positive, since at high temperatures the state $\xi = 0$ certainly corresponds to a stable equilibrium. In the region of the critical pressure, $\beta = b(p - p_{cr})$ where b can have either sign. γ can in any case be assumed to be constant.

Thus the thermodynamic potential assumes the form:

$$\Phi = \Phi_0 + a t \xi + \beta \xi^2 + \frac{1}{3} \gamma \xi^3. \quad (2)$$

In the ordered state we have:

$$\frac{\partial \Phi}{\partial \xi} = a t + 2\beta \xi + \gamma \xi^2 = 0, \quad \frac{\partial^2 \Phi}{\partial \xi^2} = 2(\beta + \gamma \xi) > 0; \quad (3)$$

and thus

$$\xi = \frac{-\beta + \sqrt{\beta^2 - a \gamma t}}{\gamma}. \quad (4)$$

The extra entropy is given by

$$S = -\frac{\partial \Phi}{\partial T} = -a \xi, \quad (5)$$

and the extra specific heat by

$$C = \frac{T dS}{dT} = \frac{T_{cr} a^2}{2 \sqrt{\beta^2 - a \gamma t}}, \quad (6)$$

where the temperature T can be replaced by its critical value over a small region. If we put:

$$\frac{\beta^2}{a \gamma} - t = \tau, \quad (7)$$

which represents a shift of the temperature scale, then C assumes the simple form:

$$C = \frac{A}{\sqrt{\tau}}, \quad (8)$$

where

$$A = \frac{1}{2} T_{cr} a \sqrt{\frac{a}{\gamma}}. \quad (9)$$

Here A is roughly constant in the critical region. Using the same symbols we have:

$$\xi = -\frac{\beta}{\gamma} + \sqrt{\frac{a}{\gamma}} \sqrt{\tau}. \quad (10)$$

For positive β , Φ has no minimum for positive t and $\xi > 0$. When t is negative the equilibrium corresponds to values of ξ starting from zero. At $t = 0$ we have a Curie point.

The corresponding value of the specific heat is:

$$C_c = \frac{T_{cr} a^2}{2\beta}, \quad (11)$$

which is inversely proportional to the difference between the pressure and the critical pressure. If β is negative the curve $\Phi(\xi)$ also has a minimum for positive t . Depending on t this point may lie higher or lower than $\Phi(0)$ and this will determine the stability of the corresponding equilibrium. At the phase change, $\Phi(\xi) = \Phi(0)$; hence:

$$3at + 3\beta\xi + \gamma\xi^2 = 0. \quad (12)$$

If we eliminate t from (12) and (3), we have:

$$3\beta\xi + 2\gamma\xi^2 = 0.$$

Thus at the transition point

$$\xi_{tr} = \frac{3|\beta|}{2\gamma} \quad (13)$$

and

$$t_{tr} = \frac{3\beta^2}{4\alpha\gamma} \quad (14)$$

and finally

$$C_{tr} = \frac{T_{cr} a^2}{|\beta|} \quad (15)$$

($|\beta|$ is the absolute value of β). The specific heats are twice as big near the normal phase change as near the Curie point. The latent heat is:

$$q = T S_{tr} = a T_{cr} \xi_{tr} = \frac{3a|\beta|}{2\gamma} T_{cr}. \quad (16)$$

Thus

$$q C_{tr} = \frac{3}{2} \frac{T_{cr}^2 a^3}{\gamma} = \frac{3}{2} A^2 \quad (17)$$

is approximately independent of pressure. From comparing (14) with $t = 0$ at the Curie point we see that the pressure dependence of the phase change has no discontinuity at the critical point, but the second-order differential, $\partial^2 T / \partial p^2$, has.

The values of the constants α , β , γ can naturally not be considered in general, since they depend on specific properties of the system. Even in the case of a particular model we cannot make any quantitative calculations. However, we can at least make some qualitative limitations for α and γ in considering the thermodynamic potential at high temperatures. Here the entropy term is predominant, and Φ can be written in the form:

$$\Phi = -kT \ln Z \quad (18)$$

(Z is the number of possible combinations for the equivalent degree of order.) Let us consider the example of a mixture of two components in the ratio 1 : 1, i.e. the case where the lattice points possess two possibilities for arrangement. It is unimportant whether there are actually two kinds of molecule, or only two different orientations of one molecule. The order consists of a regular distribution of both classes over the lattice. If we call the number of molecules in an ordered state N_1 , and those in the other state N_2 , then we have for the degree of order:

$$\frac{N_1 - N_2}{N_1 + N_2}.$$

However, we can clearly see that N_1 and N_2 are symmetrical and interchangeable, since the definitions of order and disorder can equally be interchanged. Accordingly all physical magnitudes are functions of (1), and we choose ξ :

$$\xi = \left[\frac{N_1 - N_2}{N_1 + N_2} \right]^2. \quad (19)$$

If we denote the total number of molecules by $N_1 + N_2 = N$ then we have:

$$N_1 = \frac{1}{2} N (1 + \sqrt{\xi}), \quad N_2 = \frac{1}{2} N (1 - \sqrt{\xi}). \quad (20)$$

The number of possibilities which give a specified value of ξ is clearly:

$$Z = \frac{N!}{N_1! N_2!}, \quad (21)$$

whence

$$\Phi = N L T \left\{ \frac{1}{2} (1 + \sqrt{\xi}) \ln \frac{1}{2} (1 + \sqrt{\xi}) + \frac{1}{2} (1 - \sqrt{\xi}) \ln \frac{1}{2} (1 - \sqrt{\xi}) \right\}. \quad (22)$$

Expanding in powers of ξ , we have:

$$\Phi = N k T \left[\ln \frac{1}{2} + \frac{1}{2} \xi + \frac{1}{12} \xi^2 + \frac{1}{30} \xi^3 \right]. \quad (23)$$

The value of α in (1) is thus equal to $N k T / 2$ at high temperatures, and $\gamma = N k T / 10$. We can qualitatively substitute the value of $d\alpha/dT$ for α at high

temperatures, and the value of γ from equation (23) with $T = T_{cr}$. This gives:

$$\alpha = \frac{1}{2} N k, \quad \gamma = \frac{1}{10} N k T_{cr}, \quad A = \frac{1}{4} N k \sqrt{5 T_{cr}},$$

or, for one mole,

$$A = \frac{1}{4} R \sqrt{5 T_{cr}}.$$

This value agrees well with the measurements of Ruhemann on NH_4Cl , as does the $1/\sqrt{\tau}$ law.

18. ON THE THEORY OF THE DISPERSION OF MAGNETIC PERMEABILITY IN FERROMAGNETIC BODIES

The distribution of magnetic moments in a ferromagnetic crystal is investigated. It is found that such a crystal consists of elementary layers magnetised to saturation. The width of these layers is determined. In an external magnetic field the boundaries between these layers move; the velocity of this propagation is determined. The magnetic permeability in a periodical field parallel and perpendicular to the axis of easiest magnetisation is found.

1. It was pointed out by Bloch¹ and Heisenberg² that a ferromagnetic crystal consists magnetically of elementary regions, which are magnetised nearly to saturation. They presumed that these regions are thread-like; we shall show here that they should more likely be considered as elementary layers. This can possibly be brought into accord with the experimental evidence obtained by various authors³ by photographing the distribution of colloidal particles of Fe_2O_3 on a surface of a ferromagnetic crystal. In unmagnetised crystals these elementary layers are magnetised successively in opposite directions, so that the crystal has no magnetic moment as a whole. When the crystal is magnetised the boundaries between the oppositely magnetised layers move, so that the layers with one direction of magnetic moment grow at the cost of the layers with moments in the opposite direction.

Some authors (among them also F. Bloch) tried to apply statistical considerations to determine the number and dimensions of the elementary regions in a ferromagnetic body. This is, however, quite impossible, because if there were not the demagnetising influence of the surface of the body, for instance in an infinite body, there would be generally no elementary regions and such a body would be magnetised to saturation. This is quite analogous to the impossibility of determining with statistical methods the number of drops of a liquid in a condensing vapour, for in fact the liquid is formed as a continuous body. The presence of separate elementary regions, magnetised in opposite directions, is due only to the demagnetising effect of the surface, and the number and dimensions of these regions are entirely determined by the dimensions of the body.

Between two such elementary layers with oppositely directed magnetic moments there is no discrete boundary, but there is an intermediate region where the direction of the magnetic moments changes gradually from one

L. Landau and E. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, *Phys. Z. Sowjet.* 8, 153 (1935).

direction to the opposite. We shall determine here the distribution of the moments in such an intermediate region, and also the width of the elementary layers.

We shall discuss the case of a ferromagnetic crystal with one distinguished axis, which is the axis of easiest magnetisation, as, for example, a single crystal of cobalt with its hexagonal axis. Every ferromagnetic body deformed in one direction (for instance a stretched or contracted wire) is also of this kind, if the sign of the deformation is the same as the sign of the magnetostriction. We shall speak in the following of a ferromagnetic crystal, but it is to be remembered that it is not necessarily a single crystal but an arbitrary body with one direction of easiest magnetisation.

Such a crystal consists of layers, which are parallel to the distinguished axis, magnetised to saturation parallel or anti-parallel to this axis. The fact that the elementary regions in the crystal are really layers and do not have a thread-like form we shall prove later.

We find the distribution of the directions of the magnetic moments inside the crystal in the following way. The magnetic energy of a crystal consists of two parts:

(1) The energy which is due to inhomogeneity in the distribution of the directions of the magnetic moments. This energy per unit volume can be written in the form

$$\frac{1}{2} \alpha [(\nabla s_x)^2 + (\nabla s_y)^2 + (\nabla s_z)^2]$$

where s_x, s_y, s_z are the components of the magnetic moment s of unit volume (its absolute value s is constant through the whole crystal and is practically equal to the saturation moment).

(2) The magnetic-anisotropy energy, due to the presence of an axis of easiest magnetisation. If we choose the co-ordinates with Z -axis along this axis, this energy per unit volume can be written in the form

$$\frac{1}{2} \beta (s_x^2 + s_y^2),$$

showing that the minimum of energy is obtained when s is directed along this axis.

We can then find the distribution of the directions of s if we demand that the energy of the crystal must have a minimum, i.e.

$$\int \left\{ \frac{1}{2} \alpha [(\nabla s_x)^2 + (\nabla s_y)^2 + (\nabla s_z)^2] + \frac{1}{2} \beta (s_x^2 + s_y^2) \right\} dV = \min. \quad (1)$$

where the integral is taken over the whole crystal.

To find the distribution of moments between two layers with opposite magnetisation in the crystal we can neglect the effects due to the surface of the crystal. These effects are fundamental for determining the width of the layers, but can be neglected when we wish to find only the distribution of magnetic moments in the intermediate region between two layers inside the

crystal. This means that if we choose the X -axis of our co-ordinate system perpendicular to the layers, the distribution of magnetic moments will be independent of the co-ordinates y and z , the direction of s changing only with x , namely from a certain direction along the Z -axis in one layer through the intermediate region to the opposite direction in the next one. (1) can now be written as

$$\int \left[\frac{1}{2} \alpha s'^2 + \frac{1}{2} \beta (s_x^2 + s_y^2) \right] dV = \min, \quad (2)$$

where ' denotes differentiation by x . The theory is such as though we assumed the crystal to be infinite. But it is necessary to remember that really in an infinite crystal all the moments would have the same direction and the crystal would be spontaneously magnetised so that no layers would be present. The presence of the layers is due to the finite dimensions of the crystal; we now assume the crystal to be infinite only to find the distribution of the moments in the intermediate regions far from the surface of the crystal; the surface-effects will be discussed in section 2.

In such a model all magnetic moments are distributed in the YZ -plane. We shall denote the angle between the s - and Z -axis by θ . Then the components of s are

$$s_x = 0, \quad s_y = s \sin \theta, \quad s_z = s \cos \theta \quad (3)$$

and (1) takes the form (θ is a function of x only):†

$$\int \left(\frac{1}{2} \alpha s^2 \theta'^2 + \frac{1}{2} \beta s^2 \sin^2 \theta \right) dx = \min. \quad (4)$$

To find the θ which makes this integral a minimum, we write the Euler-equation

$$\alpha \theta'' - \beta \sin \theta \cos \theta = 0, \quad (5)$$

whence

$$\theta'^2 - \frac{\beta}{\alpha} \sin^2 \theta = \text{const.} \quad (6)$$

The width of the layers is large compared with the width of the intermediate region. Therefore we can take as the boundary condition to the equation which determines θ the conditions

$$\left. \begin{aligned} \theta = 0 \quad \text{for} \quad x = -\infty; \quad \theta = \pi \quad \text{for} \quad x = +\infty \\ \text{and} \\ \theta' = 0 \quad \text{for} \quad x = \pm \infty \quad \text{or for} \quad \theta = 0, \pi, \end{aligned} \right\} \quad (7)$$

showing that in two adjoining layers the directions of s are opposite. We see then that the constant in (6) is equal to zero, and we obtain

$$\theta'^2 = \frac{\beta}{\alpha} \sin^2 \theta. \quad (8)$$

† A somewhat analogous calculation was made by Bloch¹ but from a different point of view.

Integrating this equation we find the solution which satisfies (7) in the form

$$\cos \theta = -\tanh \sqrt{\frac{\beta}{\alpha}} x. \quad (9)$$

This gives the distribution of the directions of \mathbf{s} between the two layers.

The numerical value of the constant α can be obtained approximately in the following way. The energy $\frac{1}{2} \alpha s'^2$ has its maximal possible value when \mathbf{s} changes its direction after every distance equal to the lattice-constant a of the crystal, i.e. when $s'^2 \cong s^2/a^2$. This maximum must be of the order kT_c if T_c is the Curie-temperature (k is Boltzmann's constant). Thus we find approximately

$$\alpha = \frac{k T_c}{a s^2}. \quad (10)$$

The "width" of the intermediate region can be defined according to (9) as $\sqrt{\alpha/\beta}$. With (10) we obtain

$$\sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{k T_c}{a \beta s^2}}.$$

For Ni the Curie-temperature is $T_c = 630^\circ\text{K}$, the saturation moment of the unit volume $s = 480$ (for 18°C) and the lattice constant $a = 3.5 \times 10^{-8}$ cm. The anisotropy-energy constant β we take from the experiments of R. Becker and M. Kersten.⁴ They measured the magnetic susceptibility of stretched nickel wires and found the minimal value attained for a large strain to be 0.6. This corresponds to the constant $\beta = 1/0.6 = 1.7$. With this value we get for the "width" of the intermediate region 2.5×10^{-6} cm, i.e. ca. 70 lattice-constants.

2. The analysis of the preceding paragraph gives only the distribution of the directions of the magnetic moments in the intermediate regions, but gives nothing for determining the width of the layers. To find it, it is necessary to discuss the properties of the surface of the crystal. For this sake we employ here the following method. We find the distribution of magnetic moments near the surface of the crystal for a given value d of the width of the layer, and we then determine this width so that the energy of the whole crystal shall be a minimum.

Near the surface there is a magnetic field; let \mathbf{H} be the macroscopical field-strength. Inside the crystal the field \mathbf{H} and the magnetic moment must satisfy the equation

$$\operatorname{div}(\mathbf{H} + 4\pi \mathbf{s}) = 0$$

and outside

$$\operatorname{div} \mathbf{H} = 0.$$

The intermediate regions between the layers and hence the energy $\frac{1}{2} \alpha s'^2$ are of no importance for the distribution of \mathbf{s} near the surface. If the magnetic-anisotropy energy were equal to zero, i.e. $\beta = 0$, the equilibrium, i.e. the distri-

bution with the minimum of magnetic energy, would be such that H would be zero. We could write then for s inside the crystal:

$$\operatorname{div} s = 0 \quad (11)$$

with the boundary condition on the surface of the crystal

$$s_n = 0 \quad (12)$$

if s_n is the component of s normal to the surface. If β is small, we may assume that these equations are not changed appreciably, and we shall assume (11) and (12) as the equations determining the distribution of s near the surface.

Let us regard at first a surface of a crystal (of the same kind of symmetry as in sect. 1), which is perpendicular to the axis of the easiest magnetisation, the crystal being a parallelepipedon. We introduce as in sect. 1 the co-ordinate-axes with the Z -axis parallel to the distinguished axis, and the YZ -plane parallel to the magnetic layers of the crystal. The distribution of s is uniform in the direction of the Y -axis, changing with x and towards the surface with z . The

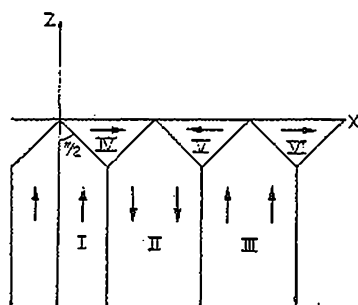


FIG. 1.

intermediate regions between the layers are of no importance for the distribution of s near the surface, as was already noted. We can therefore assume that s is everywhere situated in the ZX -plane; the angle between s and Z -axis we denote by φ , so that we now have

$$s_x = s \sin \varphi, \quad s_y = 0, \quad s_z = s \cos \varphi. \quad (13)$$

At a great distance from the surface there are regular layers, say of the width d , with opposite directions of s , i.e. φ is equal successively to zero or π , jumping from one of these values to another periodically along the X -axis with the period d . The distribution of s is then given by equation (11) with the boundary conditions (if the surface of the crystal is the XY -plane) $\varphi = \pm \pi/2$ for $z = 0$ and for $z = -\infty$; φ as a function of x must change from 0 to π after every interval d .

The solution of (11), which satisfies these conditions, can be constructed in the way shown on Fig. 1. This figure shows the distribution of the magnetic moments in the crystal in the XZ -plane, or some other plane, parallel to this

one (the arrows showing the direction of \mathbf{s}). In regions I and III the moments are directed along the Z -axis ($\varphi = 0$); in II along the same axis but in the opposite direction ($\varphi = \pi$); in IV, V, VI the moments are parallel to the surface, i.e. $\varphi = +\pi/2$. This solution evidently satisfies the boundary conditions. It satisfies also equation (11), for in each of the regions I, II etc. \mathbf{s} is constant, and on the boundaries between these regions the surface divergence is equal to zero, for the components of \mathbf{s} normal to these boundaries are everywhere equal on both sides of the boundary.

It is of course also possible to construct some other solution of (11), for instance if we introduce somewhere inside the crystal the same distribution as we have in Fig. 1 near the surface. But our solution is the one which gives for a given d the least energy of the crystal, and is therefore the only one allowed by physical considerations.

We can now calculate the energy of the crystal. The energy which is due to the distribution of magnetic moments near the surface of the crystal is per unit volume the anisotropy energy $\beta s_x^2/2$ (for $s_y = 0$). In the regions such as IV, V, VI (Fig. 1) s_x is equal to $\pm s$, i.e., the energy per unit volume is $\beta s^2/2$. Let l_1, l_2, l be the dimensions of the crystal in the directions of the X, Y, Z -axes respectively. The volume of one of the regions, like IV, V, VI (Fig. 1) is then equal to $d^3 l_2/4$. There are l_1/d such regions on each of the two opposite surfaces of the crystal, their total volume is $d l_1 l_2/2$ and therefore the energy due to the surface of the crystal is

$$E_s = \frac{1}{4} \beta d l_1 l_2 s^2. \quad (14)$$

In the inner parts of the crystal the energy is due to the presence of the intermediate regions between the layers. We can calculate it with the help of the results obtained in section 1. To find this energy we consider one of such regions. Its energy is according to section 1

$$\int \left(\frac{1}{2} \alpha s^2 \theta'^2 + \frac{1}{2} \beta s^2 \sin^2 \theta \right) dV$$

where we must insert θ from (9). θ is a function only of x . Therefore the integration over $dy dz$ gives the surface $l l_2$ of the layer, and we get the energy for one intermediate region:

$$\frac{1}{2} l l_2 s^2 \int_{-\infty}^{+\infty} (\alpha \theta'^2 + \beta \sin^2 \theta) dx = l l_2 s^2 \beta \int_{-\infty}^{+\infty} \frac{dx}{\cosh^2 \sqrt{\frac{\beta}{\alpha}} x} = 2 s^2 l l_2 \sqrt{\alpha \beta}.$$

There are l_1/d such regions in the crystal and the whole internal energy of the crystal is

$$E_i = 2 s^2 l l_1 l_2 \frac{\sqrt{\alpha \beta}}{d}. \quad (15)$$

We now find the width d of the layers from the condition that the whole energy $E = E_i + E_s$ of the crystal must be a minimum. Hence we find

$$d = 2 \sqrt{2l} \sqrt[4]{\frac{\alpha}{\beta}}, \quad (16)$$

the corresponding energy being

$$E = s^2 l_1 l_2 \sqrt{2l\beta} \sqrt[4]{\alpha\beta}. \quad (17)$$

If the surface of the crystal is not perpendicular to the axis of easiest magnetisation, the solution of (11), which satisfies the same conditions at a large distance from the surface and with s parallel to the surface on the surface of the crystal, can be constructed in the same way, and we find the distribution of s in a plane parallel to XZ as is shown on Fig. 2.

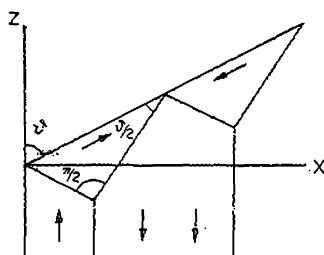


FIG. 2.

Let the angle between the Z -axis and the intersection of surface of the crystal with the plane XZ be ϑ_1 , and the same angle for the opposite surface be ϑ_2 . The angle of inclination of the surface to the plane XZ is of no importance for the width of the layers. In the same way as we obtained (16) we now find

$$d = 2 \sqrt{l \left(\frac{1}{\sin \vartheta_1} + \frac{1}{\sin \vartheta_2} \right)} \sqrt[4]{\frac{\alpha}{\beta}}. \quad (18)$$

The length l of the crystal and the angles ϑ_1 , ϑ_2 can in general change along the crystal, with them the width of the layers changes also.

Now it is easy to show that the crystal really consists of magnetic layers and not of thread-like regions. If the crystal consisted of elementary regions of spontaneous magnetisation which have the form of rectilinear prisms with the base d^2 , the distribution of the magnetic moments near the surface, satisfying (11) and (12), could be represented by Fig. 3, showing two of the elementary regions in a slit and in a plane. The energy E_i is now twice as large as (15); for the surface of the boundaries between the elementary regions is now twice as large as we had before. The surface energy E_s is now, as is easy to calculate,

$$E_s = \frac{1}{6} \beta s^2 l_1 l_2 d.$$

If we determine d so that the whole energy $E_i + E_s$ would be a minimum, we find this minimal energy

$$\sqrt{\frac{8}{3}} s^2 l_1 l_2 \sqrt{l \beta} \sqrt[4]{\alpha \beta},$$

i.e. $\sqrt{4/3}$ times larger than (17). Hence we see that this model is energetically less advantageous than a model with layers.

Numerically we get from (16) for deformed nickel with the same numbers as in section 1 the width of the layers $d \cong 5 \times 10^{-3}$ cm or *ca.* 10^5 lattice-constants (for $l = 1$ cm).

3. If the crystal is placed in an external magnetic field, which is directed parallel to the axis of easiest magnetisation, the boundaries between the layers begin to move, so that the layers with magnetic moments parallel to the field become wider. We shall now determine the velocity of this propagation.

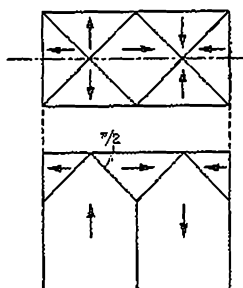


FIG. 3.

As in section 1 we consider only one intermediate region between two layers and neglect also the surface-effects. The distribution of s is given by (3) and (9) and is at rest when the field is absent. It begins to move with a velocity v along the X -axis when the field is introduced.

If a magnetic moment in the crystal were free, i.e. were not under the influence of the other moments, the variation of s with time would be determined by the external field. The influence of the interaction between the magnetic moments can be characterised by introducing an "effective field" in the following way.

If the macroscopical field-strength inside the crystal is H , the energy of the crystal can be written as

$$\int \left[\frac{1}{2} \alpha s'^2 - \frac{1}{2} \beta s_z^2 - (H \cdot s) \right] dV \quad (19)$$

(it is more convenient to write here the anisotropy energy as $-\frac{1}{2} \beta s_z^2$ instead of $\frac{1}{2} \beta (s_x^2 + s_y^2)$; both these expressions are evidently equivalent). We did not write the term $(H \cdot s)$ in the formulae of section 1, because the macroscopical field-strength inside the crystal was zero when there was no external field and all s were distributed in a ZY -plane.

In equilibrium this energy must be a minimum, i.e., the variation of (19) with \mathbf{s} must be equal to zero. This gives

$$\int ([\alpha \mathbf{s}'' + \beta s_z \mathbf{n} + \mathbf{H}] \cdot \delta \mathbf{s}) dV = 0$$

(\mathbf{n} is a unit vector in the direction of the Z -axis). But $\delta \mathbf{s}$ is always perpendicular to \mathbf{s} (because the absolute value of \mathbf{s} is constant); therefore we see that

$$\mathbf{f} = \alpha \mathbf{s}'' + \beta s_z \mathbf{n} + \mathbf{H} \quad (20)$$

must be parallel to \mathbf{s} . Hence this quantity \mathbf{f} now plays the part of the "effective field".

There are two kinds of the interaction between the magnetic moments in the crystal: exchange-interaction and relativistic interaction. The latter is in general much weaker than the former. Therefore in the presence of the field the magnetic moment would act as a free moment, i.e. would rotate around \mathbf{f} and we should have for $\dot{\mathbf{s}}$ (\cdot denotes differentiation with respect to the time) the equation

$$\frac{\dot{\mathbf{s}}}{\mu_0} = [\mathbf{f} \wedge \mathbf{s}]$$

with $\mu_0 = e/mc$ (and not $e/2mc$, because the moments \mathbf{s} in ferromagnetic bodies are spin-moments). The approach of \mathbf{s} to \mathbf{f} is due only to the relativistic interaction. Since this interaction is weaker than the exchange-interaction, we can assume that the coefficient before the term $[\mathbf{f} \wedge \mathbf{s}]$ is not altered, and we can simply add a term giving the approach of \mathbf{s} to \mathbf{f} .

$$\frac{\dot{\mathbf{s}}}{\mu_0} = [\mathbf{f} \wedge \mathbf{s}] + \lambda \left(\mathbf{f} - \frac{(\mathbf{f} \cdot \mathbf{s}) \mathbf{s}}{s^2} \right). \quad (21)$$

The second term here is a vector directed from \mathbf{s} to \mathbf{f} . The constant λ is $\lambda \ll s$ in accordance with the fact that the relativistic interaction is weak. We disregard here altogether the variation of the absolute value of \mathbf{s} .

To apply this equation to the motion of the boundary between the two layers, we must determine first the macroscopical field \mathbf{H} inside the crystal. Since the whole distribution of \mathbf{s} is independent of the coordinates y and z , the equations determining the field, i.e.

$$\text{curl } \mathbf{H} = 0, \quad \text{div}(\mathbf{H} + 4\pi \mathbf{s}) = 0$$

become

$$\frac{\partial H_y}{\partial x} = 0, \quad \frac{\partial H_z}{\partial x} = 0, \quad \frac{\partial (H_x + 4\pi s_x)}{\partial x} = 0. \quad (22)$$

If there is an external field h in the direction of the Z -axis (h can of course be a function of time), we can put

$$H_x = -4\pi s_x, \quad H_y = 0, \quad H_z = h, \quad (23)$$

if we remember that inside the crystal, far from the intermediate regions, i.e. where $s_x = 0$, the field H must be equal to the external field h . If s_x were everywhere equal to zero, and no external field were present, H would also be equal to zero, as in section 1.

We suppose that the field h is small (compared with $s\beta$). When the field is absent s is determined by (3) and (9). In the presence of the field s_x is no longer equal to zero, but if h is small, s_x , being proportional to h , is also small. Inserting now (20) and (23) in (21) and neglecting everywhere the terms of the second order in s_x and h , we obtain for the components of s the equations:

$$\left. \begin{aligned} \frac{\dot{s}_x}{\mu_0} &= \alpha(s_x s_y'' - s_y s_x'') - \beta s_y s_x - s_y h + \lambda(\alpha s_x'' - 4\pi s_x) \\ &\quad - \frac{\lambda}{s^2} [\alpha(s_y s_y'' + s_z s_z'') s_x + \beta s_x s_x^2], \\ \frac{\dot{s}_y}{\mu_0} &= \alpha(s_x s_z'' - s_z s_x'') + (4\pi + \beta) s_x s_z + \lambda \alpha s_y'' \\ &\quad - \frac{\lambda}{s^2} [\alpha(s_y s_y'' + s_z s_z'') s_y + \beta s_y s_z^2 + h s_y s_x], \\ \frac{\dot{s}_z}{\mu_0} &= \alpha(s_x'' s_y - s_x s_y'') - 4\pi s_x s_y + \lambda(\alpha s_z'' + \beta s_z + h) \\ &\quad - \frac{\lambda}{s^2} [\alpha(s_y s_y'' + s_z s_z'') s_z + \beta s_z^3 + h s_z^2]. \end{aligned} \right\} \quad (24)$$

In the absence of the field $\dot{s}_x = \dot{s}_y = \dot{s}_z = 0$, $h = 0$, $s_x = 0$ and if we put (3) into (24) we obtain just equation (5). s_x , s_y , s_z in (24) are functions of x and of the time t . We suppose that both variables enter only in a combination $x - vt$, v being the velocity of propagation of the whole distribution along the X -axis.

Then $\dot{s}_x = -s v_x'$ and the same for \dot{s}_y , \dot{s}_z , if ' now denotes differentiation by $x - vt$. To solve (24) we put

$$s_x = s, \quad s_y = s \sin(\theta + \psi), \quad s_z = s \cos(\theta + \psi), \quad (25)$$

where θ is determined from (9) (and satisfies (5) or (6)) where instead of x must now stand $(x - vt)$. ψ is small compared with θ ; s_x and $s\psi$ are both proportional to h (and are equal to zero when $h = 0$) and we can neglect the terms of the second order in s_x and ψ . We presume, as is verified by the result, that the velocity v is also proportional to h ; therefore we neglect such terms as $v s_x$ or $v s\psi$ also.

$$s(\alpha s\psi'' - \beta s\psi \cos 2\theta - h \sin \theta) + \lambda[\alpha s_x'' - (4\pi\beta \cos 2\theta) s_x] = 0,$$

$$\lambda(\alpha s\psi'' - \beta s\psi \cos 2\theta - h \sin \theta) - s[\alpha s_x'' - (4\pi + \beta \cos 2\theta) s_x] = -\frac{s v \theta'}{\mu_0}.$$

The third equation is identical with the second. Hence we find:

$$\left. \begin{aligned} \alpha s \psi'' - \beta s \psi \cos 2\theta - h \sin \theta \\ &= -\frac{s v \lambda \theta'}{\mu_0 (s^2 + \lambda^2)} = -\frac{v \lambda s}{\mu_0 (s^2 + \lambda^2)} \sqrt{\frac{\beta}{\alpha}} \sin \theta, \\ \alpha s_x - (4\pi + \beta \cos 2\theta) s_x \\ &= \frac{v \theta' s^2}{(s^2 + \lambda^2) \mu_0} = \frac{v s^2}{\mu_0 (s^2 + \lambda^2)} \sqrt{\frac{\beta}{\alpha}} \sin \theta \end{aligned} \right\} \quad (26)$$

or according to (9):

$$\theta' = \sqrt{\frac{\beta}{\alpha}} \sin \theta.$$

It we introduce θ as an independent variable instead of $x - vt$ we get from (26)

$$\left. \begin{aligned} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\psi}{d\theta} \right) + \left(2 - \frac{1}{\sin^2 \theta} \right) \psi \\ &= \frac{1}{s \beta \sin \theta} \left(h - \frac{v \lambda s}{\mu_0 (s^2 + \lambda^2)} \sqrt{\frac{\beta}{\alpha}} \right), \\ \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{ds_x}{d\theta} \right) + \left(2 - \frac{1 + 4\pi/\beta}{\sin^2 \theta} \right) s_x \\ &= \frac{v s^2}{\mu_0 \beta (s^2 + \lambda^2)} \sqrt{\frac{\beta}{\alpha}} \frac{1}{\sin \theta}. \end{aligned} \right\} \quad (27)$$

Both of these equations are of the type

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dy}{d\theta} \right) + \left(2 - \frac{m^2}{\sin^2 \theta} \right) y = f(\theta).$$

This equation has a solution only in the cases: (1) if $m = 1, 2, \dots$ and $f(\theta)$ is zero (the solution is then $P_1^m(\cos \theta)$) or orthogonal to the solutions of the homogeneous equation; (2) if m is not an integer and $f(\theta)$ is not zero. Hence we see that the equations (27) can have a solution only if the right part of the first one is equal to zero, i.e. if

$$v = \frac{\mu_0 (s^2 + \lambda^2)}{\lambda s} \sqrt{\frac{\alpha}{\beta}} h. \quad (28)$$

But $\lambda \ll s$; therefore we can write:

$$v = \frac{\mu_0 s}{\lambda} \sqrt{\frac{\alpha}{\beta}} h. \quad (29)$$

This determines the velocity of motion of the boundaries between the layers in an external magnetic field, directed along the axis of easiest magnetisation.

If the external field h is a periodical field which we write in the form

$$h = h_0 e^{i\omega t} \quad (30)$$

the magnetisation of the crystal is on the average equal to zero. The magnetic susceptibility is then defined as the ratio of the magnetisation as a function of time to $h(t)$. In the time t the boundary between any two layers moves a distance (we put (30) in (29) and integrate):

$$\frac{\mu_0 s}{i \lambda \omega} \sqrt{\frac{\alpha}{\beta}} h.$$

There are l_1/d layers in the crystal (notations as in section 2). Therefore the total magnetisation of the crystal is

$$\frac{\mu_0 s^2}{i \omega \lambda} \sqrt{\frac{\alpha}{\beta}} \frac{l l_1 l_2}{d} h.$$

Hence the magnetic susceptibility χ_l for a unit volume in the longitudinal field is:

$$\chi_l = \frac{\mu_0 s^2}{i \omega \lambda d} \sqrt{\frac{\alpha}{\beta}}. \quad (31)$$

The magnetic permeability is

$$\mu_l = 1 - \frac{4\pi i \mu_0 s^2}{\omega \lambda d} \sqrt{\frac{\alpha}{\beta}}. \quad (32)$$

The expression for d we can take from (16) or (18). It is possible then to check the dependence of d on the dimensions of the crystal, if we determine experimentally the dependence of μ_l from these dimensions.

Hence, as a function of ω , χ_l or μ_l have no proper frequencies, but a mere damping. They become infinite with $\omega = 0$ in accordance with the fact that we did not take into consideration any hysteresis-effects. For large α the variation of the absolute value of s (which we neglected here) can prevail, and this formula for μ_l can become insufficient.

4. We shall determine now the magnetic permeability in a transverse field, i.e. when the field h is directed along the X -axis. The effect of the intermediate regions is now of no importance, and we can assume that in the absence of the field all the moments are directed parallel or anti-parallel (in the different layers) to the Z -axis, i.e. $s_z = \pm s$ and $s_x = s_y = 0$. For the same reason the term $\alpha s''$ in (20) is now of no importance, and we must put for the effective field:

$$f = H + \beta s_z n. \quad (33)$$

The components of H are now

$$H_x = h, \quad H_y = H_z = 0, \quad (34)$$

and equation (21) gives for the components of \dot{s} :

$$\left. \begin{aligned} \frac{\dot{s}_x}{\mu_0} &= -\beta s_y s_z + \lambda h - \frac{\lambda}{s^2} (h s_x + \beta s_z^2) s_x \\ \frac{\dot{s}_y}{\mu_0} &= \beta s_x s_z - h s_x - \frac{\lambda}{s^2} (h s_x + \beta s_z^2) s_y \\ \frac{\dot{s}_z}{\mu_0} &= h s_y + \lambda \beta s_z - \frac{\lambda}{s^2} (h s_x + \beta s_z^2) s_z \end{aligned} \right\} \quad (35)$$

s_y and s_x are proportional to h , and if we assume, as in section 3, that h is small, we can neglect the terms of the second order in s_x , s_y , h . Instead of s_z we can now put $\pm s$ and we obtain from (35)

$$\frac{\dot{s}_x}{\mu_0} = \mp s \beta s_y + \lambda h - \lambda \beta s_x, \quad \frac{\dot{s}_y}{\mu_0} = \pm s \beta s_x \mp h s - \lambda \beta s_y. \quad (36)$$

The third equation becomes an identity.

If h is determined with (30), we can solve (36) with

$$s_x = s_{x0} e^{i\omega t}, \quad s_y = s_{y0} e^{i\omega t}.$$

We then obtain

$$s_x = \frac{\mu_0^2 \beta (s^2 + \lambda^2) + i \omega \lambda \mu_0}{(i \omega + \beta \lambda \mu_0)^2 + \beta^2 s^2 \mu_0^2} h \quad (37)$$

and

$$s_y = \pm \frac{\mu_0^2 \beta s \lambda - \mu_0 s (i \omega + \mu_0 \beta \lambda)}{(i \omega + \beta \lambda \mu_0)^2 + \beta^2 s^2 \mu_0^2} h \quad (38)$$

s_y has opposite directions in different layers and therefore gives nothing for the magnetisation of the whole crystal. From (37) we get the magnetic susceptibility (for unit volume) for the magnetisation in the direction of the X -axis in a transversal field:

$$\chi_t = \frac{\mu_0^2 \beta (s^2 + \lambda^2) + i \omega \lambda \mu_0}{(i \omega + \beta \lambda \mu_0)^2 + \beta^2 s^2 \mu_0^2}.$$

But $\lambda \ll s$ and we can write

$$\chi_t = \frac{\mu_0^2 \beta s^2 + i \omega \lambda \mu_0}{\beta^2 s^2 \mu_0^2 - \omega^2 + 2i \omega \beta \lambda \mu_0}, \quad (39)$$

and the magnetic permeability is

$$\mu_t = 1 + 4\pi \frac{\mu_0^2 \beta s^2 + i \omega \lambda \mu_0}{\beta^2 s^2 \mu_0^2 - \omega^2 + 2i \omega \beta \lambda \mu_0}. \quad (40)$$

Hence, as a function of ω , χ_t or μ_t have a proper frequency

$$\omega_0 = \mu_0 \beta s \quad (41)$$

and a damping with a decrement

$$\gamma = \mu_0 \beta \lambda. \quad (42)$$

Evidently $\gamma \ll \omega_0$, because $\lambda \ll s$. With the same numerical values as in section 1, we find this proper frequency for deformed nickel to be $1.5 \times 10^{10} \text{ sec}^{-1}$, corresponding to the wavelength 12.6 cm.

It is possible to distinguish in (40), which we can write

$$\mu_t = 1 + 4\pi \frac{\omega_0^2 + i \omega \gamma}{\beta(\omega_0^2 - \omega^2 + 2i \omega \gamma)}, \quad (43)$$

some different cases according to the value of the frequency:

$$(a) \quad \omega \ll \frac{\omega_0^2}{\gamma}$$

$$\mu_t = 1 + \frac{4\pi \omega_0^2}{\beta(\omega_0^2 - \omega^2)}, \quad (44)$$

$$(b) \quad \omega - \omega_0 \sim \gamma$$

$$\mu_t = 1 + \frac{2\pi \omega_0}{\beta(\omega_0 - \omega + i \gamma)}, \quad (45)$$

$$(c) \quad \omega \gg \omega_0$$

$$\mu_t = 1 - 4\pi \frac{\omega_0^2 + i \omega \gamma}{\beta \omega^2}. \quad (46)$$

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19. ON THE RELATIVISTIC CORRECTION OF THE SCHRÖDINGER EQUATION FOR THE MANY-BODY PROBLEM

THE first derivation of the interaction terms of order v^2/c^2 was made by Breit¹. The same formula was subsequently derived more rigorously by Bethe and Fermi² from quantum electrodynamics. Breit's formula contains however, as he himself emphasised, a term proportional to e^4/r^2 which contains no \hbar . Thus in the limiting classical case, this is not in agreement with classical electrodynamics. It is also in contradiction to experiments. Breit³ has indeed later shown that the superfluous term can be eliminated by a different treatment. In this approach, however, the use made of quantum electrodynamics cannot be said to be free from objection. I should like to point out that Breit's formula can be obtained in a much more natural manner simply by writing all the equations to the order of magnitude v^2/c^2 . The Dirac equation for an electron is in this case:

$$E \psi_1 = \left\{ \frac{p^2}{2m} + e \varphi - \frac{p^4}{8m^3 c^2} - \frac{i e \hbar}{4m c^2} (V \cdot r) + \right. \\ \left. + \frac{1}{2c} ([v \wedge E] \cdot \mu) - \frac{e}{m c} \frac{(v \cdot A) + (A \cdot v)}{2} - (\mu \cdot H) \right\} \psi_1 \quad (1)$$

(μ is the matrix vector of the magnetic spin). Now we can calculate φ , A , E , H for the second electron from the current and charge density. In the relevant approximation, we have:

$$\varphi = \int \frac{\varrho}{r} dv, \quad A = \frac{1}{c} \int \frac{j + (j \cdot n) n}{r} dv \quad (2)$$

(n is a unit vector in the direction of r). We can now substitute for ϱ and j from the expressions for the Dirac charge density and current for the second electron:

$$\varrho = e \left\{ |\psi_2|^2 + \frac{\hbar^2}{4m^2 c^2} |\nabla \psi_2|^2 \right\} + \frac{i n}{2m c} + (\nabla \psi^* \cdot [\mu \wedge \nabla \psi_2]), \quad (3)$$

$$j = \frac{e \hbar^2}{2im} (\psi_2^* \nabla \psi_2^* - \psi_2 \nabla \psi_2^*) + c \operatorname{curl} (\psi_2^* \mu \psi_2). \quad (4)$$

L. Landau, Über die relativistische Korrektur der Schrödingergleichung für das Mehrkörperproblem, *Phys. Z. Sowjet.* 8, 487 (1935).

After substituting from (3) and (4) in (2) and then (1) the Breit formula without the undesired term is obtained directly.

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20. ON THE THEORY OF THE ACCOMMODATION COEFFICIENT

It is shown that the accommodation coefficient must be treated entirely classically in the most important case. In this situation a $T^{3/2}$ proportionality is found for not too high temperatures. For hydrogen and helium at low enough temperatures, a quantum boundary case can arise, which leads to a T^3 law.

THE theory of accommodation coefficients has been treated by several authors^{1,2,3}. However, in some cases unjustified simplifications were made (e.g. considering a one-dimensional model) and in others the orders of magnitude of the quantities involved were not considered, so that no final formula could be written down. Let us consider the collision of an atom with the surface of a condensed body; an important part is played by the order of magnitude of the collision frequency, i.e. of the reciprocal of the duration of the time of interaction between the atom and the surface. As a result of the collisions only those oscillations can arise which have a frequency which is at most of the same order of magnitude as the collision frequency. Important is therefore the relation between the collision frequency and the greatest oscillation frequency of the condensed body. A reasonable theoretical treatment can only be undertaken in the case for which the collision frequency is significantly less than the "reststrahl" value, so that only slow, i.e. long wave, vibrations can be induced. Such vibrations can be treated as being acoustic.

Since the interaction between the surface and the atom must anyway be idealised, we shall only consider the case of a liquid surface, although the results can certainly be applied with the same qualitative accuracy to the case of a solid body, in particular that of a crystal. Let us consider a characteristic vibration in the form of a plane wave reflected from the surface. Since the pressure must be zero at the surface, it can be written in the form:

$$p = A \{ (\cos[\omega t - (\mathbf{k}_1 \cdot \mathbf{r}_1)] - \cos[\omega t - (\mathbf{k}_2 \cdot \mathbf{r}_2)]) \} \quad (1)$$

where A is the amplitude, ω is the frequency, and \mathbf{k}_1 and \mathbf{k}_2 are the wave vectors of the incident and reflected waves. The origin of the co-ordinates is chosen on the surface plane.

The hydrodynamic equations are expressed by

$$\frac{\partial v}{\partial t} = - \frac{\text{grad } p}{\rho} \quad (2)$$

(v velocity, ρ density).

L. Landau, Zur Theorie des Akkommodationskoeffizienten, *Phys. Z. Sowjet.* 8, 489 (1935).

From this,

$$v = \frac{A}{\rho \omega} \{k_2 \sin[\omega t - (k_2 \cdot r)] - k_1 \sin[\omega t - (k_1 \cdot r)]\}. \quad (3)$$

In order to determine the motion of the surface, consider the normal component of the velocity for a point on the surface, for example, the origin. We have

$$v_z = - \frac{2A k_z \sin \omega t}{\rho \omega}, \quad (4)$$

where

$$k_z = k_{1z} = -k_{2z}.$$

If we now put

$$v_z = \frac{d\zeta}{dt},$$

then we have for the movement of a point on the surface:

$$\zeta = \frac{2A k_z}{\rho \omega^2} \sin \omega t. \quad (5)$$

The quantity ζ can be considered to be the co-ordinate of the characteristic oscillation. To relate it to the normal co-ordinate we must calculate the total energy of the vibration.

The total acoustic energy is equal to twice the mean kinetic energy of the liquid:

$$\overline{\int \rho v^2 dV}. \quad (6)$$

Since we have two plane waves, the energy can be considered as the sum of the two waves. Substituting (3) for v , we have

$$\overline{\int \rho v^2 dV} = \frac{A^2 k^2}{\rho \omega^2} V, \quad (7)$$

where V denotes the entire volume. (k is the absolute value of k_1 and k_2 .)

This quantity is by definition equal to the mean square of the time derivative of the normal co-ordinate \dot{q}^2 . From (5) follows

$$\overline{\dot{\zeta}^2} = \frac{2A^2 k_z^2}{\rho^2 \omega^2}, \quad (8)$$

or, considering (7)

$$\overline{\dot{\zeta}^2} = \frac{2k_z^2}{k^2 \rho V} \overline{q^2}. \quad (9)$$

From this it follows that we can put

$$\zeta = \sqrt{\frac{2}{M}} \frac{k_z}{k} q \quad (10)$$

where M is the total mass of the liquid. Let us now consider an atom which impinges onto the surface in the neighbourhood of the origin. Its distance from the surface is $z - \zeta$, and the corresponding potential energy is $U(z - \zeta)$, or for small ζ ,

$$U(z - \zeta) = U - \zeta \frac{dU}{dz}. \quad (11)$$

Here the interaction with every characteristic mode can be considered separately.

The perturbation energy is thus equal to

$$- \sqrt{\frac{2}{M}} \cos \theta \cdot q \frac{dU}{dz} \quad (12)$$

(θ is the angle between the direction of the wave vector and the normal to the surface). To calculate the corresponding excitation probabilities, we should have to specify the matrix elements for dU/dz . This is not even possible with special assumptions about U . However, we can consider two cases, for which the calculation is much simplified.

First case: \hbar times the collision frequency is small compared with $k_B T$ (k_B : Boltzmann constant) i.e. the energy given up is small compared with the energy of the atoms. We will discuss later the condition necessary for this situation. We merely remark now that these conditions are satisfied for all gases except hydrogen and helium throughout the entire temperature range in which they exist as gases. When the energy given up is small compared with that of the atoms, the problem can be considered classically. In the first place we calculate the mean energy given up to a non-excited mode of vibration at a collision. This is given by

$$\frac{1}{2} \left| \int F e^{i\omega t} dt \right|^2, \quad (13)$$

where $F = -\sqrt{2/M} \cos \theta dU/dZ$ is the effective force. To obtain exact formulae, we must make a specific assumption about the form of the potential energy, U . We put

$$U = A e^{-z/a}. \quad (14)$$

The constant, A , here depends on the choice of the origin of the co-ordinates, and can be replaced by an arbitrary quantity. Since the movement in the z -direction can be considered separately, because of the translational symmetry in the x and y directions (long waves!) we have

$$\frac{m \dot{z}^2}{2} = \frac{m v_z^2}{2} - A e^{-z/a}, \quad (15)$$

where v_z is the z -component of the velocity at infinity. We now choose $A = m v_z^2/2$. We have

$$\dot{z} = dz/dt = v_z \sqrt{1 - e^{-z/a}}, \quad (16)$$

$$t = \frac{1}{v_z} \int \frac{dz}{\sqrt{1 - e^{-z/a}}} = \frac{2a}{v_z} \operatorname{arc} \cosh e^{z/2a} \quad (16a)$$

whence

$$e^{z/2a} = \cosh \frac{v_z t}{2a}. \quad (17)$$

The force is

$$F = -\sqrt{\frac{2}{M}} \cos \theta \cdot \frac{dU}{dz} = \sqrt{\frac{2}{M}} \cos \theta \cdot \frac{m v_z^2}{2a} e^{-z/a} \quad (18)$$

or

$$F = \sqrt{\frac{2}{M}} \frac{m v_z^2}{2a \cosh^2 \frac{v_z t}{2a}} \cos \theta. \quad (18a)$$

The Fourier component of the force is then equal to

$$\sqrt{\frac{2}{M}} \cos \theta \cdot \frac{m v_z^2}{2a} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} dt}{\cosh^2 \frac{v_z t}{2a}}. \quad (19)$$

If we put

$$\tanh \frac{v_z t}{2a} = u,$$

then (19) becomes

$$\sqrt{\frac{2}{M}} \cos \theta \cdot m v_z \int_{-1}^1 du \left(\frac{1+u}{1-u} \right)^{-i\omega a/v_z} \quad (20)$$

But it can be shown that

$$\int_{-1}^1 \left(\frac{1+u}{1-u} \right)^n du = \frac{2n\pi}{\sin \pi n}, \quad (21)$$

and we have

$$\int F e^{-i\omega t} dt = \sqrt{\frac{2}{M}} \cos \theta \cdot \frac{2\pi m \omega a}{\sinh \frac{\pi \omega a}{v_z}}. \quad (22)$$

The energy given to a mode of vibration is thus

$$\frac{1}{M} \cos^2 \theta \cdot \frac{4\pi^2 m^2 \omega^2 a^2}{\sinh^2 \frac{\pi \omega a}{v_z}}. \quad (23)$$

If we take the average of this expression over all directions of the modes, then for a mode of frequency, ω , we have

$$\frac{4\pi^2 m^2 \omega^3 a^2}{3M \sinh^2 \frac{\pi \omega a}{v_z}}. \quad (24)$$

To calculate the total energy ε given up to all modes, we must integrate this expression for all the modes. The number of modes for a liquid is given by

$$V \frac{4\pi \omega^2 d\omega}{c^3} \quad (25)$$

(c is the sound velocity), and ε is then given by

$$\varepsilon = \int \frac{16\pi^3 m^2 a^2}{3\rho c^3} \frac{\omega^4 d\omega}{\sinh^2 \frac{\pi \omega a}{v_z}}. \quad (26)$$

Since we have assumed throughout that the frequencies of the modes which are excited are small compared with $k_B T/\hbar$, we can take the integral from 0 to ∞ . The integral

$$\int_0^\infty \frac{x^4 dx}{\sinh^2 x}$$

after being partially integrated, becomes

$$\int_0^\infty 4(\cosh x - 1) x^3 dx = \int_0^\infty \frac{8x^3 dx}{e^{2x} - 1} = \frac{1}{2} \int_0^\infty \frac{y^3 dy}{e^y - 1}.$$

The last integral is known from the theory of heat radiation, and is equal to $\pi^4/15$. Thus we obtain finally

$$\varepsilon = \frac{8\pi^2 m^2 v_z^3}{45\rho c^3 a^3}. \quad (27)$$

Now the number of atoms with velocity v_z falling per unit time on unit area is

$$v_z e^{-m^2/2 k_B T} dv_z.$$

The mean energy given up by one atom of gas to an unexcited lattice is thus

$$\bar{\varepsilon} = \int_0^\infty \varepsilon v_z e^{-m^2/2 k_B T} dv_z / \int_0^\infty v_z e^{-m^2/2 k_B T} dv_z, \quad (29)$$

which comes simply to

$$\varepsilon = \frac{(2\pi k_B T)^{5/2}}{3\rho c^3 a^3 m^{1/2}}. \quad (29a)$$

If the lattice has a finite temperature T' , it loses energy to the gas. In this case, the energy given by a mode is proportional to the energy of this mode, or since it is assumed that $\hbar\omega \ll k_B T'$, proportional to T' . The total energy given up by the lattice is thus proportional to T' . The coefficient of proportionality can be calculated from the condition that at thermal equilibrium the two energies must be equal.

The energy given up by the lattice is thus $\varepsilon T'/T$ and the difference,

$$\varepsilon \left(1 - \frac{T'}{T} \right) = \frac{2\pi k_B (2\pi k_B T)^{3/2}}{3\varrho c^3 a^3 m^{1/2}} (T - T'). \quad (30)$$

We normally denote by the accommodation coefficient, α , the ratio of this energy to the greatest possible amount which can be transferred. Since the mean energy of the incident atom is equal to $2k_B T$, then this greatest possible amount is $2k_B(T - T')$. The accommodation coefficient is equal to

$$\alpha = \frac{\pi}{3\varrho m^{1/2}} \left(\frac{2\pi k_B T}{a^2 c^2} \right)^{3/2}. \quad (31)$$

Thus a $T^{3/2}$ law is found. In place of the velocity of sound we can introduce the Debye temperature, Θ , which is related to the sound velocity by:

$$\frac{4\pi}{3} V \frac{k^3 \Theta^3}{\hbar^3 c^3} = N \quad (32)$$

(N is the number of atoms in the lattice and \hbar Planck's constant $= 2\pi \hbar$). α is then given in the form

$$\alpha = \frac{N}{4V \varrho m^{1/2}} \left(\frac{2\pi \hbar^2 T}{a^2 k_B \Theta^2} \right)^{3/2} \quad (33)$$

or

$$\alpha = \frac{1}{4\mu m^{1/2}} \left(\frac{2\pi \hbar^2 T}{a^2 k_B \Theta^2} \right)^{3/2}, \quad (33a)$$

where μ is the mass of an atom in the lattice.

It is necessary for the validity of this formula, as was mentioned above, that the collision frequency is small. To establish the limits of its application, let us consider the opposite case of a large collision frequency. If the mass of the gas atom is not small compared to that of a lattice atom, then there is no reason in this case for the accommodation coefficient to be reduced, and it will in fact be of the order unity. The temperature limit for the validity of (33) is thus

$$\frac{1}{4\mu m^{1/2}} \left(\frac{2\pi \hbar^2 T}{a^2 k_B \Theta^2} \right)^{3/2} \ll 1. \quad (34)$$

If $m \ll \mu$, then at high temperatures the accommodation coefficient will still be small compared to unity. It is easy to see that then $\alpha \sim m/\mu$. We have

$$\frac{1}{4\mu m^{1/2}} \left(\frac{2\pi \hbar^2 T}{a^2 k_B \Theta^2} \right)^{3/2} \ll m/\mu. \quad (35)$$

If we put as usual, $a \sim 3 \times 10^{-8}$ cm, we obtain from equation (35)

$$T \ll m \Theta^2 / 30,000 \quad (36)$$

(m is here the atomic weight of the gas). In the case of (34) m must be replaced by $\sqrt[3]{m \mu^2}$. Comparison shows that these conditions have not been satisfied in cases so far observed, so that a comparison with theory cannot be made. These cases in fact apply to conditions in which the accommodation coefficient depends on the "form" of the atoms.

Second case: \hbar times the collision frequency is large compared with $k_B T$. The energy given up is then of the order of the energy of the incident atom. We can consider the surface of the body here as a fixed wall as far as the interaction with the gas atoms is concerned. From perturbation theory in the continuous spectrum, the probability of a change of state during the collision is

$$\frac{2\pi}{\hbar} |\eta|^2, \quad (37)$$

where η is the matrix element of the exchange energy, for which the wave functions of the initial states are normalised to a particle flux of one, and those of the end states are normalised with respect to the energy. The matrix elements for $-dU/dz$, i.e. for the effective force on the gas atoms, can best be obtained as the derivative of the matrix elements of the momentum. In accordance with our assumptions and the boundary conditions, $\psi = 0$ for $z = 0$, we have for the wave function, ψ , of the initial state:

$$\psi = \frac{2}{\sqrt{v}} \sin \frac{m v}{\hbar} z. \quad (38)$$

v is here naturally the z -component of the velocity. For the wave function of the end state, we have analogously

$$\psi' = \frac{2}{\sqrt{\pi \hbar v'}} \sin \frac{m v'}{\hbar} z. \quad (39)$$

Now the matrix element of the momentum is given by

$$\begin{aligned} -i \hbar \int \psi'^* \frac{d\psi}{dz} dz &= -\frac{4i m v}{\sqrt{\pi \hbar v'}} \int \cos \frac{m v}{\hbar} z \cdot \sin \frac{m v'}{\hbar} z \cdot dz \\ &= -\frac{2i m v}{\sqrt{\pi \hbar v'}} \int \left[\sin \frac{m}{\hbar} (v' - v) z + \sin \frac{m}{\hbar} (v + v') z \right] dz \\ &= \frac{2i m v}{\sqrt{\pi \hbar v'}} \left\{ \frac{\hbar}{m(v' - v)} \cos \frac{m}{\hbar} (v' - v) z \right. \\ &\quad \left. + \frac{\hbar}{m(v + v')} \cos \frac{m}{\hbar} (v + v') z \right\} \Big|_0^L. \end{aligned} \quad (40)$$

In these integrals, as is normal, the value of the cosine is averaged out at the upper limit. Substitution of $z = 0$ gives

$$\frac{4i}{\sqrt{\pi}} \frac{\sqrt{\hbar v v'}}{(v^2 - v'^2)}. \quad (41)$$

The matrix element of the force is obtained from this by multiplication with

$$i \frac{E' - E}{\hbar} = i \frac{m}{2\hbar} (v'^2 - v^2) \quad (42)$$

and is thus equal to

$$2m \sqrt{v v' / \pi \hbar}. \quad (43)$$

The matrix element of the normal co-ordinate, q , of the mode, we know to be equal to $\sqrt{n \hbar / 2\omega}$ when the number of excited quanta changes from n to $n - 1$ or vice versa.

The matrix element of the perturbation energy is given by

$$\eta = \sqrt{2/M} \cos \theta \sqrt{n \hbar / 2\omega} 2m \sqrt{v v' / \pi \hbar}. \quad (44)$$

For the transition probability, we obtain now from (7) and (30)

$$8 \cos^2 \theta n m^2 v v' / M \hbar \omega. \quad (45)$$

Integrating for all directions of the sound quanta, we obtain for the transition probability when $v' > v$:

$$\frac{m^2 v v' V 4\pi \omega^2 d\omega}{3\pi^3 M \hbar \omega c^3 (e^{\hbar \omega / k_B T'} - 1)}. \quad (46)$$

Substituting

$$\hbar \omega = m(v'^2 - v^2)/2,$$

we obtain

$$\frac{1}{\hbar \omega} \cdot \frac{V m^5 v v' (v'^2 - v^2) v' dv'}{3\pi^2 M \hbar^3 c^3 (e^{m(v'^2 - v^2)/2k_B T'} - 1)}. \quad (47)$$

We multiply this by the expression for the velocity distribution of the impinging atoms,

$$\frac{m}{k T} e^{-m v^2 / 2 k_B T} v dv$$

and by $\hbar \omega$ and obtain for the energy transferred from the lattice to the gas,

$$d\varepsilon_1 = \frac{m^6 (v'^2 - v^2)^2 e^{-m v^2 / 2 k_B T} v^2 dv v'^2 dv'}{3\pi^2 \rho \hbar^3 c^3 k_B T (e^{m(v'^2 - v^2)/2k_B T} - 1)}. \quad (48)$$

If we consider the reverse transition from velocity v' to v , n is to be replaced by $n + 1$. The energy given up by the gas to the lattice is thus equal to

$$d\varepsilon_2 = \frac{m^6 (v^2 - v'^2)^2 e^{-m v'^2 / 2 k_B T} v^2 dv v'^2 dv'}{3\pi^2 \rho \hbar^3 c^3 k_B T (1 - e^{m(v^2 - v'^2)/2k_B T})}. \quad (48a)$$

For the total energy transfer, we have

$$d\varepsilon = d\varepsilon_2 - d\varepsilon_1 = \frac{m^6(v^2 - v'^2)^2 v^3 dv v'^2 dv'}{3\pi^2 \rho \hbar^3 c^3 k_B T [e^{\frac{mv'^2}{2k_B T}} - e^{\frac{mv^2}{2k_B T}}]} \left[e^{\frac{mv'^2}{2k_B} \left(\frac{1}{T'} - \frac{1}{T} \right)} - e^{\frac{mv^2}{2k_B} \left(\frac{1}{T'} - \frac{1}{T} \right)} \right]. \quad (49)$$

If $T = T'$, then $d\varepsilon = 0$, as we should expect, and when $T - T'$ is small, we have

$$d\varepsilon = \frac{m^7(v'^2 - v^2)^3 v^2 dv v'^2 dv' (T - T')}{6\pi^2 \rho \hbar^3 c^3 k_B^2 T^3 [e^{\frac{mv'^2}{2k_B T}} - e^{\frac{mv^2}{2k_B T}}]}. \quad (49a)$$

To get the accommodation coefficients, we divide by $2k_B(T - T')$ and integrate:

$$\alpha = \iint \frac{m^7(v'^2 - v^2)^3 v^2 dv v'^2 dv'}{12\pi^2 \rho \hbar^3 c^3 k_B^2 T^3 [e^{\frac{mv'^2}{2k_B T}} - e^{\frac{mv^2}{2k_B T}}]}. \quad (50)$$

Let us write

$$mv^2/2k_B T = x, \quad mv'^2/2k_B T = y,$$

then

$$\alpha = \frac{4m(k_B T)^3}{3\pi^2 \rho \hbar^3 c^3} \int_0^\infty dx \int_0^\infty \frac{(y-x)^3 \sqrt{xy} dy}{e^y - e^x}. \quad (51)$$

This definite integral can be evaluated approximately. We have

$$\alpha \cong 1.7 \frac{m(k_B T)^3}{\rho \hbar^3 c^3}. \quad (51a)$$

Thus at sufficiently low temperatures, α varies as T^3 .

Comparing (51) and (31) shows that the condition for (51) is

$$4T \ll k_B \hbar^2 / ma^2 \quad (52)$$

or, substituting numerical values,

$$T \ll 2000/m. \quad (53)$$

This requirement shows that, except in the cases of helium and hydrogen, we are always dealing with the "classical" accommodation coefficient.

If we use the Debye temperature from (32), (51) assumes the simple form

$$\alpha \cong \frac{50m}{\mu} \left(\frac{T}{\Theta} \right)^3. \quad (54)$$

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21. ON THE THEORY OF THE PHOTOELECTROMOTIVE FORCE IN SEMICONDUCTORS

The electromotive force that appears in a circuit containing a semiconductor illuminated from one side is calculated. Two cases are regarded: a semiconductor with conducting electrons and a semiconductor with conducting electrons and "holes".

1. INTRODUCTION

If we have a circuit which contains a semiconductor in contact with a metal on both sides and one of the contacts is illuminated, an electromotive force (e.m.f.) appears in the circuit. This effect is known as the Dember-effect.

The light which falls on the semiconductor is absorbed inside it. By the absorption of a quantum of light the non-conducting electrons are thrown up into the conducting zone of energy levels, and so new conducting electrons appear in the semiconductor.

The light in general penetrates into the semiconductor to a distance of the order of some wavelengths, which is much greater than the mean free path of the electrons. Therefore we can assume that in every volume element of the semiconductor there is a uniform intensity of light, and the number of the conducting electrons created by the light in unit time in this element is proportional to this intensity (it is equal to the number of absorbed quanta). The number of conductivity electrons created in unit time in unit volume, we shall denote as J . Along the semiconductor there is, consequently, a gradient ∇J (practically J becomes zero at a distance of some wavelengths).

Under the influence of illumination the density of the (conducting) electrons varies along the semiconductor. If the velocity distribution function of the electrons remained a Boltzmann-function as in the case of no illumination, this change of the electron-density along the semiconductor could not give any e.m.f. in the circuit. Indeed, if the distribution of the charges along a closed circuit is not uniform, the potential differences which then appear in the circuit must cancel out along the whole circuit. In the opposite case a current would flow in the circuit, i.e. it would be equivalent to a perpetual mobile, for there are no exterior sources of energy. In particular, in our case the potential difference, which appears in the semiconductor owing to the

L. Landau and E. Lifshitz, On the theory of the photoelectromotive force in semiconductors, *Phys. Z. Sowjet.* 9, 477 (1936).

non-uniformity of the distribution of electrons along it, is compensated with the contact potential differences in both contacts of the semiconductor with the metal. The e.m.f. in the circuit appears only as a consequence of the deviation of the distribution function from Boltzmann's law.

All this refers only to the case when only the electrons take part in conduction. In some cases the non-conducting electrons, falling into the conduction zone levels, leave in the places of their former states "holes", which also can take part in the conduction as "positive" electrons. In such a case the e.m.f. appears as a result of the non-uniformity in the distribution of the electrons and "holes" along the semiconductor. This is quite analogous to Nernst's concentration cell. We shall first regard the e.m.f. in a semiconductor with conduction electrons only, and in section 5 a semiconductor with electrons and "holes".

Under the influence of the gradient in the intensity of illumination, which causes the non-uniformity in the distribution of electrons along the semiconductor, there appears in the latter an electrical field. To determine the total e.m.f. appearing in the circuit, we must calculate the potential difference between the ends of the open circuit, i.e. when there is no current. Therefore the above-mentioned electric field must be determined from the condition that the current is equal to zero. To do this we must find the velocity-distribution function of the electrons in the presence of the electric field and a gradient ∇J , calculate with its aid the current and set it equal to zero. Integrating the field along the whole length of the semiconductor we find the potential difference, between both ends of the latter and, adding the contact potential differences, we find the total e.m.f.

If the semiconductor is placed in a magnetic field, an e.m.f. also appears in a direction perpendicular to the gradient of the intensity of the light (the Kikoin-Noskov effect). This effect will be considered in another paper.

The theories of the photo-e.m.f. published up to date cannot be regarded as satisfactory. Fröhlich¹ calculated the distribution function but made some erroneous assumptions. He assumes for instance, that the probability of an electron transition by a collision with the lattice does not depend on the energy of the electron and of the phonon. As to Frenkel's article², it was obviously not intended to be serious. For example, we can hardly explain otherwise the sentence: "leaving aside a more detailed consideration of this question we shall assume in the following (3a) to hold not only in the case of small additional concentrations, but also in the case of large ones" (*loc. cit.* p. 188). This means nothing less, than the assumption that $(\bar{n} + q)^2 = \bar{n}^2 + 2\bar{n}q$ holds for large q !

2. THE DISTRIBUTION FUNCTION

In the absence of an electric field and illumination the conductivity electrons have a Boltzmann-distribution. To find the distribution function in the presence of the electric field E and of a gradient ∇J (both having the same

direction), we write the kinetical equation in the form:

$$\begin{aligned}
 & e E \nabla_p f(\theta, \varepsilon) + \frac{p}{m} f \nabla(\theta, \varepsilon) \\
 &= - \int W_1 \sqrt{\frac{\varepsilon - \hbar \nu_1}{\varepsilon}} \{f(\theta, \varepsilon)(N_1 + 1) - f(\theta', \varepsilon - \hbar \nu_1) N_1\} d\Omega' \\
 & - \int W_2 \sqrt{\frac{\varepsilon + \hbar \nu_2}{\varepsilon}} \{f(\theta, \varepsilon) N_2 - f(\theta', \varepsilon + \hbar \nu_2)(N_2 + 1)\} d\Omega' \\
 & - \lambda(f(\theta, \varepsilon) - f_0(\varepsilon)).
 \end{aligned} \tag{1}$$

The distribution function is now a function not only of the energy ε , but also of the angle θ between the momentum p of the electron and the direction of ∇J (or E). As J changes along the semiconductor, f is also an implicit function (by means of J) of the co-ordinate z along this direction.

The right-hand side of (1) represents the balance of the transitions of electrons by collisions with the lattice. The first integral contains two terms: the first is due to transitions from a given state with the energy ε and the angle θ between p and E to another with the energy $\varepsilon - \hbar \nu_1$, an elastical vibration quantum (phonon) $\hbar \nu_1$ being simultaneously absorbed by the lattice. The second term of the first integral is due to the opposite transitions, the quantum $\hbar \nu_1$ being absorbed by the electron. The integration is over all possible directions of the electron with the energy $\varepsilon - \hbar \nu_1$ ($d\Omega'$ is the space angle element in this direction). The second integral takes account of the transitions from the state with an energy ε and the angle θ to states with energy $\varepsilon + \hbar \nu_2$, the quantum $\hbar \nu_2$ being absorbed by the electron, and vice versa.

N is the distribution function of the phonons; N_1 and N_2 are then the numbers of phonons with the energy $\hbar \nu_1$ and $\hbar \nu_2$. $W_1(N_1 + 1)$ and $W_1 N_1$ are the probabilities of the transitions of an electron with the emission or absorption of a phonon $\hbar \nu_1$; correspondingly $W_2(N_2 + 1)$ and $W_2 N_2$ are the probabilities of the same transitions with a phonon $\hbar \nu_2$. N is the Bose distribution function, i.e.

$$N = \frac{1}{e^{\hbar \nu / k T} - 1}. \tag{2}$$

In the absence of the field and of illumination the left side of (1) is equal to zero and the equation is satisfied with a Boltzmann-function

$$f_0(\varepsilon) = a_0 e^{-\varepsilon / k T}. \tag{3}$$

The constant a_0 is determined by the number n of the electrons in unit volume without illumination according to $a_0 = n(2\pi m k T)^{-3/2}$.

The probabilities W_1 and W_2 are, as is known, proportional to the square root of the energy of the electron and also to ν_1 and ν_2 respectively. As to the energy of the electron, it is in both cases (W_1 and W_2) either the least energy of the transition (i.e. $\varepsilon - \hbar \nu_1$ and ε in W_1 and W_2 respectively), or in both cases the greater (ε and $\varepsilon + \hbar \nu_2$ in W_1 and W_2). It is convenient in both cases to

take the arithmetic mean of both these energies: $\varepsilon - \hbar \nu_1/2$ in W_1 and $\varepsilon + \hbar \nu_2/2$ in W_2 . We then obtain the probabilities W_1 and W_2 in the form

$$W_1 = A \nu_1 \sqrt{\varepsilon - \frac{\hbar \nu_1}{2}}, \quad W_2 = A \nu_2 \sqrt{\varepsilon + \frac{\hbar \nu_2}{2}}, \quad (4)$$

where A is a constant.

The third term $\lambda(f - f_0)$ from the right-hand side determines the diminution of the number of electrons due to their capture, i.e. for instance to their recombination with the free levels in the lower nearly completed zone of energy levels. It may be noted that the conductivity electrons can be captured not only in this way, but also directly by the lattice or by impurities in the lattice. The number of electrons of given energy in unit volume, captured in unit time, is proportional to their number in this volume, i.e. to the distribution function f , and is equal to λf , where λ is the probability of capture. λ is in general a function of the energy of the electron. But the conductivity electrons are at the same time created by the thermal motion and so the diminution of the number of electrons is $\lambda f - \text{const}$. To find the constant we note that without illumination the number of those captured and of those created by thermal motion are equal. Without illumination $f = f_0$ and so $\text{const} = \lambda f_0$, and we come to the term $\lambda(f - f_0)$.

The first term from the left-hand side of (1) gives the variation of the number of electrons in the state ε, θ under the influence of the field E . (∇_p is a gradient in the p -space). The second term determines the variation of this number owing to the variation of f along the semiconductor, i.e. under the influence of ∇J .

The mean velocity of the electron divided by λ is evidently the "Schublänge" of the electron, i.e. the mean distance traversed up to its capture.

We assume in the following that the temperature is high enough so that the energy $\hbar \nu$ of the phonons which can exchange momenta with the electrons and so come into consideration is much smaller than $k_B T$:

$$\hbar \nu / k_B T \ll 1.$$

By a transition of an electron with the emission or absorption of a phonon the momentum must be preserved. Before the transition the momentum was equal to p ; after the phonon $\hbar \nu_1$ (or $\hbar \nu_2$) was absorbed by the electron or given up to the lattice let the momentum be p_1 (or p_2). The momentum of the phonon is $\hbar \nu_1/w$ (or $\hbar \nu_2/w$) if w is the velocity of sound. The law of conservation then gives:

$$(p_1 - p_0)^2 = (\hbar \nu_1/w)^2, \quad (p_2 - p_0)^2 = (\hbar \nu_2/w)^2.$$

Using $\hbar \nu_1/\varepsilon \ll 1$, $\hbar \nu_2/\varepsilon \ll 1$ we find

$$p_0 = \sqrt{2m\varepsilon}, \quad p_1 = \sqrt{2m(\varepsilon - \hbar \nu_1)} = \sqrt{2m\varepsilon} \left(1 - \frac{\hbar \nu_1}{2\varepsilon}\right),$$

$$p_2 = \sqrt{2m(\varepsilon + \hbar \nu_2)} = \sqrt{2m\varepsilon} \left(1 + \frac{\hbar \nu_2}{2\varepsilon}\right)$$

and the law of conservation gives

$$\begin{aligned} 2m\varepsilon + 2m(\varepsilon - \hbar\nu_1) - 4m\varepsilon\left(1 - \frac{\hbar\nu_1}{2\varepsilon}\right)\cos\vartheta_1 &= \left(\frac{\hbar\nu_1}{w}\right)^2, \\ 2m\varepsilon + 2m(\varepsilon + \hbar\nu_2) - 4m\varepsilon\left(1 + \frac{\hbar\nu_2}{2\varepsilon}\right)\cos\vartheta_2 &= \left(\frac{\hbar\nu_2}{w}\right)^2 \end{aligned}$$

(ϑ_1 and ϑ_2 are the angles between \mathbf{p} and \mathbf{p}_1 or \mathbf{p}_2 respectively).

Introducing $\nu_1 + \nu_2 = 2\nu$ and $\nu_1 - \nu_2 = \Delta\nu$ we find approximately:

$$\left. \begin{aligned} (\hbar\nu)^2 &= 8m\varepsilon w^2 \sin^2 \frac{1}{2}\vartheta, \\ \hbar\Delta\nu &= -4m w^2 \sin^2 \frac{1}{2}\vartheta \end{aligned} \right\} \quad (5)$$

ϑ being the angle between \mathbf{p} and the momentum after the transition (just the same \mathbf{p}_1 or \mathbf{p}_2). From (5) it is evident that $\Delta\nu/\nu \sim \hbar\nu/\varepsilon \ll 1$.

As $\hbar\nu_1 \ll \varepsilon$ and $\hbar\nu_2 \ll \varepsilon$ we can write (2) approximately in the form

$$N_1 = \frac{k_B T}{\hbar\nu_1} - \frac{1}{2}, \quad N_2 = \frac{k_B T}{\hbar\nu_2} - \frac{1}{2} \quad (6)$$

and (4) in the form:

$$W_1 = A\nu_1 \sqrt{\varepsilon} \left(1 - \frac{\hbar\nu_1}{4\varepsilon}\right), \quad W_2 = A\nu_2 \sqrt{\varepsilon} \left(1 + \frac{\hbar\nu_2}{4\varepsilon}\right). \quad (7)$$

We now insert (6) and (7) in the right-hand side of (1) and expand $f(\theta', \varepsilon - \hbar\nu_1)$ and $f(\theta', \varepsilon + \hbar\nu_2)$ in powers of ν_1 and ν_2 up to terms of the order $(\hbar\nu)^2/k_B T$ or $\hbar\Delta\nu$, which are of the same order. The right-hand side of (1) then becomes:

$$\begin{aligned} & A\sqrt{\varepsilon} \left\{ \int f(\varepsilon, \theta) \left[-\frac{2k_B T}{\hbar} + \Delta\nu \left(\frac{3k_B T}{4\varepsilon} - \frac{1}{2} \right) + \frac{\hbar\nu^2}{4\varepsilon} \left(3 - \frac{k_B T}{\varepsilon} \right) \right] d\Omega' \right. \\ & + \int f(\varepsilon, \theta') \left[\frac{2k_B T}{\hbar} + \Delta\nu \left(-\frac{3k_B T}{4\varepsilon} - \frac{1}{2} \right) + \frac{\hbar\nu^2}{4\varepsilon} \left(3 + \frac{k_B T}{\varepsilon} \right) \right] d\Omega' \\ & + \int f'(\varepsilon, \theta') \left[-k_B T \Delta\nu + \hbar\nu^2 \left(\frac{3k_B T}{2\varepsilon} + 1 \right) \right] d\Omega' \\ & \left. + \int f''(\varepsilon, \theta') [k_B T \hbar\nu^2] d\Omega' - \lambda[f(\varepsilon, \theta) - f_0(\varepsilon)] \right\}. \end{aligned} \quad (8)$$

We write the function $f(\varepsilon, \theta)$, as is always done, in the form

$$f(\theta, \varepsilon) = f(\varepsilon) + \cos\theta \cdot g(\varepsilon) \quad (9)$$

where $g(\varepsilon)$ is small compared with $f(\varepsilon)$. Inserting (5) and (8) in (7) and integrating over $d\Omega'$ we find:

$$\begin{aligned} & \frac{A\sqrt{\varepsilon}}{h} 16\pi m w^2 k_B T \left[\frac{2f(\varepsilon)}{k_B T} + f'(\varepsilon) \left(2 + \frac{\varepsilon}{k_B T} \right) + \varepsilon f''(\varepsilon) \right] \\ & - \frac{A\sqrt{\varepsilon}}{h} 8\pi k_B T g(\varepsilon) \cos\theta - \lambda(f - f_0) - \lambda g \cos\theta \end{aligned} \quad (10)$$

$g(\varepsilon)$ is a small correction to $f(\varepsilon)$, which is proportional to ∇J . The gradient ∇J is, of course, to be considered as small, for at a distance of the order of a mean free path of the electron the light is almost not absorbed. Therefore in the first two terms on the left-hand side of (1) we must only put $f(\varepsilon)$ for $f(\theta, \varepsilon)$ (the electric field E caused by ∇J is also proportional to ∇J). In the first term we have:

$$e E \nabla_p f(\varepsilon) = e E \frac{p}{p} \frac{\partial f}{\partial p} = e E \cos\theta \frac{\partial f}{\partial p} = e E \cos\theta \sqrt{\frac{2\varepsilon}{m}} \frac{\partial f}{\partial \varepsilon}.$$

In the second we can write

$$p \nabla f = p \nabla J \frac{\partial f}{\partial J} = p \cos\theta J' \frac{\partial f}{\partial J}$$

(here ' denotes differentiation by the co-ordinate z along the direction of ∇J). We finally obtain for the left-hand side of (1):

$$\cos\theta \sqrt{\frac{2\varepsilon}{m}} e E \frac{\partial f}{\partial \varepsilon} + \sqrt{\frac{2\varepsilon}{m}} \frac{\partial f}{\partial J} J' \cos\theta. \quad (11)$$

Equalising (10) and (11) we find from the terms without $\cos\theta$ an equation to determine $f(\varepsilon)$:

$$\frac{A\sqrt{\varepsilon}}{h} 16\pi m w^2 k_B T \left[\frac{2f}{k_B T} + f'(\varepsilon) \left(2 + \frac{\varepsilon}{k_B T} \right) + \varepsilon f''(\varepsilon) \right] = \lambda(f - f_0).$$

This equation can be written in the form:

$$\sqrt{\frac{2}{m\varepsilon}} \frac{d}{d\varepsilon} (\varepsilon s) = \lambda(f - f_0) \quad (12)$$

with

$$s = \frac{2^{7/2} A \pi m^{3/2} w^2 \varepsilon}{h} (f + k_B T f'). \quad (13)$$

Equation (12) is then the continuity equation

$$\operatorname{div}_p s = \nabla_p s = \lambda(f - f_0)$$

and s is the flow of the electrons in p -space. In the following we shall use (12) with (13) in the form

$$\frac{d}{dx} [x^2(f + f')] = \mu(f - f_0) \quad (14)$$

with $x = \varepsilon/k_B T$ and

$$\mu = \frac{\hbar}{16\pi m w^2 A \sqrt{k_B T}} \lambda \sqrt{x} \quad (15)$$

and ' denotes differentiation by x .

The terms of (10) and (11) containing $\cos\theta$ give an equation for $g(x)$:

$$g(x) = -\frac{\sqrt{2}\hbar}{8\sqrt{m\pi k_B T} A} \left[\frac{eE}{k_B T} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial J} J' \right]. \quad (16)$$

We dropped the term λg in comparison with $(A\sqrt{\varepsilon}/\hbar) 8k_B T \pi g$, for their ratio is, as will be shown, of the order of the ratio of the mean free path of the electron to the "Schublänge", which is much smaller than unity.

The current density i can now be calculated according to

$$i = e \int \frac{p_z}{m} \cos\theta g(\varepsilon) d\tau_p = \frac{8\pi e m k_B^2 T^2}{3} \int g(x) x dx$$

or, if we insert (16):

$$i = -\frac{\sqrt{2} m e \hbar k_B T}{3 A} \int_0^\infty \left(\frac{eE}{k_B T} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial J} \nabla J \right) x dx. \quad (17)$$

If we determine E so that the current i is zero, we get from (17)

$$\frac{eE}{k_B T} = -\nabla J \frac{\frac{d}{dJ} \int_0^\infty x f dx}{\int_0^\infty \frac{\partial f}{\partial x} x dx}. \quad (18)$$

In the absence of illumination the distribution function is $f_0 = a_0 e^{-x}$ (2) with $a_0 = n(2\pi m k_B T)^{-3/2}$. Taking the first term in (17) we find the conductivity σ of the semiconductor in the absence of illumination:

$$\sigma = \frac{e^2 n \hbar}{6(\pi k_B T)^{3/2} m A}.$$

According to a well-known formula the same σ is also equal to:

$$\sigma = \frac{4e^2 l n}{3 \sqrt{2\pi m k_B T}}$$

(l is the mean free path of the electron). Comparing both expressions for σ , we find

$$A = \frac{h}{4 \sqrt{2\pi m k_B T} l}. \quad (19)$$

With the aid of this expression it is easy to convince ourselves that the ratio of λg to $A \sqrt{\varepsilon} 8\pi g k_B T/h$ is of the order of the ratio of l to $\sqrt{\varepsilon/m} 1/\lambda = l_s$, i.e. of l to the "Schublänge".

We now proceed to the solution of equation (14). The "Schublänge" of the electron is in general much greater than its mean free path. We shall assume that their ratio, which is of the order of $\sqrt{k_B T/m} \times 1/\lambda l$, is so great that

$$\sqrt{\frac{k_B T}{m}} \frac{1}{\lambda l} \gg \frac{k_B T}{m w^2}.$$

In this case we have $\mu \ll 1$ and therefore we can solve (14) by means of successive approximations.

To find the zero-approximation we solve (14) without the right-hand side. As a first integral we then find

$$f + f' = \frac{\text{const}}{x^2}.$$

This is a simple linear equation of the first order; its general solution is

$$f = a e^{-x} + b e^{-x} \int \frac{e^x}{x^2} dx$$

where a and b are two constants. This is to be put into the right-hand side of (14) and the equation solved again. This is easily done and we obtain the distribution function in the first approximation in the form:

$$\begin{aligned} f(x) = & a e^{-x} + b e^{-x} \int \frac{e^x}{x^2} dx + (a - a_0) e^{-x} \int \left(\frac{e^x}{x^2} \int \mu e^{-x} dx \right) dx \\ & + b e^{-x} \int \left[\frac{e^x}{x^2} \int \left(\mu e^{-x} \int \frac{e^x}{x^2} dx \right) dx \right] dx. \end{aligned} \quad (20)$$

As we shall immediately show, the constant $(a - a_0)$ is of the first order in μ . As we intend to restrict ourselves to the first approximation, we must leave in the expression for $f(x)$ only the terms of zero and first order in μ ,

i.e. we must drop the fourth term on the right side. It is to be noted that the choice of the constant limits in the integrals in (20) is arbitrary, until the constants a and b are determined from the conditions which we shall now discuss.

It is to be mentioned that all the conductivity electrons created by the monochromatic illumination have (immediately after their creation) nearly the same energy. We shall assume that they all have the same energy and call it ε_0 . ε_0 is practically equal to the energy of the absorbed light quantum. As the energy of a light quantum from the visible region of the spectrum is ca. $100 k_B T$, the quantity $x_0 = \varepsilon_0 / k_B T$ is a very large number.

The distribution function is different for $x > x_0$ and $x < x_0$. We shall attempt to determine it in the form

$$f(x) = a_1 e^{-x} + b_1 e^{-x} \int_1^x \frac{e^y}{y^2} dy + (a_1 - a_0) e^{-x} \int_0^x \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy \quad (21)$$

for $x < x_0$, and

$$f(x) = a_2 e^{-x} + b_2 e^{-x} \int_1^x \frac{e^y}{y^2} dy + (a_2 - a_0) e^{-x} \int_0^x \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy \quad (22)$$

for $x > x_0$. After we have chosen the limits of integration, we must determine the constants a_1, b_1, a_2, b_2 from the following conditions.

The flow s of the electrons in p -space is given by (13). The flow of the electrons through a sphere (in p -space), corresponding to a given energy ε , is obviously equal to the product of s and $4\pi p^2 = 8m\pi\varepsilon = 8m\pi k_B T x$:

$$8\pi m k_B T x s = \frac{2^{13/2} A \pi^2 m^{5/2} w^2 (k_B T)^2 x^2}{h} (f + f') = \frac{1}{B} (f + f') x^2 \quad (23)$$

with

$$B = \frac{h}{2^{13/2} A \pi^2 m^{5/2} w^2 (k_B T)^2} = \frac{l}{4\pi m^2 w^2 k_B T} \quad (24)$$

(A is to be taken from (19)).

This flow is directed towards the diminution of energy. For $x = 0$ it must obviously be equal to zero. This is equivalent to the condition that $f(x)$ must have no singular point for $x = 0$. If we insert (21) in (23), we find the flow in the form

$$\frac{1}{B} \left[b_1 + (a_1 - a_0) \int_0^x \mu e^{-x} dx \right].$$

Equating it to zero for $x = 0$ we get: $b_1 = 0$.

Now, for $x = x_0$ the values of both functions (21) and (22) must be equal. This gives

$$a_1 + (a_1 - a_0) A_1 = a_2 + b_2 \frac{e^{x_0}}{x_0^2} + (a_2 - a_0) A_1 \quad (25)$$

with

$$A_1 = \int_0^{x_0} \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy$$

and we have written for $\int_1^{x_0} \frac{e^y}{y^2} dy$ its asymptotical value e^{x_0}/x_0^2 . The asymptotical value of A_1 is

$$A_1 = \int_0^{x_0} \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy \cong \frac{e^{x_0}}{x_0^2} A$$

with

$$A = \int_0^{\infty} \mu e^{-x} dx. \quad (26)$$

For very large energies the distribution function must be a Boltzmann function. To make use of this condition, we write the asymptotical expression for large x for $f(x)$ (22). Asymptotically

$$\int_1^x \frac{e^x}{y^2} dy \cong \frac{e^x}{x^2}, \quad \int_0^x \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy \cong A \int_1^x \frac{e^y}{y^2} dy \cong A \frac{e^x}{x^2}$$

and therefore

$$f(x) = a_2 e^{-x} + \frac{b_2}{x^2} + \frac{(a_2 - a_0)A}{x^2}.$$

Hence we find that it must be

$$b_2 = -(a_2 - a_0) A.$$

Then $f(x)$ is a Boltzmann-function for large x . If we insert this expression for b_2 in (25) and note that $A_1 \cong A e^{x_0}/x_0^2$, we find

$$a_2 = a_1 + (a_1 - a_0) A_1.$$

The distribution function for $x > x_0$ is hence asymptotically equal to

$$f(x) = a_2 e^{-x} = [a_1 + (a_1 - a_0) A_1] e^{-x}. \quad (27)$$

In unit time J electrons with energy ϵ_0 appear in a unit volume under the influence of illumination. Correspondingly, in the p -space for $x = x_0$ the sum of the flows downwards (towards diminution of energy) and upwards (towards increase of energy) must be equal to J . If we put (21) into (23) we find the

flow downwards for $x = x_0$:

$$\frac{1}{B} (a_1 - a_0) \int_0^{x_0} \mu e^{-x} dx$$

(for $b_1 = 0$). But

$$\int_0^{x_0} \mu e^{-x} dx \cong \int_0^{\infty} \mu e^{-x} dx = A$$

and the flow is equal to $(a_1 - a_0) A/B$. The flow upwards is zero, for the distribution function for $x > x_0$ (27) is a Boltzmann-function and $f + f'$ is, consequently, zero. Of course, this is only approximately valid, for (27) is only an asymptotical expression. But the next terms in the asymptotical series give expressions which are quite negligible.

Hence we get from the above condition

$$a_1 - a_0 = \frac{BJ}{A}$$

and the distribution function (21) for $x < x_0$ becomes

$$f(x) = \left(a_0 + \frac{BJ}{A}\right) e^{-x} + \frac{BJ}{A} e^{-x} \int_0^x \int_0^y \frac{e^{y-z}}{y^2} \mu(z) dz dy. \quad (28)$$

For $x > x_0$ the distribution function (27) becomes

$$f(x) = \left(a_0 + \frac{BJ}{A}\right) e^{-x} + \frac{BJ}{A} A_1 e^{-x}.$$

From this expression we see that $f(x)$ for $x > x_0$ is quite unessential. Indeed, it is valid only from $x = x_0$ and therefore the first term contains in any case e^{-x_0} and is therefore very small. The second term is $(BJ e^{x_0}/x_0^2) e^{-x}$ (for $A_1 \cong (e^{x_0}/x_0^2) A$). When $x = x_0$ it does not contain e^{-x_0} , but it contains a superfluous factor A/x_0^2 in comparison with the corresponding term $(BJ/A) e^{-x}$ in (28) and is therefore also of no importance.

In the following we shall need an asymptotical expression for (28) for large x . This is

$$f(x) = \left(a_0 + \frac{BJ}{A}\right) e^{-x} + \frac{BJ}{x^2} \quad (29)$$

3. THE CONTACT POTENTIAL DIFFERENCE

We now proceed to calculate the potential differences in the contacts of the semiconductor with the metal. It is to be found from the condition that in the absence of a current through the contact the number of electrons passing

in unit time from the semiconductor into the metal and backwards must be equal.

In the metal the electrons have a Fermi-distribution. The conduction zone begins from the bound-energy of this distribution; below this energy all the states are occupied and above it the distribution becomes a Boltzmann-function (Fig. 1). In the semiconductor the conduction zone begins so high that the distribution is of course a Boltzmann-function.

The number of electrons passing in unit time from the semiconductor to the metal is equal to an integral of the product of the distribution function with the velocity and transparency coefficient for a transition from the semiconductor to the metal, over all velocities, directed from the semiconductor

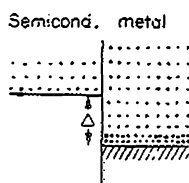


FIG. 1

to the metal. The transparency coefficient determines the ratio of the number of electrons passing to the metal to the whole number of electrons approaching the boundary of the semiconductor. This coefficient is in general a function of the energy ϵ of the electron, and also of the direction of its velocity. After integration over the directions, we obtain an "averaged" transparency coefficient $D(\epsilon)$, which is a function of the energy only. The number of electrons passing to the metal is then proportional to

$$\int_0^{\infty} x f(x) D(x) dx$$

($f(x)$ is the distribution function in the semiconductor).

In the absence of illumination $f = f_0$ and the number of electrons passing to the metal is (apart from a constant factor) equal to

$$\int_0^{\infty} x f_0 D(x) dx.$$

The same is also true of the number of electrons passing from the metal to the semiconductor.

Now we wish to compute the sum of the contact potential differences from both sides (illuminated and not illuminated) of the semiconductor. In the absence of illumination both these differences would be equal with opposite signs, and their sum would be zero. It is therefore enough to determine the change of the potential difference produced in the illuminated contact by illumination.

Under the influence of illumination Δ changes (Fig. 1). Δ is (apart from an additive constant) the contact potential difference, and its change is equal to the change of this difference produced by illumination, i.e. to the sum V_k of contact potential differences from both sides of the semiconductor (multiplied by e).

The electrons in the metal, which can pass to the semiconductor (i.e. have energies larger than Δ) have a Boltzmann-distribution. Therefore the number of electrons passing to the semiconductor changes by illumination $e^{-eV_k/k_B T}$ times (we neglect the change of the transparency coefficient and of the velocities of the electrons in the metal under the influence of the displacement of the limits of the conducting zones by illumination). In the absence of illumination this number was proportional to

$$\int_0^{\infty} f_0 x D dx;$$

now it is proportional to

$$e^{-eV_k/k_B T} \int_0^{\infty} f_0 x D dx.$$

The number of electrons passing from the semiconductor to the metal is now proportional to

$$\int_0^{\infty} f x D dx$$

where f is the distribution function in the illuminated semiconductor. So we obtain the condition

$$\int_0^{\infty} f x D dx = e^{-eV_k/k_B T} \int_0^{\infty} f_0 x D dx$$

and hence

$$\frac{eV_k}{k_B T} = - \ln \frac{\int_0^{\infty} f x D dx}{\int_0^{\infty} f_0 x D dx}.$$

It is convenient to introduce conditionally a "field" E_k , differentiating V_k by the co-ordinate along the direction of ∇J . V_k will then be equal to the integral of E_k along the semiconductor from the illuminated side to the unilluminated:

$$\frac{eE_k}{k_B T} = - \nabla J \frac{\frac{d}{dJ} \int_0^{\infty} D x f dx}{\int_0^{\infty} D x f dx}. \quad (30)$$

Now we shall determine D as a function of the energy ε of the electron in the semiconductor, assuming that the contact is good, i.e. that there is no potential barrier between the semiconductor and the metal. We are interested only in a qualitative result and therefore discuss a one-dimensional case. The problem is then equivalent to the following. A flux of electrons, moving in some direction, comes from a region with a larger to a region with a smaller potential energy (the kinetic energy hence becomes larger; Fig. 2). It is necessary to determine the transparency coefficient, i.e. the ratio of the density

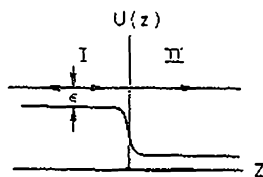


FIG. 2

of the electrons which passed to the second region to the whole density of electrons approaching the boundary between the two regions. We shall assume that the kinetic energy ε of the electron in the first region is small. The function $U(z)$ in the intermediate region between I and II is unknown. One cannot assume it to be a rectangular barrier, for this might give even qualitatively wrong results.

In the first region the wave function ψ of the electron is

$$\psi = c_1 e^{ikz} + c_2 e^{-ikz} \quad (31)$$

$c_1 e^{ikz}$ is the wave approaching the boundary between both regions; $c_2 e^{-ikz}$ is the reflected wave propagated in the opposite direction.

We shall now investigate the properties of this wave function in the vicinity of the transition of the second region, i.e. for small z ; namely for z small compared with the wavelength, but so large that the energy of the electron is still small. This function must be determined so that it would be transferred to the second region in a wave propagated along the z -axis in the positive direction. As the energy of the electron is small, we can use for the investigation of the properties of the wave function in the first region (for small z) the Schrödinger-equation for $\varepsilon = 0$. This equation is $d^2\psi/dz^2 = 0$ and hence

$$\psi = d_1 + i d_2 z$$

(d_1 and d_2 are in general complex). The ratio $d_1/d_2 = d$ is determined by the form of the function $U(z)$ in the intermediate region.

For small ε (i.e. small k), (31) can be written as follows:

$$\psi = (c_1 + c_2) + i k z (c_1 - c_2).$$

In order that it may pass in the second region into a propagating wave, we can require that the ratio $(c_1 + c_2)/i k (c_1 - c_2)$ be equal to $d_1/i d_2$ which is already

determined in a correct way:

$$\frac{c_1 + c_2}{k(c_1 - c_2)} = \frac{d_1}{d_2} = d, \quad \text{or} \quad c_2 = c_1 \frac{k d - 1}{k d + 1}$$

$|c_1|^2 = c_1 c_1^*$ determines the density of the electrons approaching the boundary between both regions, and $c_2 c_2^* = |c_2|^2$ the density of the electrons reflected back to the first region. Hence the transparency coefficient is proportional to

$$\frac{|c_1|^2 - |c_2|^2}{|c_1|^2} = \frac{2(d + d^*) k}{(1 + k d)(1 + k d^*)}$$

As k is assumed to be small, we get

$$\frac{|c_1|^2 - |c_2|^2}{|c_1|^2} \cong 2k(d + d^*).$$

The wave-vector k is proportional to the velocity of the electron (in the first region), i.e. to the square root of its kinetic energy ε . Hence

$$D(\varepsilon) = \text{const.} \sqrt{\varepsilon}. \quad (32)$$

4. THE PHOTOELECTROMOTIVE FORCE

To find the total electromotive force we must integrate the sum of (18) and (30) along the whole length of the semiconductor. Adding (30) to (18) we find

$$\frac{e(E + E_k)}{k_B T} = -\nabla J \frac{\int_0^\infty D x f dx \cdot \frac{d}{dJ} \int_0^\infty x f dx + \int_0^\infty \frac{\partial f}{\partial x} x dx \cdot \frac{d}{dJ} \int_0^\infty D x f dx}{\int_0^\infty D x f dx \cdot \int_0^\infty \frac{\partial f}{\partial x} x dx}. \quad (33)$$

We must here insert for $f(x)$ the function (28) and integrate in all integrals not up to ∞ but to x_0 , for this expression for $f(x)$ is valid for $x \leq x_0$. Of course, by integrating the first term in (28) we can take the upper limit to be ∞ for, e.g.

$$\int_0^{x_0} x e^{-x} dx \cong \int_0^\infty x e^{-x} dx = 1.$$

We also introduce the notation:

$$\int_0^{x_0} x D(x) e^{-x} dx \cong \int_0^\infty x D e^{-x} dx = D_0. \quad (34)$$

We shall see that by integrating the second term in (28) large x will turn out to be of importance. Therefore we can use instead of (28) the asymptotical expression (29).

$\lambda(x)$, and therefore also μ , we assume to be independent of the illumination, i.e. independent of J .

We now insert (29) in the numerator of (33). The terms which originate only from the Boltzmann-term $(\alpha^0 + BJ/\Lambda) e^{-x}$ in (29) then give zero (this shows that a Boltzmann-distribution could not give an e.m.f., as was pointed out already in section 1). The terms which originate only from the term BJ/x^2 in (29) are small compared with those which originate from both terms of (29) (for Λ is small). Therefore we leave only the latter.

It is also to be noted that we must insert in

$$\int_0^{x_0} \frac{\partial f}{\partial x} x dx$$

only the first term of (28). The second would give $2BJ/x^3$ in $\partial f/\partial x$, i.e. the next power of x in the denominator. As we always use only the greatest term in the asymptotical series, such a term, being the next in these series, must be neglected.

Finally, when we insert the term BJ/x^2 in the integrals of (33) they become divergent for large x ; this verifies our assumption that large x are of importance. The integrals become divergent also for small x ; but this depends only on the fact that the asymptotical expression for $f(x)$ which we use is valid only for large x . Therefore we must take the value of the integral only for the upper limit. Instead of the lower we write conditionally 1; the value of the integral for the lower limit must be neglected all the same.

Having all this in mind, we obtain for the numerator of (33):

$$-a_0 B \int_1^{x_0} \frac{D - D_0}{x} dx + \frac{B^2 J}{\Lambda} D_0 \ln x_0.$$

When we put (28) into the denominator of (33), we can leave only the largest terms (i.e. the first term of (28)), for they do not give zero as in the numerator. Hence we find:

$$\frac{e(E + E_k)}{k_B T} = \frac{-a_0 B \int_1^{x_0} \frac{D - D_0}{x} dx + \frac{B^2 J}{\Lambda} D_0 \ln x_0}{\left(\alpha_0 + \frac{BJ}{\Lambda}\right)^2 D_0} \nabla J$$

According to (32) $D(x) = \text{const} \sqrt{x}$. This is strictly valid for small x only, but in our approximation we can always use this expression.

Therefore:

$$D_0 = \int_0^\infty D x e^{-x} dx = \text{const} \int_0^\infty x^{3/2} e^{-x} dx = \text{const} \frac{3}{4} \sqrt{\pi},$$

$$\int_1^{x_0} \frac{D}{x} dx = \text{const} 2\sqrt{x_0}, \quad \int_1^{x_0} \frac{D_0}{x} dx = \text{const} \frac{3}{4} \sqrt{\pi} \ln x_0.$$

Hence:

$$\frac{e(E + E_k)}{k_B T} = \frac{-2a_0 B \sqrt{x_0} + \left(a_0 + \frac{BJ}{A}\right) B \ln x_0 \cdot \frac{3}{4} \sqrt{\pi}}{\frac{3}{4} \sqrt{\pi} \left(a_0 + \frac{BJ}{A}\right)^2} \nabla J. \quad (35)$$

Integrating this over the whole length of the semiconductor, i.e. over dJ from $J = J_0$ (on the illuminated side) up to zero on the opposite side, we find the electromotive force V :

$$V = - \frac{8B \sqrt{x_0} k_B T}{3 \sqrt{\pi} e \left(a_0 + \frac{BJ_0}{A}\right)} J_0 + \frac{k_B T}{e} A \ln x_0 \ln \left(1 + \frac{BJ_0}{a_0 A}\right). \quad (36)$$

For weak illumination, i.e. small J_0 , we can neglect the second term in V compared with the first, and in the first BJ_0/A compared with a_0 . We then get an electromotive force

$$V = - \frac{8B \sqrt{x_0} k_B T}{3 \sqrt{\pi} e a_0} J_0 = - \frac{2^{5/2}}{3} \sqrt{\frac{k_B T}{m}} \frac{l}{nw^2} \sqrt{x_0} J_0, \quad (37)$$

increasing linearly with J_0 .

For strong illumination, i.e. for large J_0 , we can, on the contrary, neglect the first term and a_0 as compared with BJ/A in the second. $A a_0/B$ can be interpreted as the value J_l of J_0 after which V is no more a linear function of J_0 . Then we obtain the e.m.f. in the form

$$V = \frac{k_B T}{e} A \ln x_0 \ln \frac{J_0}{J_l}, \quad (38)$$

i.e. V is proportional to $\ln(J_0/J_l)$. We note also that for a definite value of J_0 the e.m.f. changes its sign, i.e. its direction. As to A , we find from (15) and (26) that it is of the order $k_B T/mw^2(l/l_s)$ if l_s is the "Schublänge" of the electrons.

All the cases observed experimentally up to the present are apparently not of the kind just discussed, but are such that the "holes" take part in the conductivity of the semiconductor as well as the electrons. This case is discussed in the next section.

5. THE PHOTOELECTROMOTIVE FORCE IN A SEMICONDUCTOR WITH CONDUCTING "HOLES"

We now discuss a semiconductor in which the "holes" take part in the conductivity as well as the electrons. The dependence of V on J_0 is in this case different from the preceding case. It is also necessary to note that if only the "holes" take part in the conductivity the results of the former section are valid with a change in the sign of e .

The limit of the filled zone of states in the metal lies between the upper limit of the filled zone and the lower limit of the conduction-zone of the semiconductor (Fig. 3). The separate free levels in the filled zones are the "holes", which can take part in the conductivity as electrons with a positive charge.

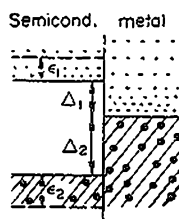


FIG. 3

The energy of the conducting-electrons in the metal is counted from the upper limit of the filled zone; in the semiconductor (ϵ_1) from the beginning of the conduction zone. The energy ϵ_2 of the "hole" in the semiconductor is defined as that of the corresponding free level, counted downwards from the upper limit of the filled zone. Analogously the energy of a "hole" in the metal is defined.

The distribution function for the "holes" in the semiconductor is of the same form as for the electrons, but the constant coefficients in these functions have different values. In particular differences occur in the constants a_0 , which are proportional to the density of the electrons or "holes" in the absence of illumination. The constants B are also different, for the mean free path is different for electrons and "holes". Finally the Λ are different, for the "Schub-längen" are different.

In (33) the Boltzmann-part of the distribution function gave zero, showing that such a function can give no effect alone. We shall see that this is not the fact in the present case. Therefore we can now take only the greatest terms in the distribution function, i.e., only the Boltzmann-function. Hence the distribution functions for the electrons (1) and the "holes" (2) are:

$$\begin{aligned} f_1(x_1) &= \left(a_{01} + \frac{B_1 J}{\Lambda_1} \right) e^{-x_1} = a_1 e^{-x_1}, \\ f_2(x_2) &= \left(a_{02} + \frac{B_2 J}{\Lambda_2} \right) e^{-x_2} = a_2 e^{-x_2} \end{aligned} \quad (39)$$

with

$$x_1 = \frac{\varepsilon_1}{k_B T}, \quad x_2 = \frac{\varepsilon_1}{k_B T}$$

and

$$a_1 = a_{01} + \frac{B_1 J}{A_1} = a_{01} + \beta_1 J, \quad a_2 = a_{02} + \frac{B_2 J}{A_2} = a_{02} + \beta_2 J. \quad (40)$$

The whole current i is equal to the sum of the electron-current i_1 and the "hole"-current i_2 . If we remember that the "charge" of the "hole" is opposite to that of the electron, we find from (17) and (39):

$$i_1 = -c_1 \left(-\frac{eE}{k_B T} a_1 + a'_1 \nabla J \right), \quad i_2 = c_2 \left(\frac{eE}{k_B T} a_2 + a'_2 \nabla J \right),$$

$$\left(a'_1 = \frac{da_1}{dJ}, \quad a'_2 = \frac{da_2}{dJ} \right)$$

c_1 and c_2 denote the positive values of the coefficients in (17) for the electrons and the "holes" respectively.

Now E must be determined so that the total current $i_1 + i_2 = 0$. This gives:

$$\frac{eE}{k_B T} = \frac{c_1 a'_1 - c_2 a'_2}{c_1 a_1 + c_2 a_2} \nabla J.$$

The potential difference V_J between the illuminated and the opposite side of the semiconductor, arising under the influence of ∇J , is equal to the integral of E taken from $J = J_0$ (on the illuminated side) up to $J = 0$ (on the opposite side): Using (40) we find:

$$\frac{eV_J}{k_B T} = \frac{c_1 \beta_1 - c_2 \beta_2}{c_1 \beta_1 + c_2 \beta_2} \ln \frac{c_1 a_{10} + c_2 a_{20} + (c_1 \beta_1 + c_2 \beta_2) J_0}{c_1 a_{10} + c_2 a_{20}}. \quad (41)$$

The numbers of electrons and "holes" in the absence of illumination are, in general, of quite a different order of magnitude. Therefore it is always possible to neglect one of the a_{01} and a_{02} compared with the other. If for instance $a_{10} \gg a_{20}$, i.e. in the absence of illumination, the number of electrons is much greater than that of the "holes" (the "holes" appear under the influence of the illumination), we can write instead of (41):

$$\frac{eV_J}{k_B T} = \frac{c_1 \beta_1 - c_2 \beta_2}{c_2 \beta_2 + c_2 \beta_2} \ln \left[1 + \frac{c_1 \beta_1 + c_2 \beta_2}{c_1 a_{01}} J_0 \right]. \quad (42)$$

We must calculate now the contact potential difference. The current from the semiconductor to the metal consists of the electron-current and the "hole"-current. As the "charge" of the "hole" is opposite to that of the electron, we find (as in section 1) that the total current is proportional to

$$\int_0^\infty x_1 f_1(x_1) D_1(x_1) dx_1 - \int_0^\infty x_2 f_2(x_2) D_2(x_2) dx_2$$

(D_1 and D_2 are the transparency coefficients for the electrons and the "holes" respectively). If we insert (39), we find

$$a_1 D_{01} - a_2 D_{02} \quad (43)$$

with

$$D_{01} = \int_0^{\infty} D_1 e^{-x} x dx, \quad D_{02} = \int_0^{\infty} D_2 e^{-x} dx.$$

It was already pointed out that in the absence of illumination it is always possible to assume that the number of electrons, for instance, is much greater than that of the "holes": $a_{01} \gg a_{02}$. The distance Δ_2 (Fig. 3) from the upper limit of the filled zone in the metal to the upper limit of the filled zone in the semiconductor is much larger than the distance Δ_1 to the lower limit of the conduction zone in the semiconductor. Δ_1 is of the order of the activation energy of the electron in the semiconductor, which must be small compared with the whole distance $\Delta_2 + \Delta_1$; in the opposite case we should have an insulator instead of a semiconductor. Therefore from the metal to the semiconductor can pass practically only electrons and not "holes".

As was pointed out in section 3 it is enough to calculate only the change of the contact potential difference under the influence of illumination. This change is equal to the sum V_k of the contact potential differences in both contacts. In the absence of illumination the current from the semiconductor to the metal is proportional to $a_{01} D_{01}$ (for $a_{01} \gg a_{02}$). The same is true of the current from the metal to the semiconductor. As in section 3 we conclude that with illumination the current from the metal to the semiconductor becomes proportional to

$$e^{-e V_k / k_B T} a_{01} D_{01} \quad (44)$$

(according to our assumption the "holes" do not pass from the metal to the semiconductor).

Comparing (43) and (44) we find the condition for no current through the contact in the form:

$$a_1 D_{01} - a_2 D_{02} = a_{01} D_0 e^{-e V_k / k_B T}.$$

Hence we find (with (40)):

$$\frac{e V_k}{k_B T} = -\ln \left[1 + \frac{\beta_1 D_{01} - \beta_2 D_{02}}{D_{01} a_{01}} J_0 \right].$$

Adding V_k to V_J (41) we finally find the electromotive force:

$$V = \frac{k_B T}{e} \frac{c_1 \beta_1 - c_2 \beta_2}{c_1 \beta_1 + c_2 \beta_2} \ln \left[1 + \frac{c_1 \beta_1 + c_2 \beta_2}{c_1 a_{01}} J_0 \right] - \frac{k_B T}{e} \ln \left[1 + \frac{\beta_1 D_{01} - \beta_2 D_{02}}{a_{01} D_{01}} J_0 \right]. \quad (45)$$

For a very strong illumination, i.e. for very large J_0 , the difference

$$a_1 D_{01} - a_2 D_{02} = a_{01} D_{01} + (\beta_1 D_{01} - \beta_2 D_{02}) J_0$$

can become negative (if $\beta_1 D_{01} - \beta_2 D_{02} < 0$) and (45) will be no longer valid. In this case the limits of the zones (Fig. 3) are so displaced that Δ_1 becomes of the order of magnitude of the whole distance $\Delta_1 + \Delta_2$. Then the transition of the "holes" from the metal to the semiconductor becomes of importance. The contact potential difference V_k (and hence also the electromotive force) becomes very large, namely of the order $(\Delta_1 + \Delta_2)/e$. However, we shall not discuss this case in detail, for it is apparently not observed experimentally.

All the constants $c_1, c_2, \beta_1, \beta_2$ which enter in (45) can be expressed by means of universal constants, the mean free path of the electrons (or "holes") etc.; this is however of no special interest. It is important only to note the following. If we write

$$\frac{c_2 \beta_2}{c_1 \beta_1} = \kappa, \quad \frac{c_1 a_{01}}{c_1 \beta_1 + c_2 \beta_2} = C_1, \quad \frac{a_{01} D_{01}}{\beta_1 D_{01} - \beta_2 D_{02}} = C_2 \quad (46)$$

we can write (45) in the form:

$$V = \frac{k_B T}{e} \left[\frac{1 - \kappa}{1 + \kappa} \ln \left(1 + \frac{J_0}{C_1} \right) - \ln \left(1 + \frac{J_0}{C_2} \right) \right]. \quad (47)$$

We see that the coefficient before the first logarithm can have a value between ± 1 . Both the constants C_1 and C_2 are proportional to the number of electrons in unit volume without illumination, for they are proportional to a_{01} . The electron density depends on the temperature mainly by a factor of the form $e^{-q/T}$. Therefore C_1 and C_2 are also $\sim e^{-q/T}$.

All the results are quite analogous if in the absence of illumination the number of "holes" is much greater than the number of electrons, i.e. if $a_{02} \gg a_{01}$. Instead of (45) we should then have:

$$V = \frac{k_B T}{e} \frac{c_1 \beta_1 - c_2 \beta_2}{c_1 \beta_1 + c_2 \beta_2} \ln \left[1 + \frac{c_1 \beta_1 + c_2 \beta_2}{c_2 a_{02}} J_0 \right] + \frac{k_B T}{e} \ln \left[1 + \frac{\beta_2 D_{02} - \beta_1 D_{01}}{a_{02} D_{02}} J_0 \right] \quad (48)$$

and the constants c_1 and c_2 in (47) would be proportional to the density of the "holes" in the absence of illumination.

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22. ON THE THEORY OF SOUND DISPERSION

It is shown that sound dispersion at temperatures which are sufficient for the excitation of several quanta of vibration is given by the same formula as at temperatures for which at the most one quantum is excited. The transfer of this energy and its dependence on molecular velocity is also considered. It is established in this case that only high velocities play any part, and the phenomenon behaves classically. The dependence of the effect on temperature is determined.

1. A number of investigations undertaken in recent years have shown that the velocity of sound in many gases depends on the frequency. This effect only occurred when molecular vibrations were excited. It could be explained by assuming that the probability of energy exchange is very small between the translational and rotational energies on the one hand, and the vibration energy on the other. With the propagation of sound are associated adiabatic density changes, which also implies temperature changes. Now when this occurs at very high frequencies, the molecular vibrations will no longer be affected. This is to say that the specific heat and also the speed of sound will depend on the frequency.

The dispersion of sound, and the related absorption, have been treated by several authors, and particularly thoroughly by Rutgers.¹ Rutgers limits himself to the derivation of the dispersion formula for low temperatures, at which a vibration quantum is only excited in a few molecules. The number of molecules with more than one quantum excited he sets equal to zero. Furthermore he assumes that only the excitation of a particular normal vibration comes in question. If we put the number of molecules per cm^3 with 0 and 1 excited quantum equal to n_0 and n_1 , respectively, and the total number equal to N , then the rate of change of n_1 is given by the equation

$$\frac{dn_1}{dt} = K(k_{01}n_0N - k_{10}n_1N). \quad (1)$$

Here K gives the number of collisions per unit time which a molecule would make if the gas density were one molecule per cm^3 , k_{01} and k_{10} are respectively the probabilities that an unexcited molecule becomes excited, and vice versa, on collision. We have

$$\frac{k_{01}}{k_{10}} = \exp\left[-\frac{h\nu}{k_B T}\right], \quad (2)$$

where ν is the frequency of the particular molecular vibration. In the case treated by Rutgers, $h\nu \gg k_B T$, and

$$n_0 \gg n_1, \quad n_{10} \gg n_{01}.$$

If we now use $N = n_0 + n_1$, we can replace (1) approximately by

$$\frac{dn_1}{dt} = K(k_{01} N^2 - k_{10} n_1 N). \quad (1a)$$

If we now assume that the temperature, T , is a periodic function of the time, then the quantities occurring in (1a), in particular n_1 , will also vary periodically in time. If we multiply the change in n_1 by the energy of a vibration quantum, i.e. by $h\nu$ and divide it by the temperature change, the specific heat can be obtained, and finally the velocity of sound.

We should like to find out what form the dispersion takes when the Rutgers condition, $h\nu \gg k_B T$, no longer holds. For the time being, however, we will retain the condition, so that only one normal vibration contributes to the sound dispersion. There are molecules with several, say l , quanta available, the number of which molecules we shall denote by n_l . We shall have a complicated system of equations to solve, describing the simultaneous changes in the numbers, n_l . We should expect that certain quantum transitions occur in a shorter time, and thus at higher frequencies, whilst others will correspond to slower vibrations, so that the absorption curve should spread over a wider frequency range.

It will subsequently be shown, however, that with certain plausible assumptions, a simple solution of the problem is reached, which contradicts the two conjectures just laid down. In calculating the dispersion it suffices to solve only one differential equation, and the dispersion formula remains essentially the same as at low temperatures.

In order to find the simplest route to this result, we make the following assumptions:

(1) There is only one molecular vibration which contributes to the sound dispersion. We assume that this vibration is not degenerate.†

(2) The excitation of molecular vibrations on collision can be calculated with the help of first order perturbation theory. This assumption is justified in those cases in which dispersion occurs, since the probability of excitation of vibration is then small. This is so since we can propose a model for the motion in zeroth approximation, for which no excitation of vibration results, but which differs only little from the real motion.

(3) The interaction energy responsible for the excitation depends linearly on the normal co-ordinate of the vibration in question. This assumption is justified if the amplitude of the vibration is small compared with the distances at which the interaction energy changes appreciably.

† This assumption is only made to simplify the derivation. The results can easily be generalised later.

(4) The molecular vibrations are harmonic. To find the probability of excitation of the vibration quanta, we must first of all consider the matrix elements of the perturbation energy. Since by assumption (3) the perturbation energy depends linearly on the normal co-ordinate of the vibration, and by (4) the vibration is harmonic, the matrix elements of the perturbation energy will be proportional to the matrix elements of a harmonic oscillator. Hence all matrix elements vanish for which the initial and final states do not differ by one vibration quantum. The matrix element which corresponds to the transition from the l th to the $(l+1)$ st, or from the $(l+1)$ st to the l th quantum state is, on the other hand, proportional to $\sqrt{l+1}$. The transition probability between two quantum states, l and i of a molecule, is given apart from a factor given by the square of the matrix elements. We have accordingly:

$$k_{01} : k_{12} : k_{23} : \dots = k_{10} : k_{21} : k_{32} : \dots = 1 : 2 : 3 : \dots \quad (3)$$

$$k_{li} = 0 \quad \text{when} \quad l - i \neq \pm 1.$$

Equation (2) is again valid for the relation k_{01}/k_{10} .

Now in order to get one single equation instead of a system of differential equations the quantities, n_l , it is helpful to introduce the number, Z , of vibration quanta contained in one cm^3 . This quantity will also be of immediate significance in the calculation of the specific heat. We have

$$\sum_l l n_l = Z, \quad \sum_l n_l = N. \quad (4)$$

The time derivative of Z is given by

$$\frac{dZ}{dt} = KN \left[\sum_l k_{l, l+1} n_l - \sum_l k_{l, l-1} n_l \right]. \quad (5)$$

Transforming (5) and using (3) and (4), we have

$$\begin{aligned} \frac{dZ}{dt} &= K[N k_{01} \sum_l (l+1) n_l - N k_{10} \sum_l l n_l] \\ &= K[2k_{01} N^2 - (k_{10} - k_{01}) ZN]. \end{aligned} \quad (6)$$

As we can see, (6) has the same form as (1 a). It is not a fundamental difference that instead of the quantity n_1 in (1 a) Z appears in (6). Thus we must multiply the change in the quantity Z by $h\nu$, and divide by the temperature change, in order to obtain the specific heat; in a similar manner as in the case at lower temperature, we took the change in the quantity, n_1 . Following the same steps as Rutgers, we obtain from (6) a dispersion formula which differs from the Rutgers result only as follows: we must replace the quantity, k_{10} , which occurs in the Rutgers dispersion formula (it went into the derivation as the factor before $n_1 N$ in (1 a) by $(k_{10} - k_{01})$ (i.e. by the factor before ZN in (6))†.

† Compare also equation (7) below, in which k_{10} only occurs in the combination $k_{10} - k_{01}$.

If $h\nu \gg k_B T$ and thus $k_{10} \gg k_{01}$, we can ignore k_{01} compared with k_{10} , and have the Rutgers formula again.†

In order to be able more easily to undertake the generalisation to the case in which several normal vibrations contribute to the sound dispersion, we must follow the argument of the Rutgers calculation in some detail. We put

$$\left. \begin{aligned} T &= T_0 + T_1 e^{i\omega t}, \\ Z &= Z_0 + Z_1 e^{i\omega t}. \end{aligned} \right\}$$

Here $\omega/2\pi$ is the sound frequency. The periodic changes of the other quantities we need not express explicitly. With the assumptions $T_1 \ll T_0$, we have from (6), ignoring quantities of the second order,

$$Z_1 [i\omega + KN(k_{10} - k_{01})] = PT_1.$$

Here P is dependent on the quantities occurring in (6), and on their temperature derivatives, but contains neither ω nor Z . The contribution of the vibration to the specific heat is then

$$c_{\text{osc}} = \frac{P h \nu}{i\omega + KN(k_{10} - k_{01})}. \quad (7)$$

A direct calculation of P can be avoided by letting ω tend to 0. In this case c_{osc} must take on the value following from the Einstein function for T_0 . We denote this by c_{osc}^0 . We have this

$$c_{\text{osc}} = c_{\text{osc}}^0 \frac{KN(k_{10} - k_{01})}{i\omega + KN(k_{10} - k_{01})} = c_{\text{osc}}^0 \frac{1}{1 + i\frac{\omega}{\omega_0}},$$

where we have set

$$KN(k_{10} - k_{01}) = \omega_0. \quad (8)$$

The fact that we have found that this contribution to the specific heat has a complex value means that the change in the energy of the vibration does not take place with the same phase as the temperature change. From (7) we can calculate c_p and c_v and finally the velocity of sound, $\sqrt{\alpha \cdot c_p / c_v}$, which is also complex. (α is the derivative of the pressure with respect to the density for isothermal compression.) The real part gives the phase velocity of the sound wave, whilst from the imaginary part the absorption coefficient can be calculated.

If now several harmonic normal vibrations contribute to the sound dispersion, and we assume that the interaction energy is a linear function of all the normal co-ordinates, then these vibrations give rise to additive terms of the type (7) in the specific heat, since with a linear perturbation energy the excita-

† It is to be remarked, on the other hand, that the observations of Kneser (*Ann. Phys.* 16, 360, 1933) do not lead to a consistent approximation. He ignores k_{01} compared with k_{10} , but he replaces k_{10} not by $(k_{10} - k_{01})$, but by $(k_{10} + k_{01})$. The mistake in his derivation is that he ignores the transitions between one and two quanta states, which is not consistent, if we wish to draw a distinction between these two possibilities, whose percentage difference is of the order $e^{-h\nu/k_B T}$.

tion probabilities of the normal vibrations are independent of one another. The formula for the sound dispersion is thus calculated from the sum of the specific heats, c_p and c_v .

It should be noted here that the Rutgers formula is also valid for the limiting classical case, since it remains valid for arbitrarily high temperatures. It can also be derived by means of classical perturbation theory. Naturally the basic assumptions made here are important, especially the harmonicity of the vibration, and the linearity of the perturbation.

2. In this section we shall calculate the characteristic frequency of the Rutgers formula $\omega_0 = KN(k_{01} - k_{10})$, and in particular determine its temperature dependence.

Here KN is the number of collisions made by a molecule per unit time, and can best be determined from the thermal conductivity. The actual expression from the well-known formulae of kinetic theory of gases gives, as an order of magnitude,

$$KN = \frac{pc}{\vartheta}, \quad (9)$$

where p is the pressure, c the specific heat, and ϑ the thermal conductivity of the gas.

There remains still $k_{10} - k_{01}$, or essentially the excitation probability of the vibration, to be determined. In accordance with assumption (2), we presume that the excitation of the vibration occurs significantly slower than the exchange of the translational energy. Otherwise we should not be able to distinguish the absorption of the sound waves due to the vibrations from that which depends on internal friction and thermal conductivity, and the sound waves would not behave anomalously.

This improbability of excitation of the vibration energy can only be due to the fact that the duration of the vibration is small compared with that of the collision, so that the collision behaves adiabatically with respect to the vibration. The condition for this is that the amplitude of the vibration is small compared with the distance over which the forces of interaction change significantly. This has already been specified in assumption (3). In the adiabatic case we obtain exponentially small expressions for the transition probability³.

We wish to limit ourselves to the calculation of the exponential. We consider only those collisions for which the energy of the colliding particles is large compared with the vibration quantum, since we shall see subsequently that for all temperatures in question, it is only such collisions that play an important part. For this type of collision the excitation of vibrations can be considered purely classically, i.e. we can set the excitation probability proportional to the square of the Fourier components of the force. Let us consider the integral

$$\int_{-\infty}^{+\infty} F(t) e^{2\pi i \nu t} dt \quad (10)$$

in the complex plane, and displace the path of integration as far into the positive imaginary region as the singular points of the function $F(t)$ permit. We

can then easily see that the order of magnitude of the integral (10) is determined by the exponential $e^{-2\pi vt'}$ where t' denotes the imaginary part of the singular point nearest to the real axis. Since the interaction of the molecules cannot be quantitatively defined, we limit ourselves to a one dimensional movement in order to discover the qualitative behaviour. The formula

$$\frac{dx}{dt} = \sqrt{\frac{2}{M(E-U)}},$$

then shows that the singular points of $x(t)$ coincide with those of $U(x)$, and hence also with those of $F(x)$. (It can easily be shown that the value $U(x) = E$ is indeed a singular point for the function $t(x)$, but not for the function $x(t)$.) t' is thus the imaginary part of the time necessary to get to the singular point of the function $x(t)$. Furthermore we ascribe to U the form $Ae^{-x/a}$, which certainly has no quantitative meaning, but which expresses well the qualitative character of the atomic interaction. The singular point of this function occurs at $x = \infty$. Since only the imaginary part of the time is of interest to us, we need only integrate the expression

$$t = \int \frac{dx}{\sqrt{\frac{2}{M}(E-U)}}$$

in the region $U > E$. The constant A depends on the choice of origin, and can thus be selected arbitrarily. For convenience we put $A = E$, and after substituting v for the speed at infinity, we have for t' the expression

$$t' = \frac{1}{v} \int_{-\infty}^0 \frac{dx}{\sqrt{e^{-x/a} - 1}}$$

or

$$t' = \frac{\pi a}{v}$$

and the probability k_{01} is proportional to†

$$e^{-\frac{4\pi^2 a v}{v}}.$$

This expression diminishes very rapidly as the speed becomes less. Thus for the excitation of a vibration only collisions of particles whose velocity lies in the "tail" of the Maxwellian distribution play a part, and not the ordinary collisions. We have for the total exponential factor,

$$e^{-\frac{4\pi^2 a v}{v} - \frac{Mv^2}{2k_B T}}.$$

† This formula has been derived several times from different starting points, e.g. C. ZENER, *Phys. Rev.* 38, 277 (1931).

The most favourable velocity makes the exponential a maximum. We have thus

$$\frac{4\pi^2 a v}{v} = \frac{M v^2}{k_B T}$$

or

$$v^3 = \frac{4\pi^2 a v k_B T}{M}.$$

This quantity satisfies the condition

$$\frac{M v^2}{2} \gg h v, \quad (11)$$

which we took as the basis of the classical behaviour in the calculation for all temperatures at which the degrees of freedom of the vibration gave rise to any specific heat. The exponential factor is obtained finally in the form

$$k_{01} \sim e^{-\sqrt[3]{\frac{54\pi^2 a^2 v^2 M}{k_B T}}}. \quad (12)$$

By M we understand the effective collision mass, thus for a homogeneous gas about half the molecular mass. From (11) we have

$$k_{10} = e^{\frac{h v}{k_B T}} k_{01} \sim k_{01}$$

k_{01} and k_{10} , and also their difference, are thus of the same order of magnitude, and we obtain from (8), (9), and (12),

$$\omega_0 \sim \frac{p c}{\vartheta} e^{-\sqrt[3]{\frac{(2\pi^2 a v)^2 M}{2 k_B T}}} \frac{1}{\sqrt[3]{T}}. \quad (13)$$

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23. ON THE THEORY OF UNI-MOLECULAR REACTIONS

An investigation is made of the decomposition velocity of large molecules. Such reactions follow a uni-molecular course at high pressures, but a bimolecular course at low pressures. It is shown that there is a large intermediate zone between these limiting cases, in which the order of reaction moves gradually from the first condition into the second. Formulae are derived which connect the reaction velocity under various conditions with the thermodynamic functions of the gas.

IN uni-molecular reactions spontaneous decomposition of molecules occurs. In the course of this, those molecules decompose which have sufficient energy for the purpose, the so-called activated molecules. The reaction only proceeds in a truly uni-molecular way, however, if the number of activated molecules diminishes slowly in the course of their decomposition. Under the influence of the mutual collisions of the molecules, a thermal equilibrium is set up. The number of activated molecules, that is, those which possess a sufficiently large energy, is determined by these collisions. If the activated molecules decompose so rapidly that the thermal equilibrium cannot be established, the number of such molecules diminishes rapidly. Further decomposition can then only occur spontaneously when activated molecules are created by collision. Thus the decomposition of the molecules comes to be connected with the impact of two molecules with each other: that is, the reaction proceeds in a bimolecular fashion.

In order that the reaction shall proceed in a uni-molecular manner, it is absolutely necessary that the decomposition shall take place slowly, so that the time during which the activated molecules decompose shall be long compared with the time between two collisions, so that the establishment of thermal equilibrium shall be possible. These conditions are only fulfilled for small molecules if the decomposition would violate a selection rule, and involves a transition not conserving the total spin. If there is no forbidden transition, such a molecule would decompose within the time required for one oscillation, which is much shorter than the time between two collisions.

The condition necessary for uni-molecularity can be met, however, in the case of larger molecules, apart from the case where spin is not conserved as it is then necessary for this decomposition that the activation energy should be concentrated in a single place in the molecule. In view of the comparatively small probability of such a process, the decomposition time of a large molecule increases greatly, and can become greater than the time between two colli-

sions. The object of the present work is to consider the decomposition of such large molecules.

The decomposition of large molecules follows a uni-molecular course only at sufficiently high pressures. For very low pressures the time between two collisions is large. It is usual to suppose that the transition from a uni-molecular reaction to a bimolecular reaction takes place within a small pressure range. We designate as the reaction rate the ratio between the number of molecules decomposing in a unit of time, to the total number of molecules present at the given instant. In uni-molecular reactions this rate does not depend on the pressure, while in bimolecular reactions it is proportional to the pressure. The relationship between the rate and the pressure is usually written in the form $\alpha p/(b + p)$, so that it is constant for large values of p , and proportional to the pressure for small values of p . In this paper it is shown that, in reality, the transition from the uni-molecular to the bimolecular reaction occurs in a different fashion.

We assume that the molecule possesses an energy E , sufficiently large for the molecule to be regarded as activated. That is, E is greater than the activation energy δ . We shall in what follows consider only activated molecules. In order that this molecule shall decompose, it is absolutely necessary that an energy, at least ϵ , is concentrated at one place. In view of the size of the molecules under consideration, they may be regarded as macroscopic systems, so that statistical concepts may be employed. Normally, the total energy E of the molecule is distributed equally amongst all its degrees of freedom. The concentration of the energy E at one spot (that is, the distribution of the energy ϵ over a small number of the degrees of freedom) is a fluctuation possessing only small probability. The entropy of the molecule may be designated $S(E)$ for uniformly distributed energy. When the energy ϵ is concentrated in a few degrees of freedom, the entropy of the remaining parts of the molecule $S(E - \delta)$ may be regarded as practically equal to the entropy of the entire molecule, since the entropy of the part where the energy is concentrated may, because of the small size of this part, be neglected. The molecule is a closed system. Hence the probability of the fluctuation considered, or (what is the same thing) the probability of the decomposition of the molecule, is given by:

$$W(E) = A \exp[S(E - \epsilon) - S(E)]/k \quad (1)$$

where k is the Boltzmann constant. The coefficient A has dimensions sec^{-1} . It has the order of magnitude of an oscillation period of the molecule, that is, around 10^{12} to 10^{13} sec^{-1} . The value of A is in reality dependent on that of E , but this relationship can be neglected in relation to the exponential factor, and A may be regarded as constant.

We shall designate by $n(E)$ the fraction of the total number of molecules whose energy lies within a very small range around the value E . We now investigate the variation of $n(E)$ with time: that is, we determine the derivative dn/dt . We must remember that we consider only those values of E for which the molecule can be considered as activated. The value of $n(E)$ changes, in the first place, under the influence of the mutual collisions of the molecules

and, secondly, under the influence of the decomposition. We consider first principally the first of these two factors. Through collisions with non-activated molecules, the molecules with energy E lose a portion of this energy, and cease to be activated. The reduction of the number of molecules with energy E under the influence of these collisions is clearly proportional to $n(E)$. On the other hand, non-activated molecules may become activated through collisions. In a single unit of time there will arise from this cause a certain number of activated molecules, which is clearly independent of the pre-existing total of such molecules. This number, divided by the total number of molecules, will be denoted by l . In this way we obtain, for the effect of collisions, the relation:

$$\left(\frac{dn}{dt}\right)_{\text{coll}} = -\omega n(E) + l,$$

where ω is the number of collisions in the time unit. This is clearly proportional to the pressure.

When thermal equilibrium exists, $(dn/dt)_{\text{coll}} = 0$, and the equilibrium value is given by $n(E) = n_0(E)$, for which evidently $l = \omega n_0(E)$, and also:

$$\left(\frac{dn}{dt}\right) = \omega [n_0(E) - n(E)].$$

In addition to this effect, the number $n(E)$ of activated molecules is reduced by their decomposition. Let $W(E)$ be the probability of decomposition per unit of time for the molecule with energy E . We then obtain finally:

$$\frac{dn}{dt} = \omega [n_0(E) - n(E)] - n(E) \cdot W(E). \quad (2)$$

The quantity dn/dt can never be precisely equal to zero, since the total number of molecules must necessarily diminish in the course of decomposition. Nevertheless, for slow decomposition a practically stationary state is attained, for which dn/dt is approximately zero.

From equation (2) we thus obtain:

$$n(E) = \frac{n_0 \omega}{\omega + W}.$$

The number of molecules decomposing in the unit of time, possessed of energy E (in relation to the total number of molecules) is $n(E) \cdot W(E)$. We designate this number $\lambda(E)$, given by:

$$\lambda = \frac{n_0}{\frac{1}{\omega} + \frac{1}{W}}. \quad (3)$$

The distribution $n_0(E)$ is given by the Gibbs distribution:

$$n_0(E) = \exp \left[\frac{F_0 - E}{k T_0} \right] \delta \Gamma.$$

Here F_0 and T_0 designate the free energy and temperature in the equilibrium state, while δF is the region of the phase-space of the molecule corresponding to a state with an energy close to E . The logarithm of δF is the entropy of the molecule in the same state, given by:

$$S(E) = k \cdot \ln \delta F.$$

Thus we obtain:

$$n_0(E) = \exp \left[\frac{F_0 - E}{k T_0} + \frac{S(E)}{k} \right]. \quad (4)$$

Inserting (1) and (4) into (3), we obtain:

$$\lambda(E) = \frac{\exp \left[\frac{F_0 - E}{k T_0} + \frac{S(E)}{k} \right]}{\frac{1}{\omega} + \frac{1}{A} \exp \left\{ \frac{S(E) - S(E - \varepsilon)}{k} \right\}}. \quad (5)$$

This expression has, generally speaking, a rather sharp maximum, attributable to the exponential functions of which it consists. Because of this, the principal role in decomposition is played by those molecules which have an energy corresponding to the maximum λ . The maximum of λ also determines the rate of the decomposition. In what follows we shall use λ to designate this maximum value, and consider the energy as that for which λ has a maximum value.

Three possible cases must be considered. If λ is a maximum, the first term in the denominator of (5) can be considerably smaller than the second, or the second considerably smaller than the first, or they can both be of the same order of magnitude. We proceed to consider these possibilities in turn.

$$1. \text{ Let: } \frac{1}{\omega} \ll \frac{1}{A} \exp \left\{ \frac{S(E) - S(E - \varepsilon)}{k} \right\}. \quad (6)$$

It follows then from (5) that:

$$\lambda = A \exp \left[\frac{F_0 - E}{k T_0} + \frac{S(E - \varepsilon)}{k} \right]. \quad (7)$$

This is by definition to correspond to a maximum, so that the derivative $d\lambda/dE$ must be zero. As we have already indicated, A may be regarded as a constant. The derivative dS/dE is, however, the reciprocal of the temperature. If we then differentiate equation (7) in this manner, and set $d\lambda/dE$ equal to zero, we obtain:

$$-\frac{1}{k T_0} + \frac{1}{k T(E - \varepsilon)} = 0$$

$T(E - \varepsilon)$ is the temperature of the molecule with energy $E - \varepsilon$. From this equation it follows that $T(E - \varepsilon) = T_0$, and, further, that:

$$E - \varepsilon = E_0 \quad (8)$$

in which E_0 represents the mean thermal energy of the molecule. In this way it is found that the principal role in the decomposition is not played by those molecules which have energy equal to the activation energy ε , but by those whose energy is greater than ε by an amount E_0 .

If we now insert (8) into (7), taking into account the fact that $F_0 = E_0 - T_0 S_0$, where $E_0 = E(T_0)$ and $S_0 = S(T_0)$, being the energy and entropy corresponding to the equilibrium at temperature T_0 ; while according to equation (8) $E = E_0 + \varepsilon$ and so $S(E - \varepsilon) = S_0$, we find that:

$$\lambda = A e^{-\varepsilon/kT_0}, \quad (9)$$

which is the familiar formula for the dependence of reaction rate on temperature. The reaction rate λ is in this case independent of the pressure, and so the reaction proceeds in a uni-molecular manner. Condition (6) now takes the form:

$$\omega \gg A \cdot \exp \left\{ \frac{S_0 - S(E_0 + \varepsilon)}{k} \right\}. \quad (10)$$

Since ω is proportional to the pressure, it follows that the case under consideration corresponds to high values of the pressure.

2. Here we assume that, for maximum value of λ ,

$$\frac{1}{\omega} \gg \frac{1}{A} \exp \left\{ \frac{S(E) - S(E - \varepsilon)}{k} \right\}. \quad (11)$$

Equation (5) then takes the following form:

$$\lambda = \omega \exp \left[\frac{F_0 - E}{k T_0} + \frac{S(E)}{k} \right]. \quad (12)$$

Since λ is still regarded as being maximum, $d\lambda/dE$ is zero, that is:

$$\frac{-1}{k T_0} + \frac{1}{k T(E)} = 0$$

(taking the alteration of ω with the energy to be negligible in comparison with the exponential factor). It follows further from this that,

$$T(E) = T_0,$$

or the energy related to this temperature,

$$E = E_0 \quad (13)$$

E is, however, greater than ε , since we are dealing only with activated molecules. Hence (13) may be valid if E_0 is greater than ε , that is, if the mean thermal energy is greater than the activation energy. This case will be discussed later, and for the present we confine ourselves to the case in which E_0 is less than ε . In such circumstances $d\lambda/dE$ is never equal to zero, but always negative (since E is always greater than E_0 and $T(E)$ than T_0). Hence λ has the greatest

possible value for the smallest possible value of E , that is, for

$$E = \varepsilon \quad (14)$$

(We set the smallest energy of the molecule arbitrarily as equal to zero. If this is not the case, we should replace (14) by $E = \varepsilon + E_{\min}$). In this way molecules will decompose when they possess the smallest possible energy for which decomposition is in principle possible, that is, when their energy is equal to the activation energy.

When we insert (14) in (12), we obtain:

$$\lambda = \omega \exp \left[\frac{F_0 - \varepsilon}{k T_0} + \frac{S(\varepsilon)}{k} \right]. \quad (15)$$

In this case λ is proportional to the pressure, and so the reaction is bimolecular. The dependence of λ on the temperature T_0 is therefore different from that in the uni-molecular decomposition (equation (9)).

The heat of activation is usually given by the magnitude:

$$q = - \frac{d \ln \lambda}{d \left(\frac{1}{k T_0} \right)}.$$

If λ is determined by equation (9), the heat of activation is identical with the activation energy ε . In the case at present considered, we have used equation (15), neglecting the temperature dependence of $\ln \omega$, to obtain the following relation:

$$q = - \frac{d(F_0/T_0)}{d(1/T_0)} + \varepsilon_0.$$

However,

$$\frac{d(F_0/T_0)}{d(1/T_0)} = F_0 - T_0 \frac{dF_0}{dT_0} - E_0$$

and thus we obtain finally:

$$q = \varepsilon - E_0. \quad (16)$$

Thus the activation heat is in this case smaller than the activation energy. Equation (15) can here be written in the following form:

$$\lambda = \omega \exp \left[\frac{S(\varepsilon) - S_0}{k} - \frac{q}{k T_0} \right]. \quad (17)$$

Thus the factor by which $\exp(-q/kT_0)$ is multiplied is appreciably greater than ω , since $S(\varepsilon)$ is greater than S_0 (when ε is greater than E_0). The determining equation (11), when expression (14) is inserted into it, then takes the following form:

$$\frac{1}{\omega} \gg \frac{1}{A} \exp \left[\frac{S(\varepsilon) - S(0)}{k} \right].$$

From the Nernst theorem, however, we know that $S(0) = 0$, and hence,

$$\omega \ll A \exp \left[-\frac{S(\varepsilon)}{k} \right]. \quad (18)$$

This case therefore implies that the pressures are low.

A large region lies between the first and second cases considered. It is easy to see that the upper boundary of (18) is appreciably smaller than the lower boundary of (10):

$$A \exp \left[-\frac{S(\varepsilon)}{k} \right] \ll A \exp \left[\frac{S_0 - S(\varepsilon + E_0)}{k} \right].$$

In fact, the derivative of the expression $S(E) - S(\varepsilon + E)$ with respect to the energy is equal to $[1/T(E)] - [1/T(E + \varepsilon)]$, which is clearly always positive, since T always increases with increase in energy; and so the value of this expression for $E = 0$ is equal to $-S(\varepsilon)$, which is smaller than its value for $E = E_0$. The region between cases 1 and 2 will be discussed now in section 3.

3. We consider finally the intermediate case, for which the denominator of (5), for maximum λ , satisfies

$$\frac{1}{\omega} \sim \frac{1}{A} \exp \left[\frac{S(E) - S(E - \varepsilon)}{k} \right]. \quad (19)$$

We can then retain one of the two terms in the denominator, the second, for example:

$$\lambda = A \exp \left[\frac{F_0 - E}{k T_0} + \frac{S(E - \varepsilon)}{k} \right]. \quad (20)$$

This expression has a form identical with (12). E is here, however, a complex function of ω , which is determined by (19).

The reaction is bimolecular if λ depends on p , and uni-molecular when λ is independent of p . The order of reaction can in general be defined in terms of an exponent α of p , if we write: $\lambda = \text{const } p^\alpha$. It is clear that

$$\alpha = \frac{d \ln \lambda}{d \ln p},$$

or, since ω is the only quantity dependent on p and directly proportional to it, it follows that:

$$\alpha = \frac{d \ln \lambda}{d \ln \omega}.$$

If we now insert (20), remembering that the magnitude E in (20) is not a constant, but is connected with ω by equation (19), we obtain:

$$\alpha = \left[-\frac{1}{k T_0} + \frac{1}{k T(E - \varepsilon)} \right] \frac{d E}{d \ln \omega}, \quad (21)$$

since the derivative of the entropy with respect to the energy is equal to the reciprocal of the temperature. If we insert the = sign in (19) instead of \sim and differentiate, we find:

$$1 = \left[\frac{1}{k T(E - \varepsilon)} - \frac{1}{k T(E)} \right] \frac{dE}{d \ln \omega}.$$

If we now determine $dE/d \ln \omega$ from this and insert it in (21), we obtain:

$$a = \frac{\frac{1}{T(E - \varepsilon)} - \frac{1}{T_0}}{\frac{1}{T(E - \varepsilon)} - \frac{1}{T(E)}}. \quad (22)$$

In accordance with the alteration of E from $E_0 + \varepsilon$ (case 1) to ε (case 2) the value of a varies gradually from 0 to 1, passing through all intermediate values.

The heat of activation, q , takes in this case the following form:

$$q = \frac{d \ln \lambda}{d \frac{1}{k T_0}} = E - E_0, \quad (23)$$

if the dependence of E on T_0 which is expressed in (19) is neglected, which is permissible since it is expressed by $\ln \omega$. If E varies from $E_0 + \varepsilon$ to ε , the value of q then changes from ε to $\varepsilon + E_0$.

If $E_0 > \varepsilon$ then, as we have already shown, the treatment of case 2 given above is not applicable. In this case, for small pressures, $E = E_0$, that is, the essential role in the decomposition is played by the molecules which are preponderantly present in the gas. In accordance with this, we must insert in the expression $\lambda = n(E) W(E)$, instead of $n(E)$ the number unity, so obtaining:

$$\lambda = W(E_0)$$

or, according to (1):

$$\lambda = A \exp \left[\frac{S(E_0 - \varepsilon) - S(E_0)}{k} \right]. \quad (24)$$

In this expression ω does not appear. Hence the reaction remains uni-molecular even when E_0 is greater than ε , and even for small pressures. However, the significance of the decomposition rate is now altered. For the activation energy at small pressures, we obtain from (24) the expression:

$$q = \frac{d \ln \lambda}{d \frac{1}{T_0}} = T_0^2 \left[\frac{1}{T(E_0 - \varepsilon)} - \frac{1}{T_0} \right] \frac{dE_0}{dT_0} = T_0^2 c_0 \left[\frac{1}{T(E_0 - \varepsilon)} - \frac{1}{T_0} \right], \quad (25)$$

where c_0 is the specific heat of the molecule.

The boundary between cases 2 and 3 is not given by condition (18) alone, but by the fact that in (19) the energy is changed to E_0 . Hence

$$\omega \sim A \exp \left[\frac{S(E_0 - \varepsilon) - S(E_0)}{k} \right]. \quad (26)$$

Thus, in the case where E_0 is greater than ε the quantity α , which determines the order of reaction in (22), increases with reduction in pressure slowly from 0 to 1 (when the values of E are close to that of E_0), and then falls with ω , which is determined from (26), comparatively rapidly to zero.

The practical application of the formulae obtained is made difficult by the fact that the function $S(E)$ is unknown, especially for large values of E . Nevertheless, the formulae may be used in reverse order, by using the known course of uni-molecular reactions to determine the thermodynamic function of the gas.

24. THE TRANSPORT EQUATION IN THE CASE OF COULOMB INTERACTIONS

A transport equation is derived for a system consisting of charged particles taking their interactions into account. The order of magnitude of the mean free path of the particles in such a system is determined. The rate at which the temperatures of the ions and electrons in the plasma become equal is evaluated.

In the case of Coulomb interactions there appear, in the formulae for the kinetic theory of gases, integrals which are divergent when the distances between the particles are large. This means that an important role is played by those collisions in which the distances between the colliding particles are large. But at large distances the particles are only scattered through small angles with small changes in velocity. Thus collisions in which the velocity vector is only slightly changed are important.

Let $n(p_i)$ be the distribution function in momentum space. It is a function of the three components of the momentum of the particle ($i = x, y, z$). The change in the momentum during a collision we shall denote by Δ_i where $\Delta_i \ll p_i$ in all the collisions. Further, let dW be the probability (per unit time) of a collision between particles with momentum p_i and a particle with momentum p'_i , such that p'_i is changed to $p_i + \Delta_i$ and p'_i to $p'_i + \Delta'_i$. Because of momentum conservation $\Delta_i = -\Delta'_i$. We shall not, however, use this fact for the moment, in order that we may obtain formulae which are valid in the general case. The number of such collisions will then be

$$dW n(p) n'(p')$$

(for simplicity we shall omit the indices on p_i and Δ_i in $n(p_i)$ and so on).

The number of collisions changing particle momenta $p_i + \Delta_i$ and $p'_i + \Delta'_i$ back to p_i and p'_i will equal

$$dW n(p + \Delta) n(p' + \Delta'),$$

since according to the Liouville theorem the probabilities of forward and reverse transitions are equal.

Let us express the probability dW as a function of the half-sum and half-difference of the momenta in the initial and final states. Then the probability of a forward transition will be

$$dW \left(p + \frac{\Delta}{2}, p' + \frac{\Delta'}{2}, \Delta, \Delta' \right),$$

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and for the reverse transition

$$dW\left(p + \frac{\Delta}{2}, p' + \frac{\Delta'}{2}, -\Delta, -\Delta'\right).$$

Since these probabilities are equal, $dW(p, p', \Delta, \Delta')$ is an even function of Δ_i and Δ'_i .

Hence the number of particles with momentum p_i is changed, due to collisions, in unit time by

$$\int dW\left(p + \frac{\Delta}{2}, p' + \frac{\Delta'}{2}, \Delta, \Delta'\right)\{n(p)n'(p') - n(p+\Delta)n'(p'+\Delta')\}.$$

The probability dW we write in the form

$$dW = w\left(p + \frac{\Delta}{2}, p' + \frac{\Delta'}{2}, \Delta, \Delta'\right) d\tau' d\tau_\Delta,$$

where $d\tau' = dp'_x dp'_y dp'_z$ and $d\tau_\Delta$ is the product of the differentials of the parameters which define the collision.

Thus the change in the number of particles with momentum p'_i is:

$$\int d\tau' d\tau_\Delta w\left(p + \frac{\Delta}{2}, p' + \frac{\Delta'}{2}, \Delta, \Delta'\right)\{n(p)n'(p') - n(p+\Delta)n'(p'+\Delta')\}. \quad (1)$$

Let us expand the expression under the integral in a series in powers of Δ_i and Δ'_i (w should of course be expanded only with respect to Δ_i , appearing in $p_i + \Delta_i/2$ and $p'_i + \Delta'_i/2$). The zero order terms cancel each other and the terms of the first order are

$$-\int d\tau' d\tau_\Delta \left(w n' \frac{\partial n}{\partial p_i} \Delta_i + w n \frac{\partial n'}{\partial p'_i} \Delta'_i \right),$$

where $w = w(p, p', \Delta, \Delta')$ (summation is everywhere implied over indices which are repeated twice). But w is an even function of Δ_i and Δ'_i . Therefore the integral written above is equal to zero.

The second order terms are the following

$$\begin{aligned} & -\int d\tau' d\tau_\Delta w \left\{ \frac{\Delta_i \Delta_k}{2} n' \frac{\partial^2 n}{\partial p_i \partial p_k} + \Delta_i \Delta'_k \frac{\partial n'}{\partial p'_k} \frac{\partial n}{\partial p_i} + \frac{\Delta'_i \Delta'_k}{2} n \frac{\partial^2 n'}{\partial p'_i \partial p'_k} \right\} \\ & -\int d\tau' d\tau_\Delta w \frac{1}{2} \left(\Delta_i \frac{\partial w}{\partial p_i} + \Delta'_i \frac{\partial w}{\partial p'_i} \right) \left(\Delta_k n' \frac{\partial n}{\partial p_k} + n \Delta'_k \frac{\partial n'}{\partial p'_k} \right). \end{aligned} \quad (2)$$

Let us integrate two of these terms by parts over $d\tau'$, namely:

$$\begin{aligned} & -\frac{1}{2} \int d\tau' d\tau_\Delta \Delta'_i \Delta_k n' \frac{\partial w}{\partial p'_i} \frac{\partial n}{\partial p'_k} = \frac{1}{2} \int d\tau' d\tau_\Delta \Delta'_i \Delta_k w \frac{\partial n'}{\partial p'_i} \frac{\partial n}{\partial p_k} \\ & -\frac{1}{2} \int d\tau' d\tau_\Delta \Delta'_i \Delta'_k \frac{\partial w}{\partial p'_i} \frac{\partial n'}{\partial p'_k} n = \frac{1}{2} \int d\tau' d\tau_\Delta \Delta'_i \Delta'_k w \frac{\partial^2 n'}{\partial p'_i \partial p'_k} n \end{aligned}$$

Since the integration is performed over the whole of p' space, the surface integral is equal to zero, because $n' = 0$ at infinity.

As a result the second-order terms give

$$\begin{aligned} & - \int d\tau' d\tau_d w \left\{ \frac{\Delta_i \Delta_k}{2} n' \frac{\partial^2 n}{\partial p_i \partial p_k} + \frac{\Delta_i \Delta'_k}{2} \frac{\partial n}{\partial p_i} \frac{\partial n'}{\partial p'_k} \right\} \\ & - \int d\tau' d\tau_d w \left\{ \frac{\Delta_i \Delta_k}{2} \frac{\partial w}{\partial p_i} \frac{\partial n}{\partial p_k} n' + \frac{\Delta_i \Delta'_k}{2} n \frac{\partial w}{\partial p_i} \frac{\partial n'}{\partial p'_k} \right\}. \end{aligned}$$

This can be re-written in the form

$$- \frac{\partial}{\partial p_i} \int d\tau' d\tau_d w \left\{ \frac{\Delta_i \Delta_k}{2} n' \frac{\partial n}{\partial p_k} + \frac{\Delta_i \Delta'_k}{2} n \frac{\partial n'}{\partial p'_k} \right\}.$$

Thus the integral (1), defining the change due to collisions in the number of particles with given momentum is expressed, as it should be, as the divergence $\partial j_i / \partial p_i$ in momentum space, of the flow vector j_i in momentum space. The components of this flow equal

$$j_i = - \int d\tau' d\tau_d w \left\{ \frac{\Delta_i \Delta_k}{2} n' \frac{\partial n}{\partial p_k} + \frac{\Delta_i \Delta'_k}{2} n \frac{\partial n'}{\partial p'_k} \right\}.$$

As was already noted at the beginning, $\Delta_i = -\Delta'_i$. Therefore in our case the flow is

$$j_i = \int d\tau' d\tau_d \left\{ \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \int \frac{\Delta_i \Delta_k}{2} w d\tau_d \right\}.$$

If the system consists of different types of particles, then the flow j_i for a given type of particle is equal to

$$j_i = \sum \int d\tau' \left\{ \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \int \frac{\Delta_i \Delta_k}{2} w d\tau_d \right\}, \quad (3)$$

where the summation is performed over all the kinds of particles in the system, unprimed variables being related to the given type of particle and primed variables to each type of particle in turn (in this number, of course, is included the given type).

Let us apply the formulae thus obtained to the case of a system of particles with Coulomb interactions, which we are considering. For this system let us determine the change in the momenta of two particles with charges, e and e' and momenta p_i and p'_i moving at some distance from one another. Let ϱ be the impact parameter, i.e. the distance at which the two particles would pass each other if there were no interaction between them, and u_i their relative velocity. Let us consider this collision in the co-ordinate system in which the particle e' is at rest, with the x -axis along the direction of motion of the particle e , which has velocity u . We consider the scattering angle to be small. Because of this the momentum along the x -axis does not change to this approximation,

and only the momentum in a direction perpendicular to the x -axis (along the y -axis) changes. This change equals

$$\Delta_y = \int_{-\infty}^{+\infty} -\frac{\partial U}{\partial y} dt,$$

where $U = e e' / r$ is the energy of interaction between the particles.

Since the scattering is considered to be small it is possible to consider, in the integral, that the motion is unperturbed, i.e. directed along the x -axis. Then

$$\Delta_y = \int_{-\infty}^{+\infty} \frac{e e' \varrho dt}{(\varrho^2 + u^2 t^2)^{3/2}} = \frac{2e e'}{\varrho u}.$$

Going back to an arbitrary co-ordinate system, and noticing that the vector of the change in momentum is directed along the direction of ϱ_i we find

$$\Delta_i = \frac{2e e'}{u} \frac{\varrho_i}{\varrho^2}. \quad (4)$$

Let us now calculate the integrals

$$\alpha_{ik} = \int \frac{\Delta_i \Delta_k}{2} w d\tau_d = \int \frac{2e^2 e'^2}{u^2} \frac{\varrho_i \varrho_k}{\varrho^4} w d\tau_d,$$

appearing in (3). $n' dW = \omega n' d\tau' d\tau_d$ is the number of collisions per unit time with particles e' , undergone by the particle e with momentum p_i , in which its momentum changes by the given value Δ_i . In other words this is the number of collisions in which particles e and e' pass a definite distance ϱ_i apart, the particles e' having definite momentum p'_i (Δ_i is completely determined for given p'_i and ϱ_i). Denote by v_i and v'_i the velocities of the particles e and e' . Their relative velocity $u_i = v_i - v'_i$ has absolute value u . The number of collisions of the particle e which take place at a given distance ϱ_i with the given relative velocity u_i is obviously

$$u \varrho d\varrho d\varphi u' d\tau',$$

where φ is the angle determining the direction of ϱ_i (at the given velocity u_i all the possible ϱ_i lie in one plane which is perpendicular to u_i ; φ is the angle in that plane).

Hence we can change $w d\tau_d$ to $u \varrho d\varrho d\varphi$ in the integrals α_{ik}

$$\alpha_{ik} = \frac{2e^2 e'^2}{u} \int \frac{\varrho_i \varrho_k}{\varrho^3} d\varrho d\varphi.$$

In order to perform the integration, introduce, temporarily, co-ordinate axes with the x -axis directed along u_i . Then $\varrho_x = 0$ since $\varrho_i \perp u_i$. Because of

this $\alpha_{xx} = \alpha_{xy} = \alpha_{xz} = 0$. Also $\alpha_{yz} = 0$ since the integral of $\varrho_y \varrho_z = \varrho^2 \sin \varphi \cos \varphi$ over all angles φ vanishes.

Thus for α_{yy} and α_{zz} , which are not equal to zero, we find (substituting $\varrho_z = \varrho \sin \varphi$, $\varrho_y = \varrho \cos \varphi$ and integrating with respect to $d\varphi$)

$$\alpha_{yy} = \alpha_{zz} = \frac{2\pi e^2 e'^2}{u} \int_{\varrho_1}^{\varrho_2} \frac{d\varrho}{\varrho}. \quad (5)$$

The integral appearing here diverges logarithmically. The divergence at small ϱ is due to the fact that for small ϱ the scattering angle of the particles in the collision is large, and hence all the previous formulae are no longer valid. If the exact formulae are used then there would, of course, be no divergence (at small ϱ).

Since a logarithm is insensitive to small changes in its argument, we can take in (5), as the lower limit ϱ_1 , that value ϱ at which the scattering angle becomes of the order of unity, i.e. the interaction energy $e e' / \varrho$ becomes of the order of the mean kinetic energy $\bar{\varepsilon}$ of the particles:

$$\varrho_1 = \frac{e e'}{\bar{\varepsilon}}.$$

As far as the upper limit ϱ_2 in (5) is concerned, two cases must be distinguished. If the total charge on the particles in the system is not equal to zero, then as the upper limit one must take the linear dimension R of the region in which these particles lie. In the most interesting case, when the total charge of the system is zero, the charges are screened and as ϱ_2 one should take the Debye-Hückel screening radius. This radius is $1/\kappa$ where κ is the coefficient in the screened Coulomb law $e^{-\kappa r}/r$ and is determined by the well-known equation

$$\kappa^2 = \sum \frac{N_i e_i^2}{k T}.$$

Here the summation is taken over all types of particles in the system and N_i is the number of particles of the i th kind in 1 cm^3 . To an order of magnitude $\kappa \cong \sqrt{\bar{N} e^2 / k T}$ where \bar{N} is the number of particles in 1 cm^3 . But $k T \cong \bar{\varepsilon}$ so that $\kappa = \sqrt{\bar{N} e^2 / \bar{\varepsilon}}$. Thus we can take for the upper limit in (5),

$$\varrho_2 = \sqrt{\frac{\bar{\varepsilon}}{\bar{N} e^2}}.$$

Substituting ϱ_1 and ϱ_2 in (5) we find

$$\alpha_{yy} = \alpha_{zz} = \frac{\pi e^2 e'^2}{u} L,$$

where

$$L = \ln \frac{1}{\bar{N}} \left(\frac{\bar{\varepsilon}}{e^2} \right)^3. \quad (6)$$

Returning now to an arbitrary co-ordinate system we can write, in tensor form,

$$\alpha_{ik} = \pi e^2 e'^2 L \frac{u^2 \delta_{ik} - u_i u_k}{u^3},$$

where

$$\delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Substituting this expression into (3) we find the flow of particles e in momentum space in the form

$$j_i = \pi e^2 L \sum e'^2 \int \left\{ n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right\} \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau'. \quad (7)$$

The transport equation in the presence of a temperature gradient and an external electric field E_i has the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial T} v_i \frac{\partial T}{\partial x_i} + e E_i \frac{\partial n}{\partial p_i} + \frac{\partial j_i}{\partial v_i} = 0. \quad (8)$$

The Maxwellian distribution makes j_i zero, as it should do.

It would, in principle, be possible to determine from this equation the electrical and thermal conductivity of the gas consisting of the charged particles. This, however, meets considerable mathematical difficulties. We restrict ourselves to a qualitative determination of the conductivities, namely, we determine, to within an order of magnitude, the mean free path l of the particles, from which it is possible to find the electrical and thermal conductivities by the use of well-known formulae.

Let \bar{N} be (to an order of magnitude) the number of particles in 1 cm^3 , e the charge of the particles and T the temperature of the gas. As is seen from (7), when it is substituted into (8), \bar{N} and e appear in the formulae only in the combination $\bar{N} L e^4$. Therefore, the mean free path of the particles should be determined only in terms of the quantities $e^4 L \bar{N}$, kT and the mass of the particles. From these one can construct only one combination having the dimensions of a length, namely $(kT)^2 / (e^4 L \bar{N})$. To within an order of magnitude the mean free path will be equal to just this ratio

$$l \cong \frac{k^2 T^2}{e^4 L \bar{N}}. \quad (9)$$

This result disagrees with Gabor's formulae¹, which points to the incorrectness of his assumptions.

Let us consider a gas consisting of electrons and ions. Because of the large difference in masses between the electrons and ions, the exchange of energy by the electrons amongst themselves and the ions amongst themselves will take place much more rapidly than the exchange of energy between the electrons and ions (in a collision between a very heavy particle and a very light one, the energy of each of them is almost unchanged). Because of this the equilib-

rium in the energies of the electrons amongst themselves and the ions amongst themselves will be established much sooner than the equilibrium between the unlike groups. Let us consider that such an equilibrium is already established, i.e. the electrons and the ions both have a Maxwellian distribution, but the temperatures of these distributions, T' and T , are different. Let us find the rate at which the equilibrium between the electrons and ions is established, i.e. the rate of equalisation of the temperatures T' and T .

Let us work out the energy transmitted by the electrons to the ions in unit time (in 1 cm³) by collisions between them. Let e , m and e' , m' be the charges and masses of the ions and electrons and n and n' their distributions:

$$n = N(2\pi m k T)^{-3/2} e^{-\varepsilon/kT}, \quad n' = N'(2\pi m' k T')^{-3/2} e^{-\varepsilon'/kT'}; \quad (10)$$

N and N' are the numbers of ions and electrons in 1 cm³ and ε and ε' are their energies. The flow of ions in momentum space is, according to (7):

$$j_i = \pi e^2 e'^2 L \int \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau' \quad (11)$$

(all primed variables correspond to the electrons, unprimed variables to the ions). In the sum in (7) only one term remains, since the term which corresponds to the collisions of ions one with another vanishes, because the distribution of the ions is Maxwellian.

The change per unit time in the number of ions with given momenta due to collisions with electrons is $-\partial j_i / \partial p_i$. Thus the change in their energy is

$$-\int \varepsilon \frac{\partial j_i}{\partial p_i} d\tau,$$

or, integrating by parts

$$-\int \varepsilon \frac{\partial j_i}{\partial p_i} d\tau = \int j_i \frac{\partial \varepsilon}{\partial p_i} d\tau = \int j_i v_i d\tau$$

($\partial \varepsilon / \partial p_i = v_i$). Since the integration is taken over all momentum space, the surface integral disappears.

Substitute the distributions (10) into (11). We have

$$\frac{\partial n}{\partial p_k} = -\frac{n}{kT} \frac{\partial \varepsilon}{\partial p_k} = -\frac{n v_k}{kT}, \quad \frac{\partial n'}{\partial p'_k} = -\frac{n' v'_k}{kT'}.$$

Then we find

$$\begin{aligned} j_i &= \pi e^2 e'^2 L \int n n' \left(\frac{v_k}{kT} - \frac{v'_k}{kT'} \right) \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau' \\ &= \pi e^2 e'^2 L \int n n' \left[v_k \left(\frac{1}{kT} - \frac{1}{kT'} \right) + \frac{u_k}{kT'} \right] \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau'. \end{aligned}$$

But

$$u_k \frac{u^2 \delta_{ik} - u_i u_k}{u^3} = 0$$

and therefore

$$j_i = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'} \right) \int n n' \frac{u^2 v_i - (u_i v_i)^2}{u^3} d\tau d\tau'.$$

The change in energy, which we are seeking, is then equal to

$$\int j_i v_i d\tau = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'} \right) \iint n n' \frac{u_i^2 v_i^2 - (v_i u_i)^2}{u^3} d\tau d\tau'.$$

Since the mass of the electrons is much less than the mass of the nuclei, their velocity v'_k is much larger than the velocity of the ions v_k . Because of this one may consider that $u_i \cong v'_i$. Then

$$\int j_i v_i d\tau = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'} \right) \iint n n' \frac{v_i^2 v_i'^2 - (v_i v_i')^2}{v'^3} d\tau d\tau'.$$

Averaging over the angles between v_i and v'_i we find

$$\int j_i v_i d\tau = \frac{2}{3} \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'} \right) \int n v^2 d\tau \int \frac{n'}{v'} d\tau'.$$

Substituting (10) we have:

$$\begin{aligned} \int n v^2 d\tau &= N \frac{3kT}{m}, \quad \int \frac{n'}{v'} d\tau' = 4\pi N' \left(\frac{m'}{2\pi kT'} \right)^{3/2} \int_0^\infty e^{-\frac{m'v'^2}{2kT'}} v' dv' \\ &= 2N' \sqrt{\frac{m'}{2\pi kT'}}. \end{aligned}$$

As a result we find:

$$\int v_i j_i d\tau = \frac{2N N' e^2 e'^2 (2\pi m')^{1/2} L}{m k^{1/2} T'^{3/2}} (T' - T).$$

If there are ions of different types in the gas, the total energy transmitted by the electrons to the ions per unit time is

$$\frac{2N' e'^2 (2\pi m')^{1/2} L}{k^{1/2} T'^{3/2}} (T' - T) \sum \frac{N e^2}{m} \quad (12)$$

(\sum is over all types of ions).

The energy of the electrons in 1 cm³ is equal to $3N' kT'/2$. Dividing the energy (12), lost by the electrons in unit time, by $3N' k/2$, we obtain the rate of change of the electron temperature T' :

$$\frac{dT'}{dt} = -\frac{4}{3} \frac{e'^2 (2\pi m')^{1/2} (T' - T)}{(kT')^{3/2}} L \sum \frac{N e^2}{m}. \quad (13)$$

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25. ON THE PROPERTIES OF METALS AT VERY LOW TEMPERATURES

In the first part of the paper the influence of the interaction of electrons regarded as a small perturbation on the resistance of metals is analysed. The resistance produced by the interaction of electrons is proportional to T^2 .

In the second part an expression is derived for the thermo-e.m.f. which is in agreement with the Onsager-Thomson relationship.

1. The existing theory of metals does not take into account the interaction of the electrons of the metal with one another, although the latter must be very large and cannot be ignored to any extent whatever. However, since a strict examination of the problem is impossible with the existing state of the theory, the interaction of electrons must at least be examined as a small perturbation. Such an examination at least provides the possibility of elucidating the limits of applicability of the existing theory. At the same time it appears that at low temperatures of the order of a few degrees absolute the electrical resistance produced by the interaction of electrons cannot be considered as small since the effects caused by it are comparable in magnitude to or greater than the effects produced by the interaction of electrons with the thermal vibrations of the lattice. The increase in resistance produced by the interaction of electrons is proportional to T^2 and at low temperatures is greater than the normal resistance which is proportional to T^5 . The relationships existing here are analogous to the relationships existing in the field of thermal capacity of metals where the thermal capacity of the electrons plays an important role (since it is proportional to T , while the Debye specific heat is proportional to T^3) at low temperatures (a few degrees absolute), in spite of the fact that at normal temperatures the electronic thermal capacity may be neglected in comparison with the thermal capacity of the lattice.

As is well known, the state of an electron in a periodic lattice is described by the wave function

$$\psi = e^{i(\mathbf{k} \cdot \mathbf{r})} \varphi(\mathbf{k}, \mathbf{r}),$$

where the components of the vector \mathbf{k} are determined only to the accuracy of the periods of the reciprocal lattice and the function $\varphi(\mathbf{k}, \mathbf{r})$ is a periodic function with periods equal to the periods of the lattice:

$$\varphi(\mathbf{k}, x + d_1, y, z) = \varphi(\mathbf{k}, x, y + d_2, z) = \varphi(\mathbf{k}, x, y, z + d_3) = \varphi(\mathbf{k}, x, y, z)$$

d_1 , d_2 and d_3 are the periods of the lattice.

Л. Ландау и И. Померанчук, О свойствах металлов при очень низких температурах, *Журнал Экспериментальной и Теоретической Физики* 7, 379 (1937).

L. Landau und I. Pomerantschuk, Über die Eigenschaften der Metalle bei sehr niedrigen Temperaturen, *Phys. Z. Sowjet.* 10, 649 (1936).

The vector $\hbar \mathbf{k}$ (the quasi-momentum) plays the role of the momentum of the free particle; $v = 1/\hbar \cdot \partial \varepsilon / \partial k$ is the velocity of the electron, ε is its energy, $\hbar \mathbf{k} = \mathbf{F}$, where \mathbf{F} is the force acting on the electron. Owing to the ambiguity in determining the vector \mathbf{k} the magnitude of the sum of the vectors \mathbf{k} for interacting electrons is conserved only to the accuracy of the periods of the reciprocal lattice since only those transitions are possible for which

$$\sum \mathbf{k}_i - \sum \mathbf{k}^i = 0, \quad \frac{2\pi}{d}, \quad \frac{4\pi}{d}, \dots;$$

\mathbf{k}_i are the vectors in the initial state, \mathbf{k}^i are the vectors in the final state.

Thus, the increase in momentum produced by an electric field can also be nullified as a consequence of the interaction of electrons with one another, and not only with thermal vibrations of the lattice.

Now let us write the conditions for the stationary state for the distribution function of the electrons in the presence of an electric field, for the time being considering only their interaction with one another. We obtain

$$\begin{aligned} -\frac{\partial n_1}{\partial \varepsilon} v e F = \int W_{k_1 k_{11} k' k''} [n_1 n_{11} (1 - n') (1 - n'') \\ - n' n'' (1 - n_1) (1 - n_{11})] d\tau_{11} d\tau'' \end{aligned} \quad (1)$$

Here F is the electric field, n_1 and n_{11} are the densities of electrons in the initial state and n' and n'' are the densities in the final state.

Owing to the laws of conservation (of energy and quasi-momentum) we integrate with respect to the quasi-momenta of the electrons in the states n_{11} and n'' since the state n' is fixed if the states n_1 , n_{11} and n'' are known:

$$\varepsilon' = \varepsilon_1 + \varepsilon_{11} - \varepsilon'', \quad \mathbf{k}' = \mathbf{k}_1 + \mathbf{k}_{11} - \mathbf{k}'', \quad \frac{2\pi m}{d}, \quad m = 0, \pm 1, \dots$$

The transition probability $W_{k_1 k_{11} k' k''}$ is determined in the usual way through the matrix element of the perturbing energy $V_{k_1 k_{11} k' k''}$. Assuming the interaction to be the Coulomb interaction, it may be expected that the region of small angles of "deviation" (i.e. of transitions for which an electron goes from a state with a certain vector \mathbf{k} to a state characterised by the vector \mathbf{k}' which is only slightly different from \mathbf{k}) will play a special role (as is well known, for the Coulomb interaction the probability of scattering is inversely proportional to $\sin^4 \vartheta/2$, where ϑ is the angle of scattering) in a metal; however, this region is in no way remarkable since small angles of deviation mean that the interacting electrons are on average at large distances from one another, as a consequence of which there is always screening by extraneous charges and the interaction will no longer be the Coulomb interaction. Let us now substitute in (1) $n + \delta n$ in place of n , where n is the Fermi equilibrium distribution function, equal to $1/[e^{(\varepsilon - \mu)/kT} + 1]$, δn is the deviation of the distribution function from its equilibrium value, assumed small. Let us put δn equal to

$(\partial n / \partial \varepsilon) \varphi(k)$, where φ is a function to be determined. Since the Fermi function turns the right-hand side of equation (1) into zero, we take on the right-hand side first-order terms, i.e. terms linear in φ . We ignore the derivatives of δn on the left-hand side. Equation (1) then becomes

$$-\frac{\partial n}{\partial \varepsilon} V_1 e F = \int W_{k, k_1, k' k''} \left\{ \varphi_1 \frac{\partial n_1}{\partial \varepsilon} [(1 - n')(1 - n'') n_{11} + n' n'' (1 - n_{11})] \right. \\ \left. + \varphi_{11} \frac{\partial n_{11}}{\partial \varepsilon} [(1 - n')(1 - n'') n_1 + n' n'' (1 - n_1)] - \text{terms where the lower in-} \right. \\ \left. \text{dices are replaced by the upper and vice versa} \right\} d\tau_{11} d\tau'' \quad (2)$$

Since n and $\partial n / \partial \varepsilon$ are functions only of $(\varepsilon - \mu) / k_B T$, δn may be assumed to be a function of $\chi = (\varepsilon - \mu) / k_B T$ and the angles in momentum space. From the variables k_x, k_y, k_z we change to the variables $(\varepsilon - \mu) / k_B T$ and the angles in momentum space. Then in place of (2) we have

$$-\frac{\partial n_1}{\partial \varepsilon} V e F = \int W_{k, k_1, k' k''} \left\{ \right\} \Delta_{11} \Delta'' d\chi_{11} d\chi'' d\omega_{11} d\omega'' (k_B T), \quad (2a)$$

where $\Delta = \partial(k_x k_y k_z) / \partial(\varepsilon \delta \varphi)$ is the Jacobian of the change to the variables $\varepsilon, \vartheta, \varphi$, and $d\omega$ is an element of solid angle. $\{ \}$ indicates the expression contained between the figured brackets in (2). The expression behind the integral sign in (2a) is a function of $\chi_1, \chi_{11}, \chi''$ and $\omega_1, \omega_{11}, \omega''$, the whole integral is a function of χ_1 and ω_1 . If now we substitute in (2a)

$$n_1 = \frac{1}{\exp \left[\frac{\varepsilon_1 - \mu}{k_B T} \right]}$$

and correspondingly in place of n_{11} and n'' their Fermi expressions, we obtain

$$-\frac{\partial n_1}{\partial \varepsilon} V e F = k T \int W_{k, k_1, k' k''} \left\{ \frac{\varphi_1 + \varphi_{11}}{(e^{-\chi_1} + 1)(e^{-\chi_{11}} + 1)(e^{\chi''} + 1)(e^{\chi''} + 1)} \right. \\ \left. - \frac{\varphi' + \varphi''}{(e^{\chi_1} + 1)(e^{\chi_{11}} + 1)(e^{-\chi''} + 1)(e^{-\chi''} + 1)} \right\} \Delta_{11} \Delta'' d\chi_{11} d\chi'' d\omega_{11} d\omega''. \quad (3)$$

In the terms containing φ_1 one can carry out both integration with respect to χ_{11} and to χ'' , but in the terms containing $\varphi_{11}, \varphi', \varphi''$ —only the one integration (in the term containing φ' integration with respect to χ'' is replaced by integration with respect to χ'). At the same time the Jacobian Δ and the probability $W_{k, k_1, k' k''}$ in our approximation may be regarded as independent of energy, i.e. of χ (since the χ are in fact functions varying slowly in comparison

to φ , which changes its order of magnitude for a change in ε from μ to $\mu \pm k_B T$. As a result of the simple but lengthy integration we obtain

$$\begin{aligned} \frac{\partial n_1}{\partial \varepsilon} V e F &= - \frac{e^{\chi_1} V e F}{(1 + e^{\chi_1})^2 k_B T} \\ &= k_B T \int W_{k_1 k_2 k' k''} \left\{ \left[2 \int_0^\infty \frac{\chi d\chi}{e^\chi + 1} + 2 \int_0^\infty \frac{\chi d\chi}{e^\chi - 1} + \frac{\chi^2}{2} \right] \frac{\varphi_1}{(e^{\chi_1} + 1)(e^{-\chi_1} + 1)} \right. \\ &\quad + \int_{-\infty}^\infty \frac{\varphi_{11}(\chi_1 - \chi'') d\chi_{11}}{(e^{-\chi_1} + 1)(e^{-\chi_{11}} + 1)(e^{\chi_1} - e^{\chi_{11}})} - \int_{-\infty}^\infty \frac{\varphi'(\chi_1 - \chi') d\chi'}{(e^{-\chi_1} + 1)(e^{-\chi'} + 1)(e^{\chi_1} - e^{\chi'})} \\ &\quad \left. - \int \frac{\varphi''(\chi_1 - \chi'') d\chi''}{(e^{-\chi_1} + 1)(e^{-\chi''} + 1)(e^{\chi_1} - e^{\chi''})} \right\} \Delta_{11} \Delta'' d\omega_{11} d\omega''. \quad (4) \end{aligned}$$

From this equation it follows in the first place that $\varphi(\chi) = \varphi(-\chi)$, since $\varphi(-\chi)$ satisfies the same equation as $\varphi(\chi)$ but with φ_{11} a function of $-\chi_{11}$, φ' of $-\chi'$ and φ'' of $-\chi''$; in the second place, if equation (4) is re-written in the form

$$\frac{\partial n_1}{\partial \varepsilon} V e F = A \varphi_1 + \int L(\chi_1 \chi) \varphi(\chi) d\chi,$$

then the operator L has a symmetrical kernel:

$$L(\chi_1 \chi) = \frac{\chi_1 - \chi}{(e^{-\chi_1} + 1)(e^{-\chi} + 1)(e^{\chi_1} - e^\chi)} = L(\chi, \chi_1);$$

in the third place, it follows from equation (4) that $\varphi = F g(\chi, \omega)/T^2$, where $g(\chi, \omega)$ is a certain even function of χ .

We find that the current is

$$\begin{aligned} i &= -2e \int V \frac{\partial n}{\partial \varepsilon} \varphi d\tau = -\frac{2e k_B}{T} \int \frac{\partial n}{\partial \varepsilon} V \Delta g(\chi, \omega) d\chi d\omega = \frac{a F}{T^2}, \\ a &= 2e \int \frac{e^\chi g(\chi, \omega)}{(1 + e^\chi)^2} \Delta V d\chi d\omega, \quad \Delta = \frac{\partial(k_x k_y k_z)}{\partial(\varepsilon \varphi)} \end{aligned}$$

Since at low temperatures $\partial n/\partial \varepsilon$ has the form of a δ -function of $\varepsilon - \mu$, i.e. is different from zero only for values of ε close to μ , it is possible in integrating with respect to $\varepsilon(\chi)$ to regard the values of $V \Delta$ as equal to their value at $\varepsilon = \mu$.

In this way the resistance produced by the interaction of electrons with one another is proportional to T^2 .

At temperatures below the Debye temperature but greater than those at which the interaction of electrons is important the resistance is proportional to T^5 (it is determined by thermal vibrations in the lattice). At very low temperatures of the order of a few degrees absolute the resistance is proportional to T^3 . In the intermediate temperature range qualitative agreement should be obtained with experiment if these two laws are combined in a formula of the type

$$R = \alpha T^2 + \beta T^5.$$

The data on the resistance of platinum up to temperatures of the order of 20°K fit well into this formula with coefficients α and β equal to 16×10^{-7} and 3×10^{-7} .

2. The existing theories of thermo-electric phenomena at low temperatures are not satisfactory. Thus, Bethe¹ derives an expression for the thermo-e.m.f. without taking into account the establishment of equilibrium with respect to directions (which proceeds more slowly at low temperatures than that with respect to energies) and therefore obtains a result in direct contradiction to the Thomson-Onsager relations² between the Peltier heat Π , the Thomson heat M and the temperature derivative of the thermo-e.m.f. E :

$$E = \Pi \frac{1}{T}, \quad M = T \frac{dE}{dT}. \quad (5)$$

That these relations are obeyed is guaranteed by the symmetry of the collision operators, which leads to the result that if one writes the electrical (i) and thermal current (s) in the form:

$$i = \alpha_{11} \frac{d(\varphi - \mu/e)}{dz} + \frac{\alpha_{12}}{T} \frac{dT}{dz}, \quad s = \alpha_{21} \frac{d(\varphi - \mu/e)}{dz} + \alpha_{22} \frac{dT}{dz},$$

where z is the co-ordinate, φ is the potential, then

$$\alpha_{12} = \alpha_{21}.$$

This identity also gives the relations (5); in Bethe's case this equality is not observed. Bethe makes use of a distribution function at low temperatures (more precisely, its deviation from the equilibrium Fermi value) of the following form:

$$\delta n = \frac{\partial n}{\partial \varepsilon} \frac{\alpha(\omega)}{T^5} \frac{d(\varphi - \mu/e)}{dz} + \frac{\partial n}{\partial \varepsilon} \left[\frac{U(\chi, \omega)}{T^3} + \frac{g(\chi, \omega)}{T^3} \right] \frac{dT}{dz}$$

α is a function only of the angles in momentum space, U and g are functions of the angles and energies, $U(\chi, \omega)$ is an odd function of $\chi = (\varepsilon - \mu)/k_B T$,

$g(\chi, \omega)$ is an even function of χ . Let us find α_{12} and α_{21} ; $(1/T)\alpha_{12}$ is the current from the terms in δn which are proportional to dT/dz , i. e.

$$\frac{1}{T}\alpha_{12} = -2e \int V_z \frac{\partial n}{\partial \varepsilon} \left[\frac{U(\chi, \omega)}{T^3} + \frac{g(\chi, \omega)}{T^2} \right] d\varepsilon d\omega; \quad \Delta = \frac{\partial(k_x k_y k_z)}{\partial(\varepsilon \partial \varphi)}.$$

In view of the fact that $\partial n/\partial \varepsilon$ has the form of a δ -function of $\varepsilon - \mu$, at low temperatures, i. e., is different from zero for ε close to μ , one can consider that $V_z \Delta = [V_z \Delta]_{\varepsilon=\mu} = \text{const}$ in the integral containing g . In the integral containing $U(\chi, \omega)$ such an approximation gives zero as the function $U(\chi, \omega)$ is odd and therefore in that case we expand $V_z \Delta$ in a series:

$$V_z \Delta = [V_z \Delta]_{\varepsilon=\mu} + \left[\frac{\partial}{\partial \varepsilon} [V_z \Delta] \right]_{\varepsilon=\mu} (\varepsilon - \mu) = [V_z \Delta]_{\varepsilon=\mu} + \chi k_B T \left[\frac{\partial}{\partial \varepsilon} [V_z \Delta] \right]_{\varepsilon=\mu}$$

Owing to this $\alpha_{12}/T = \alpha/T^2$, where

$$\begin{aligned} \alpha &= 2e \int \frac{e^x}{(1+e^x)^2} g(\chi, \omega) d\chi d\omega [V_z \Delta]_{\varepsilon=\mu} \\ &\quad + 2e \left[\frac{\partial}{\partial \varepsilon} [V_z \Delta] \right]_{\varepsilon=\mu} k_B \int \frac{e^x}{(1+e^x)^2} \chi U(\chi, \omega) d\chi d\omega, \end{aligned}$$

where α_{21} is the heat flow from the term in δn which is proportional to the electric field

$$\begin{aligned} \alpha_{21} &= 2 \int V_z (\varepsilon - \mu) \frac{\alpha(\omega)}{T^5} \frac{\partial n}{\partial \varepsilon} d\varepsilon d\omega = -2k_B \int V_z \chi \frac{e^x}{(1+e^x)^2} \frac{\Delta \alpha(\omega)}{T^4} d\chi d\omega \\ &= -2k_B \int \chi \frac{e^x}{(1+e^x)^2} \frac{\alpha(\omega)}{T^4} \left\{ [V_z \Delta]_{\varepsilon=\mu} + \chi k_B T \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu} \right\} d\chi d\omega \\ &= -\frac{2k_B^2}{T^3} \int \chi^2 \frac{e^x}{(1+e^x)^2} \alpha(\omega) d\chi d\omega \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu}. \end{aligned}$$

Thus $\alpha_{21} \sim T^{-3}$, while $\alpha_{12} \sim T^{-1}$.

In examining the terms of the transport equation corresponding to the establishing of equilibrium with respect to direction, Kroll³ neglects quantities of the same order as he retains. Due to this the transfer of heat in a metal in which no electric current is flowing turns out to be impossible. This follows from the fact that for an electric field equal to zero (a condition from which, as is well known, the thermo-e.m.f. is determined) the deviation of the distribution function from its equilibrium Fermi value disappears in Kroll's case, i. e. the distribution function in the presence of a temperature gradient remains a Fermi one, which is incapable, as is well known, of producing heat flow.

To elucidate the problem we shall examine first in a general form the equation for the stationary state in the presence of a temperature gradient. As is well

known, the change in the number of electrons in unit phase space under the influence of interaction with lattice vibrations is equal to

$$\int A_{k_1 k'} \{n(1 - n')(N + 1) - n'(1 - n)N\} df_x df_y df_z + \int B_{k_1 k'} \{n(1 - n'')N - n''(1 - n)(N + 1)\} df_x df_y df_z \quad (6)$$

$A_{k_1 k'}$ and $B_{k_1 k'}$ are the transition probabilities, which differ only in the signs of the δ -functions, viz. $\delta(\varepsilon - \varepsilon' - \hbar \omega)$ enters into $A_{k_1 k'}$ and $\delta(\varepsilon' - \varepsilon - \hbar \omega)$ into $B_{k_1 k'}$. N is the distribution function for phonons, f is the phonon wave vector. The integral containing $A_{k_1 k'}$ takes account of transitions from the state ε to the state ε' lying energetically lower than ε , and the integral containing $B_{k_1 k'}$ takes account of transitions to the state ε'' lying energetically higher than ε . Phonons taking part in transitions of an electron with a given k must satisfy the condition

$$\varepsilon(k + f) = \varepsilon(k) \pm \hbar \omega(f),$$

i.e. must lie on the surface in the momentum space of phonons

$$\varepsilon(k + f) - \varepsilon(k) \mp \hbar \omega(f) = 0.$$

Therefore the number of phonons taking part in the given transition is proportional to an element of this surface, i.e., to the quantity $|f|d|f|$, for low temperatures which excite vibrations of low frequencies for which

$$|f| \sim \omega = \frac{|\varepsilon' - \varepsilon|}{\hbar} = \frac{k_B T |\chi' - \chi|}{\hbar}$$

Finally

$$df_x df_y df_z \sim |f|^2 d|f| d\omega \sim (k_B T)^2 |\chi' - \chi|^2 d\chi' d\omega.$$

The transition probabilities $A_{k_1 k'}$ and $B_{k_1 k'}$ at low phonon frequencies are as is well known, proportional to the frequency, i.e.

$$A_{k_1 k'} = \omega A_{k_1 k'}^0 = \frac{k_B T}{\hbar} |\chi' - \chi| A_{k_1 k'}^0.$$

In place of N in equation (6) we substitute the Planck distribution function

$$\frac{1}{\exp\left(\frac{\hbar \omega}{k_B T}\right) + 1} = \frac{1}{\exp\left[\frac{|\chi' - \chi|}{k_B T}\right] + 1},$$

in place of n we put $n + \delta n$, where n is the Fermi distribution function $(\exp[(\varepsilon - \mu)/k_B T] + 1)^{-1}$, δn being a small function which we shall suppose equal to $\partial n / \partial \varepsilon \cdot \varphi(\chi, \omega)$, where φ is a certain function. In view of the smallness of δn we shall keep only terms linear in δn . Taking account of this, we

re-write equation (6) in the form:

$$\begin{aligned}
 (k_B T)^2 q \int A_{k_1 k'}^0 \Delta' \frac{\varphi(\chi_1 \omega) - \varphi(\chi' \omega')}{(e^{\chi'} + 1)(e^{-\chi} + 1)(e^{|\chi' - \chi|} - 1)} |\chi' - \chi|^2 d\chi' d\omega' \\
 + (k_B T)^2 q \int B_{k_1 k''}^0 \Delta'' \frac{[\varphi(\chi_1 \omega_1) - \varphi(\chi'' \omega'')] \cdot |\chi_1 - \chi''|^2}{(e^{\chi} + 1)(e^{-\chi''} + 1)(e^{|\chi'' - \chi|} - 1)} d\chi'' d\omega'', \quad (7) \\
 \Delta = \frac{\partial(f_x f_y f_z)}{\partial(\varepsilon, \vartheta, \varphi)}
 \end{aligned}$$

q is a numerical factor (into which enters the coefficient of proportionality between frequency and wave vector and so on). In the first integral $\varepsilon' = \varepsilon + \hbar \omega$, in the second $\varepsilon = \varepsilon' + \hbar \omega$. Since at low temperatures the wave vectors of phonons are considerably less than the quasi-momenta of electrons, i.e. $|f| \ll |k|$, the angles ω' (ω'') and ω differ from one another only by a very small amount. Therefore $\varphi(\chi', \omega')$ can be expanded in a power series in $\omega' - \omega$:

$$\varphi(\chi' \omega') = \varphi(\chi' \omega) + \frac{\partial \varphi(\chi' \omega)}{\partial \omega} (\omega' - \omega) + \frac{1}{2} \frac{\partial^2 \varphi(\chi' \omega)}{\partial \omega^2} (\omega' - \omega)^2.$$

As is well known, however, it is not possible to expand in a series with respect to the energies of phonons, since, although $\hbar \omega \ll \varepsilon$, $\varphi(\chi, \omega)$ is a rapidly varying function. $\omega' - \omega$ is proportional to the wave vector of the phonon:

$$|\omega' - \omega| \sim |f| \sim \omega \sim k_B T |\chi' - \chi| / \hbar.$$

On integrating with respect to angles (azimuth) the term containing $\omega' - \omega$ (i.e. f) vanishes (assuming spherical symmetry) and there remains the term containing the square of the wave vector. Equation (7) now becomes

$$\begin{aligned}
 q_1 (k_B T)^2 \left\{ \int \Delta' A_{k_1 k'}^0 \frac{[\varphi(\chi_1 \omega_1) - \varphi(\chi' \omega_1)] |\chi' - \chi_1|^2}{(e^{\chi'} + 1)(e^{-\chi_1} + 1)(e^{|\chi' - \chi_1|} - 1)} d\chi' d\omega' \right. \\
 \left. + q_1 (k_B T)^2 \int \Delta'' B_{k_1 k''}^0 \frac{[\varphi(\chi \omega) - \varphi(\chi'' \omega)] |\chi'' - \chi|^2 d\chi'' d\omega}{(e^{-\chi''} + 1)(e^{\chi} + 1)(e^{|\chi'' - \chi|} - 1)} \right\} \\
 + q_2 (k_B T)^4 \left\{ \int \Delta' A_{k_1 k'}^0 \frac{\frac{\partial^2 \varphi(\chi' \omega)}{\partial \omega^2} |\chi' - \chi|^4}{(e^{\chi} + 1)(e^{-\chi} + 1)(e^{|\chi' - \chi|} - 1)} d\chi' d\omega' \right. \\
 \left. + q_2 (k_B T)^4 \int \Delta'' B_{k_1 k''}^0 \frac{\frac{\partial^2 \varphi(\chi'' \omega)}{\partial \omega^2} |\chi'' - \chi|^4 d\chi'' d\omega}{(e^{-\chi''} + 1)(e^{\chi} + 1)(e^{|\chi'' - \chi|} - 1)} \right\}. \quad (8)
 \end{aligned}$$

This equation may be written as follows:

$$(k_B T)^2 L_1(\varphi) + (k_B T)^4 L_2(\varphi) \quad (9)$$

L_1 is the collision-operator contained in the first figured brackets in (8), L_2 is that in the second. Equation (8) must now be equated to the change in the distribution function under the influence of the temperature gradient, i.e.⁴

$$-\frac{\partial n}{\partial \varepsilon} V_z \chi k_B \frac{dT}{dz}.$$

For the function φ we obtain the equation

$$-\frac{\partial n}{\partial \varepsilon} V_z \chi k_B \frac{dT}{dz} = (k_B T)^2 L_1(\varphi) + (k_B T)^4 L_2(\varphi). \quad (10)$$

In L_1 only those transitions are taken into account for which the direction of the quasi-momentum does not change (i.e. transitions inside a cone in the phase space of electrons). Into L_2 enter those transitions for which the quasi-momentum changes direction. There are insufficient transitions of the first kind to establish equilibrium in the presence of a temperature gradient. To demonstrate this let us integrate both parts of equation (10) with respect to energy (i.e. over a cone in the phase space of electrons). Remembering that

$$d\varepsilon = d\chi \cdot k_B T, \quad \frac{\partial n}{\partial \varepsilon} = -\frac{e^x}{(1 + e^x)^2 k_B T},$$

we have

$$k_B \frac{dT}{dz} \int V_z \frac{-e^x \chi}{(1 + e^x)^2} \Delta_1 d\chi = (k_B T)^3 \int L_1(\varphi) \Delta d\chi + (k_B T)^5 \int \Delta L_2(\varphi) d\chi.$$

But $\int L_1(\varphi) d\chi = 0$, since the number of electrons in the cone cannot change as a result of transitions for which the electron remains in the same cone†.

But the term containing dT/dz is not equal to zero:

$$\begin{aligned} \int \chi V_z \Delta \frac{e^x}{(1 + e^x)^2} d\chi &= \int \chi \frac{e^x}{(1 + e^x)^2} \left\{ (V_z \Delta)_{\varepsilon=\mu} + \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu} \chi k_B T \right\} d\chi \\ &= \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu} k_B T \int \chi^2 \frac{e^x}{(1 + e^x)^2} d\chi \neq 0. \end{aligned}$$

In this way the temperature gradient changes the number of electrons with a given direction of quasi-momentum, while collisions with phonons, which enter into L_1 , do not change it. It is therefore also necessary to consider

† Indeed,

$$\begin{aligned} I &= \int \Delta L_1(\varphi) d\chi = \int \Delta \Delta'' \frac{\{\varphi(\chi) - \varphi(\chi'')\} B_{k_1 k''}^0}{(e^x + 1)(e^{-x''} + 1)(e^{i\chi'' - \chi} - 1)} |\chi'' - \chi|^2 d\chi'' \\ &\quad + \int \Delta \Delta' \frac{\{\varphi(\chi) - \varphi(\chi')\} A_{k_1 k'}^0}{(e^{-x} + 1)(e^{x'} + 1)(e^{i\chi' - \chi} - 1)} |\chi' - \chi|^2 d\chi'. \end{aligned}$$

On interchanging χ and χ' (χ and χ'') I changes sign, remaining equal to itself, i.e. it is equal to zero (under such a transformation $B_{k_1 k''}^0$ becomes $A_{k_1 k'}^0$).

transitions with change in direction of quasi-momentum, although they occur much more slowly than changes for which the direction is conserved.

To solve equation (10) we look for the function $\varphi(\chi, \omega)$ in the form of the sum of a function only of the angles in momentum space and a function of the angles and the energy (i.e. of χ):

$$\varphi = \beta(\omega) + \gamma(\chi, \omega).$$

We assume here that $\gamma < \beta$, so that we shall neglect γ at the same time as β . β does not enter into L_1 since it contains the difference

$$\varphi(\chi, \omega) - \varphi(\chi', \omega) = \gamma(\chi, \omega) - \gamma(\chi', \omega).$$

$\gamma + \beta$ enters into L_2 , but since we suppose $\gamma < \beta$, we only retain β . Equation (10) is re-written thus:

$$\frac{\partial n}{\partial \varepsilon} V_z \chi k_B \frac{dT}{dz} = (k_B T)^2 L_1(\gamma) + (k_B T)^4 L_2(\beta) \quad (11)$$

To solve the equation let us integrate it with respect to ε (i.e. over the cone in momentum space). The integral containing L_1 disappears in this integration, as we have seen, and there remains

$$\left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu} \frac{dT}{dz} k_B^2 T \int \chi^2 \frac{e^\chi}{(1 + e^\chi)^2} d\chi = (k_B T)^5 \int \Delta L_2(\beta) d\chi.$$

From this equation it follows that $\beta(\omega)$ is inversely proportional to the fourth power of the temperature, i.e.

$$\beta(\omega) = \frac{\beta_1(\omega)}{T^4} \frac{dT}{dz}.$$

To determine $\gamma(\chi, \omega)$ we have now the equation

$$-\frac{\partial n}{\partial \varepsilon} V_z k_B \chi \frac{dT}{dz} = (k_B T)^2 L_1(\gamma) + k_B^4 L_2(\beta_1) \frac{dT}{dz}, \quad (12)$$

or

$$\frac{e^\chi V_z k_B \chi}{(1 + e^\chi)^2 k_B T} \frac{dT}{dz} = (k_B T)^2 L_1(\gamma) + k_B^4 L_2(\beta_1) \frac{dT}{dz}. \quad (12a)$$

From this it may be concluded that $\gamma(\chi, \omega)$ consists of two functions, one of which is proportional to T^{-3} , the other to T^{-2} ; since $\beta \cong T^{-4}$, then at low temperatures β is in fact greater than γ .

In what follows we shall require only the expression for the electric current. It is obtained from β , in contrast to the heat flow, already in first approximation (i.e. replacing $\{V_z \Delta\}$ by $\{V_z \Delta\}_{\varepsilon=\mu}$). Therefore we shall not need γ .

Consequently, in the presence of a temperature gradient we obtain δn in the form

$$\delta n = \frac{\partial n}{\partial \varepsilon} \frac{\beta_1(\omega)}{T^4} \frac{dT}{dz}$$

The total deviation of the distribution function from its equilibrium value in the presence of a temperature gradient and an electric field will be†

$$\delta n = \frac{\partial n}{\partial \varepsilon} \frac{\beta_1(\omega)}{T^4} \frac{dT}{dz} + \frac{\partial n}{\partial \varepsilon} \frac{\alpha(\omega)}{T^5} \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz}$$

α is a function of the angles which is different from β_1 . We find for the current

$$\begin{aligned} i &= -2e \int V_z \frac{\partial n}{\partial \varepsilon} \left[\frac{\beta_1(\omega)}{T^4} \frac{dT}{dz} + \frac{\alpha(\omega)}{T^5} \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz} \right] \Delta d\varepsilon d\omega \\ &= \frac{dT}{dz} \frac{2e}{T^4} \int V_z \frac{e^x}{(1+e^x)^2} \beta_1(\omega) \Delta d\chi d\omega + \frac{2e}{T^5} \int V_z \frac{e^x}{(1+e^x)^2} \alpha(\omega) \Delta d\chi d\omega \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz} \\ &= \frac{2e}{T^4} \frac{dT}{dz} (V_z \Delta)_{\varepsilon=\mu} \left[\int \frac{e^x \beta_1(\omega)}{(1+e^x)^2} d\chi d\omega + \frac{1}{T} \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz} \int \frac{e^x}{(1+e^x)^2} \alpha(\omega) d\chi d\omega \right] \\ &= \frac{C_1}{T^4} \frac{dT}{dz} + \frac{C_2}{T^5} \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz} \end{aligned}$$

C_1 and C_2 are constants. From this it follows that $\alpha_{12} \cong T^{-3}$. We find the heat flow from the term $d(\varphi - \mu/e)/dz$. It is equal to

$$\begin{aligned} \int (\varepsilon - \mu) V_z \frac{\partial n}{\partial \varepsilon} \frac{\alpha(\omega)}{T^5} \Delta d\chi d\omega &= -k_B T \int \chi \frac{e^x}{(1+e^x)^2} (V_z \Delta) \frac{\alpha(\omega)}{T^5} d\chi d\omega \\ &= -k_B T \int \frac{e^x \alpha(\omega)}{(1+e^x)^2 T^5} \left\{ (V_z \Delta)_{\varepsilon=\mu} + \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right]_{\varepsilon=\mu} \chi k_B T \right\} d\chi d\omega \\ &= -\frac{1}{T^3} k_B^2 \int \chi^2 \frac{e^x \alpha(\omega)}{(1+e^x)^2} d\chi d\omega \cdot \left[\frac{\partial}{\partial \varepsilon} (V_z \Delta) \right] = \alpha_{21} \end{aligned}$$

$\alpha_{21} \cong T^{-3}$, as is α_{12} . In the presence of an electric field the distribution function consists of two terms:‡

$$\left[\frac{\partial n}{\partial \varepsilon} \frac{\alpha(\omega)}{T^5} + \frac{\partial n}{\partial \varepsilon} g \frac{\chi(\omega)}{T^3} \right] \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz},$$

where $g(\chi, \omega)$ is a function of the energy and the angles, g is an even function of $\chi = (\varepsilon - \mu)/k_B T$. In calculating the heat flow a second approximation must

† The term $\frac{\partial n}{\partial \varepsilon} \frac{\alpha}{T^5} \frac{d}{dz} \left(\varphi - \frac{\mu}{e} \right)$ was found by Bloch.

be taken in both terms, in view of which the heat flows from α and g will be in the ratio of T^{-2} to one another. Therefore at low temperatures the term $g(\chi, \omega)$ need not be considered (just as is done in calculating electrical conductivity).

Putting the current i equal to zero let us now find the derivative of the thermo e.m.f. with respect to temperature:

$$\frac{C_1}{T^4} \frac{dT}{dz} + \frac{C_2}{T^5} \frac{d\left(\varphi - \frac{\mu}{e}\right)}{dz} = 0,$$

$$\boxed{\frac{d\left(\varphi - \frac{\mu}{e}\right)}{dT} = -\frac{C_1}{C_2} T}$$

Thus, at low temperatures the derivative of the thermo-e.m.f. with respect to temperature is proportional to T , as at high temperatures. By virtue of the relationships (5) the Thomson coefficient $M \cong T$, while the Peltier heat must be proportional to T^2 . If $d(\varphi - \mu/e)/dT$ is denoted by E , then at low temperatures the ratio E/T must be constant. The existing experimental data on the thermo-e.m.f. of pure metals (the presence of impurities introduces considerable complications) at low temperatures⁶ confirms this conclusion at temperatures above 5° K (for Sn, Ag, Pt).

Up to the present time only the influence of thermal vibrations in the lattice has been taken into account. However, both electrical resistance and thermo-electric effects are also due to impurities (residual resistance) and the interaction of electrons. The action of these two factors cannot be ignored at low temperatures since at very low temperatures the magnitude of the residual resistance or the resistance due to the interaction of electrons is greater than the resistance due to lattice vibration. In considering the influence of impurities (residual resistance) on the thermo e.m.f. it turns out that the coefficient of proportionality in the relationship $E \cong T$ twice undergoes an alteration, viz. first when the electrical resistances (from impurities and lattice vibration) become equal and secondly at very low temperatures when the thermal resistances due to impurities and lattice vibration become equal. (While the ratio of the electrical resistances is proportional to T^5 , the ratio of the thermal resistances is proportional to T^3 , from which it is evident that the thermal resistances become equal at lower temperatures than the electrical resistances).

The same results are obtained on considering the interaction of electrons, i.e. the relation $E \sim T$ is valid only in those regions of temperature where one of the resistances (the electronic or the normal one) is greater than the other. If the residual resistance is comparable with the magnitude of the electronic resistance the relation $E \cong T$ ceases to be obeyed.

In measuring the thermo e.m.f. use is made of a circuit consisting of two metals with differing temperatures at which a change in the coefficient of

proportionality in the relationship $E \sim T$ takes place. Due to this the observed picture of thermo-e.m.f. is noted for great complexity at temperatures where the influence of impurities and the interaction of electrons make their appearance.

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26. SCATTERING OF LIGHT BY LIGHT

In a recent paper Euler and Kockel¹ have calculated the effective cross-section for the scattering of light by light. The calculation was carried out for the case of small frequencies ($\hbar \omega \ll m c^2$), the frequencies being taken in a frame of reference, where the total momentum of the colliding quanta vanishes.

We have calculated the cross-section for the opposite case of large frequencies ($\hbar \omega \gg m c^2$). For the integral cross section we get an expression of the form:

$$\sigma = a \alpha^4 \left(\frac{c}{\omega} \right)^2,$$

where $\alpha = e^2/\hbar c$ with a constant a , which is difficult to compute. According to Euler and Kockel, for small frequencies σ is proportional to ω^6 . Consequently σ has a maximum value in a region $\hbar \omega \sim m c^2$.

It is also difficult to compute the dependence of the differential cross-section on the angle of scattering. We find that for the small angles the polarisation of the light quanta is not altered. The differential cross-section for small angles is

$$d\sigma = \frac{\alpha^4}{\pi^2} \left(\frac{c}{\omega} \right)^2 \ln^4 \Theta d\theta,$$

Θ being the angle of scattering and $d\theta$ the solid angle. This formula is valid for small angles, but not essentially small compared with $m c^2/\hbar \omega$. In the latter case it is necessary to insert into the logarithm $m c^2/\hbar \omega$ in place of Θ .

The formula has a relative accuracy of $1/\ln \Theta$. The cross section increases with decreasing angles, but not very rapidly, and it is impossible to affirm that this region plays the main role in the integral cross-section.

The detailed calculations will appear elsewhere².

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A. Akhiezer, L. Landau, and I. Pomeranchuk, Scattering of light by light, *Nature*, **138**, 206 (1936).

27. ON THE ORIGIN OF STELLAR ENERGY

It is well known that matter consists of nuclei and electrons. Nevertheless it can be shown that in bodies of very large mass this usual "electronic" state of matter can become unstable. The reason for this lies in the fact that the "electronic" state of matter does not lead to extremely great densities, because at such densities electrons form a Fermi gas having an immense pressure. On the other hand, it is easy to see that matter can go into another state which is much more compressible—the state where all the nuclei and electrons have combined to form neutrons.¹ Even if we assume that neutrons repel each other, this repulsion can become appreciable only at densities of the order of magnitude of nuclear densities, i.e. 10^{14} g/cm³, and the pressure of a Fermi gas consisting of neutrons is much less than an electronic gas of the same density, because of the greater mass of the neutrons.

Therefore in spite of the fact that the "neutronic" state of matter is, in usual conditions, energetically less favourable, since the reaction of neutron formation is strongly endothermic, this state must nevertheless become stable when the mass of the body is large enough. In this case the gravitational energy gained in going over to the neutronic state with its greater density, compensates the losses of internal energy.

It is easy to compute the critical mass of the body for which the "neutronic" state begins to be more stable than the "electronic" state. First of all we must calculate the energy necessary to form one neutron. For instance in the reaction ${}^8_2\text{O} + 8e^- = 16^0_1n$ we find from the mass defects that to form one neutron the energy required is 0.008 mass units or 1.2×10^{-5} erg (7.5 MeV). To transform one gram of matter into neutrons we thus need 7×10^{18} erg/g.

Now we must calculate the gain in gravitational energy. The gravitational energy of the much less dense "electronic" state can, of course, be neglected. Let us assume first of all that the neutronic state has a constant density 10^{14} g/cm³. The gravitational energy of a homogeneous sphere of mass M is then

$$3 \times 10^{-3} M^{5/3} \text{ erg.}$$

For the stability of the neutronic phase we must then have

$$3 \times 10^{-3} M^{5/3} > 7 \times 10^{18} M,$$

or

$$M > 10^{32} g = 0.05 \odot$$

Л. Ландау, Об источниках звездной энергии, *Доклады Академии Наук СССР*, 17, 301 (1937).

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On the other hand, if we assume that the neutrons behave like a Fermi gas we find for the energy

$$4 \times 10^{-22} M^{7/3} \text{ erg}$$

and, hence,

$$M > 1.5 \times 10^{30} \text{ g} = 10^{-3} \odot$$

which critical value is even less than on the first assumption.

When the mass of the body is greater than the critical mass, then in the formation of the "neutronic" phase an enormous amount of energy is liberated, and we see that the conception of a "neutronic" state of matter gives an immediate answer to the question of the sources of stellar energy. The sun during its probable time of radiation (about 2×10^9 years according to general relativity theory) must have emitted something of the order of magnitude of 3×10^{50} erg. The liberation of this amount of energy requires the transition of only about 2 per cent of the mass of the sun (with the assumption of constant density) or even only $3 \times 10^{-3} \odot$ (with the Fermi gas model) into the "neutronic" phase. Even for such a bright star as β -Orionis we find for the mass of the neutronic core only about $0.1 \odot$ (with the Fermi gas model).

Thus we can regard a star as a body which has a neutronic core whose steady growth liberates the energy which maintains the star at its high temperature; the condition at the boundary between the two phases is as usual the equality of chemical potentials. The detailed investigation of such a model should make possible the construction of a consistent theory of stars.

As regards the question of how the initial core is formed, the author has shown in a previous article² that the formation of a core must certainly take place in a body with a mass greater than $1.5 \odot$. In stars with smaller mass the conditions which could make possible the formation of the initial core have yet to be made clear.

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28. ON THE ABSORPTION OF SOUND IN SOLIDS

The absorption of short wavelength sound is investigated. The phenomenon is interpreted as a result of sound quanta and heat quanta colliding. This results in a linear frequency dependence for the absorption coefficients.

The occurrence of sound absorption in solid bodies we interpret as due to collisions between the sound quanta and the Debye heat quanta. This procedure is indeed only justified when the energy and momentum of the Debye quanta can be sufficiently sharply defined that their change by the amount absorbed of the energy and momentum of the sound quantum falls in the region of the quantum mechanical uncertainty. In other words, the free path of the Debye quanta must be large in comparison with the wavelength of the sound waves, i.e. the theory is only justified so long as we do not consider too long sound waves.

At room temperature the boundary in fact lies very high, at a wavelength of 10^{-6} cm, thus far above the experimental possibilities for sound waves. At low temperatures the relative values can be more reasonable. The opposite case, wavelength large compared to the dimensions, will be treated in a future paper†.

The direction of the calculation will here be outlined. A plane sound wave with wave vector k and frequency ω , collides with a Debye heat wave (wave vector k_1 , and frequency ω_1). So long as we work to the approximation of classical elasticity theory, the principle of superposition is valid, and no interchange of energy or momentum occurs. If, however, in the expression for the energy density, the next degree of approximation is taken, and the cubic terms are allowed for and introduced as perturbations, then transitions will come about in which energy and momentum are transferred from the sound wave to the Debye wave, i.e. there is a finite absorption.

In order that, to this approximation (i.e. the inclusion of processes in which only three waves are involved), the energy and momentum of the sound wave may be absorbed by the heat wave, the following conditions must be fulfilled:

$$k_1 + k = k_2, \quad \omega_1 + \omega = \omega_2, \quad (1)$$

where the indices distinguish between the phonon which absorbs and that which results from absorption. It is easily seen that these conditions cannot

† This paper does not seem to have been published (Editor's note).

L. Landau und G. Rumer, Über Schallabsorption in festen Körpern, *Phys. Z. Sowjet.* 11, 18 (1937).

be satisfied by the sound waves and phonons having the same speed of propagation. For from

$$k \geq |k_2| - |k_1|, \quad \omega = \omega_2 - \omega_1$$

we have, since $\omega/k = C$ (velocity of sound).

$$\frac{\omega}{C'} \geq \frac{\omega_2}{C} - \frac{\omega_1}{C},$$

Thus necessarily $C' \leq C$. Since the transverse velocity, C_t , is smaller in solid bodies than the longitudinal velocity, C_l , we see that in the first approximation for our model, only transverse sound waves can be absorbed by longitudinal phonons. For a complete absorption of the longitudinal waves also to come about, more than three waves have to take part in the process, i.e. we have to include the quartic terms in the energy, and make the perturbation calculation to the second order of approximation. Here, however, we are not interested in this, and limit ourselves to the first order perturbation calculation.

1. THE PERTURBATION ENERGY

If we denote the components of the deformation tensor by $w_{\alpha\beta}$ then the density of the elastic energy can be expressed exactly up to the third order components:

$$W = A w_{\alpha\alpha}^2 + B w_{\alpha\beta}^2 + P' w_{\alpha\alpha}^3 + Q' w_{\alpha\alpha} w_{\alpha\beta}^2 + R' w_{\alpha\beta}^3,$$

$$w_{\alpha\alpha} = \text{Tr } w_{\alpha\beta}, \quad w_{\alpha\beta}^2 = \text{Tr } w_{\alpha\gamma} w_{\gamma\beta}, \quad w_{\alpha\beta}^3 = \text{Tr } w_{\alpha\gamma} w_{\gamma\delta} w_{\delta\beta}.$$

If we neglect the cubic terms, we obtain the familiar expression for the elastic energy, from which one can easily see the meaning of the coefficients A and B . For the speed of the sound we have (ρ : density)

$$C_t = \sqrt{\frac{B}{\rho}}, \quad C_l = \sqrt{\frac{2(A+B)}{\rho}}.$$

The coefficients P' , Q' , R' , characterise the degree to which the elasticity deviates from Hooke's law. Since we allow for the cubic terms in the calculation, the deformation tensor must be more exactly considered than in familiar elasticity theory normally happens.

Through a displacement, u , the point P goes over to P_1 . We have thus:

$$dx'_i = dx_i + \frac{\partial u_i}{\partial x_\alpha} dx_\alpha.$$

The square of the line element becomes correspondingly:

$$ds'^2 = ds^2 + 2w_{\alpha\beta} dx^\alpha dx^\beta,$$

where $w_{\alpha\beta}$ is the deformation tensor by definition. We get now in a straightforward manner:

$$w_{\alpha\beta} = \frac{1}{2} \left\{ \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \delta_{mn} \frac{\partial u_m}{\partial x_\alpha} \frac{\partial u_n}{\partial x_\beta} \right\}.$$

This tensor is distinguished from the classical deformation tensor by the third term in the curved brackets. If we allow for this, and in the expression for the elastic energy consider only terms of the third order in the derivations of the displacements, then we obtain the following expression for the energy density:

$$W = \{A u_{\alpha\alpha}^2 + B u_{\alpha\beta}^2\} + \{P u_{\alpha\alpha}^3 + Q u_{\alpha\alpha} u_{\alpha\beta}^2 + R u_{\alpha\beta}^3 + A u_{\alpha\alpha} v_{\alpha\beta}^2 + B u_{\alpha\beta} v_{\beta\gamma} v_{\gamma\alpha}\},$$

where

$$v_{\alpha\beta} = \frac{1}{2} \left[\frac{\partial u_\alpha}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\alpha} \right].$$

The terms in the second curved bracket we consider as a perturbation, which results in an exchange of energy and momentum between the waves which are propagating through the medium.

2. TRANSITION PROBABILITY

We express the heat energy in the Debye harmonic waves by:

$$u(r) = \sum u(k_\alpha, r); \quad u(k_\alpha, r) = e_\alpha (a_\alpha e^{i(k_\alpha \cdot r)} + a_\alpha^* e^{-i(k_\alpha \cdot r)}),$$

where the vector e_α represents the direction of polarisation, a_α the amplitude, and k_α the propagation vector of the corresponding wave. If we denote the number of different oscillations by N_α and consider an initial state, A , in which phonons are present in numbers N_1, N_2 , and where there are N sound waves then a final state, F , corresponding to the absorption of a sound wave has numbers $N_1 - 1, N_2 + 1, N - 1$. The problem is to estimate the probability for such a transition. With this in mind, we set†

$$u_{\alpha\beta} = u_{\alpha\beta}^{(1)} + u_{\alpha\beta}^{(2)} + u_{\alpha\beta}, \quad v_{\alpha\beta} = v_{\alpha\beta}^{(1)} + v_{\alpha\beta}^{(2)} + v_{\alpha\beta},$$

where the two indices refer to the phonons. We obtain for the perturbation energy, the expression

$$\begin{aligned} W = & 6P u_{\alpha\alpha}^{(1)} u_{\alpha\alpha}^{(2)} u_{\alpha\alpha} + 2Q (u_{\alpha\alpha}^{(1)} u_{\alpha\beta}^{(2)} u_{\alpha\beta} + u_{\alpha\alpha} u_{\alpha\beta}^{(1)} u_{\alpha\beta}^{(2)} + u_{\alpha\alpha}^{(2)} u_{\alpha\beta}^{(1)} u_{\alpha\beta}) \\ & + 6R u_{\alpha\beta}^{(1)} u_{\beta\gamma}^{(2)} u_{\gamma\alpha} + 2A (u_{\alpha\alpha}^{(1)} v_{\alpha\beta}^{(2)} v_{\alpha\beta} + u_{\alpha\alpha} u_{\alpha\beta}^{(1)} v_{\alpha\beta}^{(2)} + u_{\alpha\alpha}^{(2)} v_{\alpha\beta}^{(1)} v_{\alpha\beta}) \\ & + 2B (u_{\alpha\beta} v_{\beta\gamma}^{(1)} v_{\gamma\alpha}^{(2)} + u_{\alpha\beta}^{(1)} v_{\beta\gamma}^{(2)} v_{\gamma\alpha} + u_{\alpha\beta}^{(2)} v_{\beta\gamma}^{(1)} v_{\gamma\alpha}). \end{aligned}$$

In order to obtain the probability of the transition which interests us, we quantise the entire wave system, and calculate the corresponding matrix element of the perturbation energy.

† We should actually have to write

$$u_{\alpha\beta} = u_{\alpha\beta}^{(1)} + u_{\alpha\beta}^{(2)} + u_{\alpha\beta}^{(3)};$$

however, we shall reserve the index (3) for the sound wave.

The matrix elements of the displacement amplitudes are known to us from the theory of the harmonic oscillator. Only the following are different from zero:

$$(N|A_x^*|N-1) = \sqrt{\frac{\hbar N}{2m\omega_x}} e^{i\omega_x t}, \quad (N-1|A_x|N) = \sqrt{\frac{\hbar N}{2m\omega_x}} e^{-i\omega_x t}$$

where m denotes the mass of the volume V .

For the matrix elements of the deformation and rotation components, we obtain correspondingly through differentiation:

$$(N-1|u_{x\beta}|N) = \frac{i}{2} (e^\alpha k^\beta + e^\beta k^\alpha) (N-1|a|N),$$

$$(N-1|v_{x\beta}|N) = \frac{i}{2} (e^\alpha k^\beta - e^\beta k^\alpha) (N-1|a|N),$$

where the components of the vectors e and k corresponding to the x , y , or z -axes are denoted by e^α , k^α .

For longitudinal waves $e \parallel k$ and $(N-1|v_{x\beta}|N-1) = 0$. For transverse waves, the matrix element $(N|u_{x\alpha}|N-1)$ vanishes since $(e \cdot k) = 0$. Thus to calculate the matrix elements which are of interest to us, only the following terms of the perturbation energy are relevant:

$$W = 2Q \{ u_{\alpha\alpha}^{(1)} u_{\alpha\beta}^{(2)} u_{x\beta} + u_{\alpha\alpha}^{(2)} u_{\alpha\beta}^{(1)} u_{x\beta} \} + 6R u_{\alpha\beta}^{(1)} u_{\beta\gamma}^{(2)} u_{\gamma\alpha}.$$

We calculate the expressions

$$\left. \begin{aligned} q_1 &= \frac{1}{8} (e_1^\alpha k_1^\alpha + e_1^\alpha k_1^\alpha) (e_2^\alpha k_2^\beta + e_2^\beta k_2^\alpha) (e^\alpha k^\beta + e^\beta k^\alpha), \\ q_2 &= \frac{1}{8} (e_2^\alpha k_2^\alpha + e_2^\alpha k_2^\alpha) (e_1^\beta k_1^\alpha + e_1^\beta k_1^\alpha) (e^\alpha k^\beta + e^\beta k^\alpha), \\ r &= \frac{1}{8} (e_1^\alpha k_1^\beta + e_1^\beta k_1^\alpha) (e_2^\beta k_2^\alpha + e_2^\alpha k_2^\beta) (e^\gamma k^\alpha + e^\alpha k^\gamma), \end{aligned} \right\} \quad \text{where} \quad \left\{ \begin{aligned} e_1 &\parallel k_1, \\ e_2 &\parallel k_2, \\ e &\perp k, \end{aligned} \right.$$

and obtain

$$q_1 = \frac{|k_1|}{|k_2|} (k_2 \cdot k) (k_2 \cdot e), \quad q_2 = \frac{|k_2|}{|k_1|} (k_1 \cdot k) (k_1 \cdot e),$$

$$r = \frac{(k_1 \cdot k_2)}{2|k_1||k_2|} \{ (k_1 \cdot k) (k_2 \cdot e) + (k_2 \cdot k) (k_1 \cdot e) \}.$$

Since we are considering the wave number, $|k|$, of the sound wave to be very small in comparison with the wave numbers of the phonons, we can put approximately

$$k_1 \cong k_2, \quad e_1 \cong e_2,$$

and obtain

$$q_1 \cong q_2 \cong r \cong (k_1 \cdot k) (k_1 \cdot e).$$

The perturbation energy becomes

$$W = i [2Q(q_1 + q_2) + 6Rr] u^{(1)}(\mathbf{k}_1 \cdot \mathbf{r}) u^{(2)}(\mathbf{k}_2 \cdot \mathbf{r}) u(\mathbf{k} \cdot \mathbf{r}),$$

and the required matrix element is obtained as:

$$(N_1 N_2 N | W | N_1 - 1, N_2 + 1, N - 1) = -i(4Q + 6R)(\mathbf{k}_1 \cdot \mathbf{k})(\mathbf{k}_1 \cdot \mathbf{e}) \\ \times \sqrt{\frac{\hbar N_1}{2m\omega_1}} \sqrt{\frac{\hbar(N_2 + 1)}{2m\omega_2}} \sqrt{\frac{\hbar N}{2m\omega}} e^{i(\mathbf{k}_1 + \mathbf{k} - \mathbf{k}_2) \cdot \mathbf{r}}.$$

The matrix element for the energy, H , is obtained through integration over co-ordinate space, and is only different from zero when

$$\mathbf{k}_1 + \mathbf{k} - \mathbf{k}_2 = 0,$$

i.e. momentum is conserved, and we have through integration

$$(N_1 N_2 N | H | N_1 - 1, N_2 + 1, N - 1) \\ = -i(4Q + 6R)(\mathbf{k}_1 \cdot \mathbf{k})(\mathbf{k}_1 \cdot \mathbf{e}) \sqrt{\frac{\hbar N_1}{2m\omega_1}} \sqrt{\frac{\hbar(N_2 + 1)}{2m\omega_2}} \sqrt{\frac{\hbar N}{2m\omega}} V.$$

If we denote by $a(N_1 - 1, N_2 + 1, N - 1)$ the probability amplitude for the final state, then the perturbation calculation gives us the general formula:

$$|a(N_1 - 1, N_2 + 1, N - 1)|^2 \\ = \frac{2\pi}{\hbar^2} |(N_1 N_2 N | H | N_1 - 1, N_2 + 1, N - 1)|^2 \delta(\omega_1 + \omega - \omega_2)$$

Writing ϱ_1 for the number of states for which \mathbf{k}_1 lies in the element of solid angle $d\Omega$ and has a magnitude between $|\mathbf{k}_1|$ and $|\mathbf{k}_1| + d|\mathbf{k}_1|$ then we have

$$\varrho_1 d\mathbf{k}_1 d\Omega = V \frac{k_1^2 d\mathbf{k}_1 d\Omega}{(2\pi)^3}.$$

The total required probability is thus

$$\frac{1}{(2\pi)^3} \int |a(N_1 - 1, N_2 + 1, N - 1)|^2 \varrho_1 V k_1^2 d\mathbf{k}_1 d\Omega.$$

For the reverse process, we have correspondingly:

$$\frac{1}{(2\pi)^3} \int |a(N_1 + 1, N_2 - 1, N + 1)|^2 \varrho_1 V k_1^2 d\mathbf{k}_1 d\Omega.$$

As far as absorption of sound is concerned, the difference between these two is important. This is proportional to:

$$N_1(N_2 + 1) - N_2(N_1 + 1) = N_1 - N_2 = \frac{\partial N_1}{\partial \omega_1} \omega.$$

The total absorption probability thus comes to:

$$\frac{2\pi}{\hbar^2} \frac{V^3}{(2\pi)^3} \frac{2\hbar^3}{8m^3} (4Q + 6R)^2 \int \frac{(\mathbf{k}_1 \cdot \mathbf{k})^2 (\mathbf{k}_1 \cdot \mathbf{e})^2}{\omega_1^2} \frac{\partial N_1}{\partial \omega_1} \delta(\omega_1 + \omega - \omega_2) k_1^2 dk_1 d\Omega.$$

Denoting the integral which occurs by J , we may evaluate it as follows: we introduce polar co-ordinates with the k -direction along the z -axis. Thus

$$(\mathbf{k}_1 \cdot \mathbf{k})^2 = k_1^2 k^2 \cos^2 \theta, \quad (\mathbf{k}_1 \cdot \mathbf{e})^2 = k_1^2 \sin^2 \theta \cos^2 \varphi,$$

$$\omega_1 + \omega - \omega_2 = \omega - (\omega_2 - \omega_1) = \omega - \left(\frac{\partial \omega_1}{\partial k_1} \cdot \mathbf{k} \right) = \omega - k C_l \cos \theta$$

and we have

$$\begin{aligned} J &= \pi \int \frac{\sin^2 \theta \cos^2 \theta k^2 k_1^4}{\omega_1^2} \delta(\omega - k C_l \cos \theta) \frac{dN_1}{d\omega_1} k_1^2 dk_1 d(\cos \theta) \\ &= \pi k^2 \frac{\omega^2}{k^2 C_l^2} \left(1 - \frac{\omega^2}{k^2 C_l^2} \right) \frac{1}{k C_l} \frac{1}{C_l} \int \omega_1^4 \frac{dN_1}{d\omega_1} d\omega_1. \end{aligned}$$

Substituting the Planck value for N_1 ,

$$N_1 = \frac{1}{\exp\left(\frac{\hbar \omega_1}{\Theta}\right) - 1}, \quad \Theta = k_B T,$$

we obtain for the absorption per unit time, the expression

$$\frac{\pi}{80} \frac{(2Q + 6R)^2}{e^3} \left(\frac{C_t}{C_l} \right)^2 \left[1 - \left(\frac{C_t}{C_l} \right)^2 \right] \frac{\hbar k}{C_l^3} \frac{\Theta^2}{\hbar^4}.$$

This expression shows that in the case considered here, the absorption is proportional to the first power of the frequency, in contrast to long waves, for which we know that an ω^2 law must hold.

We have assumed throughout in our calculations that the temperature is below the Debye temperature, i.e. that dispersion effects are unimportant. At higher temperatures, this may not be assumed, and a quantitative calculation becomes impossible. However, it is easy to see that in this case the absorption is proportional to the first power of the temperature.

29. ON THE THEORY OF PHASE TRANSITIONS

PART I

The question of continuous phase transitions (without latent heat) have been investigated from the general thermodynamical point of view. In doing this it becomes clear that such transitions can take place when the symmetry of the lattice changes. There are two possible types of transition, namely: (1) Curie points with a discontinuity in the specific heat, which lie on a curve in the p - T diagram, (2) isolated points in the p - T diagram which lie in a certain way on intersections of curves of normal phase transitions.

UP to the present time, among all phase transitions, Curie points, and so on, only the transition between a liquid and a gas has been fully investigated. It is known that the liquid-gas equilibrium curve in the p - T diagram has an end point, and that a continuous transition between liquid and gas can be realised by going round it. As for transitions between a liquid and a crystal, or between different crystal modifications, the question about them has not been fully clarified. In a number of cases people talk about transitions connected with rotations of molecules; however it is not at all clear how rotations can lead to phase transitions, and in particular to discontinuities in the specific heat.

One even finds strange statements that there is no essential difference at all between liquids and crystals, and that continuous transitions between them are possible. However, liquids differ essentially from crystals in that they are isotropic in contrast to anisotropic crystals. Every transition from a crystal to a liquid or to a crystal of a different symmetry is associated with the disappearance or appearance of some elements of symmetry. But elements of symmetry are either present or absent; no intermediate case is possible. And so continuous transitions (in the sense that transitions between liquid and gas are continuous) connected with changes of the symmetry of the body are absolutely impossible.

Until recently the exact formulation of the very idea of the crystal lattice was lacking. Only quite recently Bethe and Peierls¹ have stressed the role of correlations at infinity in the crystal lattice.

Note that normal phase transitions between liquid and crystal or between different modifications where the state of the body, particularly the energy,

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L. Landau, Zur Theorie der Phasenumwandlungen II, *Phys. Z. Sowjet.* 11, 545 (1937).

changes discontinuously are not the main interest of the present investigation. Our main interest is in such transitions where the state of the body (particularly the energy) changes continuously even although the symmetry changes discontinuously. (See below for details.) We shall call these transitions the continuous ones. Let us emphasise once again that they are not continuous in the sense that transitions between liquid and gas are. At every moment we can tell that we have a body of this or that symmetry.

Usually the approach to this question is made difficult by the use of an idealised model of the lattice in which all atoms are placed in their positions and thermal motion is ignored.

These difficulties can be avoided if a distribution probability $\rho(xyz)$ is used, where $\rho(x, y, z) dx dy dz$ determines the probability for finding an atom in the given volume element of the body. If the body consists of different kinds of atoms then it would be possible to introduce several functions ρ_1, ρ_2, \dots , which would determine the probabilities for each kind of atom. Even in that case it would instead be possible to use only one distribution function. For instance we can determine that function as one which gives the mean

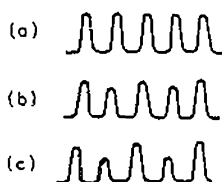


FIG. 1.

charge density at every point of the body (multiplied by $dx dy dz$ it would give the charge in that volume). In the following, we shall talk simply about the "density" $\rho(x, y, z)$, meaning by that some function which determines the distribution of atoms in the body under consideration. Note that such a method based on the function ρ also has the advantage that it is possible in quantum mechanics as well.

The important feature of the function ρ is its symmetry, i.e. that group of co-ordinate transformations with respect to which ρ is invariant. The same group also determines the symmetry of the body. It is known that there are in all 230 possible different groups of transformations, i.e. types of symmetry. In isotropic bodies (liquids) obviously $\rho = \text{const.}$

As already mentioned we shall consider here those transitions where, regardless of a discontinuity in the symmetry, the state of the body changes continuously. In other words the density ρ changes continuously. It is easy to see that such transitions are possible because even a very small change in the distribution of the atoms in the lattice is enough to change its symmetry. If for instance ρ is represented by the curve Fig. 1a (schematically drawn in one dimension) and some of the maxima decrease (Figs. 1b and 1c), then the symmetry changes as soon as the decreasing starts (the translational period of the lattice increases).

Let us consider a crystal with some density ϱ_0 which has a certain symmetry (we shall talk about the totality of symmetry transformations of ϱ_0 as the group ϱ_0). At the transition point the density starts to change and becomes $\varrho = \varrho_0 + \delta\varrho$, where $\delta\varrho$ is small compared with ϱ_0 . $\delta\varrho$ also has some symmetry (group $\delta\varrho$) which is lower than that of ϱ_0 (i.e. not all elements, that is symmetry transformations of ϱ_0 , are elements of symmetry of $\delta\varrho$; the group $\delta\varrho$ is a subgroup of the group ϱ_0). Then $\varrho = \varrho_0 + \delta\varrho$ has the same symmetry, because the sum of two functions has the same symmetry as the less symmetric term. We can therefore neglect the case where $\delta\varrho$ has a higher symmetry than ϱ_0 , since then $\varrho_0 + \delta\varrho$ would have the same symmetry as ϱ_0 , so that no change in the symmetry of the body would take place.

Symmetry transformations from the group ϱ_0 which do not belong to the group $\delta\varrho$ change $\delta\varrho$ into some other function. It is known from group theory that the function $\delta\varrho$ can be broken into a sum of functions the number of which is equal to the number of elements of the group ϱ_0 , in such a way that under every transformation of that group all these functions transform among themselves, i.e. become linear combinations of themselves.

Matrices of these linear transformations form the so-called "representation" of the group ϱ_0 . Further, all these functions into which $\delta\varrho$ is broken, can be separated into groups or "races", where all functions composing them again transformation among themselves. So we can write:

$$\delta\varrho = \sum_n \sum_i c_i^{(n)} \varphi_i^{(n)}, \quad (1)$$

where n is the number of the race and i is the number of the function in the race.

Each of these races of functions can be used as a basis for the representation of the group. That representation is realised by the transformation matrices of the functions of that race. It is known that there exists an expansion of $\delta\varrho$, into $\varphi_i^{(n)}$ where every race consists of the smallest possible number of functions (i.e. an irreducible partition, thus realising the "irreducible representation").

In (1) we shall suppose just such a partition. We could after all simply write it as $\delta\varrho = \sum_n \sum_i \varphi_i^{(n)}$, because the functions the functions $\varphi_i^{(n)}$ are not determined beforehand; in the future it will be convenient to consider the functions $\varphi_i^{(n)}$ somehow normalised.

Among all $\varphi_i^{(n)}$ there is always one function (which forms a "race" by itself) which is invariant with respect to all transformations of the group ϱ_0 . In the sum $\varrho_0 + \delta\varrho$ we shall consider this function to belong to ϱ_0 so that $\delta\varrho$ has no such function.

The thermodynamic potential Φ , of the body, is determined by the density ϱ , i.e. depends on the form of the function ϱ . In other words Φ is a functional of ϱ : $\Phi = \Phi\{\varrho\}$. Φ depends also on the temperature T and the pressure p of the body as parameters. When p and T are given the form of the function ϱ is determined from the condition that Φ should have a minimum.

Let us expand the thermodynamic potential $\Phi\{\varrho_0 + \delta\varrho\}$ in the state with density $\varrho = \varrho_0 + \delta\varrho$ in powers of $\delta\varrho$ (of course this expansion is not a normal power series; individual terms in the expansion are integral operators of $\delta\varrho$). Saying this in another way, we have an expansion in powers of $\varphi_i^{(n)}$ and $c_i^{(n)}$.

It can be seen that first order terms in the expansion are zero. The potential Φ as the quantity which characterises the physical properties of the body obviously should not change under any movements of the body, i.e. should be invariant under all possible co-ordinate transformations. If such a transformation changes ϱ_0 into ϱ'_0 and $\delta\varrho$ into $\delta\varrho'$, then

$$\Phi\{\varrho_0 + \delta\varrho\} = \Phi\{\varrho'_0 + \delta\varrho'\}.$$

From this it can be seen that if Φ is considered as a function only of $\delta\varrho$, then Φ is invariant only with respect to those transformations which do not change ϱ_0 , i.e. the transformation group ϱ_0 . Since the functions $\varphi_i^{(n)}$ under transformations of this group transform among themselves we can consider only the coefficients $c_i^{(n)}$ to change under these transformations, because the expression for Φ should be invariant with respect to transformations of these coefficients. In particular the coefficients of the powers of the $c_i^{(n)}$ in the expression of Φ will be invariants of the relevant degree. It is known that it is impossible to construct linear invariants from quantities transforming as an irreducible representation.

As to the terms of second order, they are known to separate into a sum of groups of terms consisting only of the quantities $c_i^{(n)}$ (consequently of the functions $\varphi_i^{(n)}$) belonging to one race.

The transition point is thus characterised by the fact that for a small change in T and p , an extra term $\delta\varrho$ appears in the density ϱ_0 . On one side of the transition point (which we shall call the "upper" side) terms of second order in the expansion are obviously essentially positive for all T and p . Thus the minimum is at $\delta\varrho = 0$ i.e. the state of the body corresponds to $\varrho = \varrho_0$, that is, the body has a higher symmetry. On the other ("lower") side of the transition point terms of the second order are not essentially positive and thus to the minimum of Φ corresponds some $\delta\varrho$ different from zero, which really determines the symmetry of the body. Consequently at the transition point itself the sum of all terms of the second order should be zero for any fixed $\delta\varrho$.

For that it is obviously sufficient that any group of terms of second order belonging to one race becomes zero at the transition point. On the other hand the $\delta\varrho$ which make the sum zero are just those $\delta\varrho$ which can appear at the transition point.

After the functions $\varphi_i^{(n)}$ belonging to one of the races have been chosen such that the corresponding second order terms are equal to zero, then the rest of the $\varphi_i^{(n)}$ can be taken to be equal to zero. Then $\delta\varrho = \sum_i c_i^{(n)} \varphi_i^{(n)}$ (summation only over functions of one race) is just that change of the density which makes the term of the second order vanish at the transition point, and is consequently physically realised. Therefore in future we shall only be concerned with that one race and shall drop the superscript (n) , specifying the race.

Because the functions φ_i are determined by the condition that they should make the terms of the second order vanish at the transition point, Φ can now be considered as a function only of the c_i , and the expansion in δq as an expansion in c_i where there are no terms of first order. As has been already said, the terms of the second order should form an invariant (with respect to all transitions of the group g_0). In accordance with group theory such an invariant (in an irreducible representation) is a positive definite quadratic form, which, by suitable choice of the normalisation of the c_i , can always be written as the sum of squares. In this way terms of the second order (of the given race) have the form:

$$A \sum c_i^2. \quad (2)$$

At the transition point this expression need not be zero i.e. at that point $A = 0$ (A is of course a function of p and T).

In an analogous way terms of the third, fourth, ..., order are formed correspondingly from invariants of the third, fourth, ..., order. Terms of the third order can in some cases be absent. If, for instance, in a given race only one function φ enters, then by acting with transformations of the group g_0 the coefficient c can change sign. Therefore, in that case, all invariants and consequently all terms of odd orders are equal to zero.

If at a certain point (i.e. at specified p and T) $A(p, T)$ should vanish, then, in order that this point really be a point of a continuous transition, it is necessary that the terms of third order are zero. Otherwise Φ cannot have a minimum (as a function of c_i) at that point, because that point would not correspond to a stable state of the body.

Two cases are possible:

1. Terms of the third order are identically zero (there are no invariants of third order). Transition points are determined from one condition:

$$A(p, T) = 0; \quad (3)$$

besides this terms of the fourth order should be positive definite. In that case transition points lie thus on a certain curve, which is determined by (3). This is the case of Curie points.

A physical state is realised and is determined by the coefficients c_i which correspond to the minimum of Φ (at given p and T). Define

$$\sum c_i^2 = \eta^2 \quad (4)$$

and

$$\frac{c_i}{\eta} = \gamma_i.$$

Then the expansion of Φ is written in the form

$$\Phi = \Phi_0 + A\eta^2 + B(\gamma_i)\eta^4 + \dots,$$

where all coefficients are also functions of p and T .

Because the term of the second order does not depend on γ_i the values of γ_i can be obtained by finding the minimum of $B(\gamma_i)$. Having found these values

and substituting them into $B(\gamma_i)$ we get

$$\Phi = \Phi_0 + A\eta^2 + B\eta^4 + \dots, \quad (5)$$

where $B = B(p, T)$ is the minimum value of $B(\gamma_i)$.

According to the above

$$B(p, T) > 0. \quad (6)$$

Above the Curie point $A > 0$; to the minimum of Φ corresponds $\eta = 0$, i.e. the body has the symmetry ϱ_0 . At the Curie point $A = 0$, and below it $A < 0$. From the minimisation of Φ , i.e. from $\partial \Phi / \partial \eta = 0$, we find

$$A + 2B\eta^2 = 0$$

or

$$\eta^2 = -\frac{A}{2B}. \quad (7)$$

Then

$$\Phi = \Phi_0 - \frac{A^2}{4B}.$$

The specific heat of the body is

$$C = -T \frac{\partial^2 \Phi}{\partial T^2} = C_0 + T \frac{\left(\frac{\partial A}{\partial T} \right)^2}{2B}. \quad (8)$$

Terms which vanish at the Curie point are omitted. C_0 is the specific heat of the body with the symmetry ϱ_0 , i.e. above the Curie point. Because of (8) we see that at the Curie point $C > C_0$. In this way at the Curie point the heat capacity has a discontinuity and it increases in going from a more to a less symmetric body (note, that one body is less symmetric than the other if its symmetry transformation group is a sub-group of the symmetry group of the other).

As was pointed out at the beginning of this case the coefficients γ_i are determined from $B(\gamma_i)$, i.e. they depend on the form of the terms of fourth order.

But all these terms depend also on p and T ; because of that the γ_i depend on p and T too. But the quantities γ_i determine the symmetry of $\delta\varrho$, i.e. the symmetry of the crystal. Because of that it may happen that at different parts of the Curie point curve a transition takes place from a more symmetric crystal (where $\delta\varrho = 0$) to less symmetric crystals of different symmetries (i.e. where $\delta\varrho$ has a different symmetry).

In that case in the phase diagram there is a point of intersection of the Curie curve (curve 1) with the phase transition curve (curve 2, Fig. 2); I is the most symmetric phase ($\delta\varrho = 0$); along curves AB and BC at Curie points it goes over into less symmetric phases II and III, where $\delta\varrho_{II} \neq 0$, $\delta\varrho_{III} \neq 0$.

Symmetry groups $\delta\varrho_{II}$ and $\delta\varrho_{III}$ are sub-groups of the symmetry group of the first phase. However they are not generally sub-groups of each other. Because of this the difference $\delta\varrho_{II} - \delta\varrho_{III}$ cannot become zero; consequently between

phases II and III there should not be a Curie line, but a phase transition line. At the point B all three phases are identical; along the line AB the phases I and II are identical ($\delta\varrho_{II} = 0$); along BC : $\delta\varrho_{III} = 0$.

It can further be shown, that the intersection of one Curie line with another can happen only at a point of the type shown in Fig. 3. If I is the most symmetric phase then the phases II and III have lower symmetries; their symmetry groups are sub-groups of the symmetry group of the phase I. Phase IV has even lower symmetry than II or III. Its symmetry group is simultaneously a subgroup of the symmetry groups of the phases II and III.

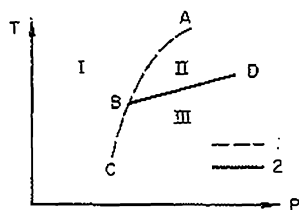
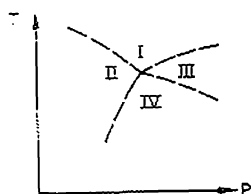


FIG. 2.



* FIG. 3.

Finally, let us consider those cases where terms of fourth order in the expansion of Φ also become zero at the transition point. For this it is necessary that the terms of the fourth order have only *one* coefficient which depends on p and T , together with which they would become zero. Otherwise the vanishing of fourth-order terms together with the condition $A(p, T) = 0$ would give more than two equations with two unknowns (p and T), which would generally have no solutions. For this it is required that only one invariant of the fourth order (formed from the c_i) exists, i.e. the terms of the fourth order are identically equal to $B(p, T) \eta^4$ for arbitrary c_i .

If terms of the fourth order are equal to zero, then for the stability of the state (i.e. for Φ to be a minimum) it is necessary for the term of the fifth order to be identically zero and the term of the sixth order to be positive. Two conditions, $A = B = 0$ then determine an isolated point. That point is a λ -point whose properties have already been investigated by the author[†]. There it has been pointed out that λ -points are the points where the Curie curve goes over into the phase transition curve. Here I shall only consider an additional intersection of the Curie curve with the phase transition curve in bodies which are mixtures of two substances. In that case it appears that the specific heat does not become infinite but, as in pure substances, experiences only a finite jump.

The fact that the body is a mixture does not introduce anything essentially new into our considerations. The symmetry of the crystal is, as before, determined by the density ϱ , and the expansion of Φ in the vicinity of a point of a continuous transition is

$$\Phi = \Phi_0 + A\eta^2 + B\eta^4 + \dots$$

[†] In this earlier paper² the quantity ξ corresponds to η^2 .

but now Φ_0 , A , B depend not only on p and T but on the concentration x of the mixture.

Let us prove that at the transition point of the Curie line into the phase transition line for mixtures (we shall in this case also call such a point a λ -point) the coefficient B in the expansion of Φ should be zero. And indeed from this it will follow that the specific heat does not become infinite at that point (see equation (8)).

Let us investigate the neighbourhood of the λ -point. First we shall write conditions for the equilibrium of two phases on the transition curve (either a phase or a continuous transition). It is known that the thermodynamic potential Φ is an additive quantity and because of that in mixtures it should be a homogeneous function of the first order of the number of particles of each kind. In particular for the mixture of two materials $\Phi = Nf(n/N)$, where n and N are the numbers of both kinds of particles. The chemical potentials of each kind of particles are

$$\frac{\partial \Phi}{\partial N} = f - x \frac{\partial f}{\partial x}, \quad \frac{\partial \Phi}{\partial n} = \frac{\partial f}{\partial x}$$

(where $x = n/N$). The equilibrium conditions are equality of the chemical potentials of both phases. In our case on one side of the transition point (where $\eta = 0$, i.e. in the more symmetric phase) $\Phi = \Phi_0$; on the other side $\Phi = \Phi_0 + A\eta^2 + B\eta^4$. If x_0 and x are the concentrations of both phases then the equilibrium conditions are

$$\frac{\partial \Phi_0}{\partial x_0} = \frac{\partial \Phi}{\partial x}$$

and

$$\Phi_0(x_0) - x_0 \frac{\partial \Phi_0}{\partial x_0} = \Phi - x \frac{\partial \Phi}{\partial x}.$$

Substituting $\Phi = \Phi_0 + A\eta^2 + B\eta^4$, we find from the first condition

$$\frac{\partial \Phi_0}{\partial x_0} = \frac{\partial \Phi_0}{\partial x} + \frac{\partial A}{\partial x} \eta^2$$

($\partial A / \partial x$, is not generally zero at the transition point and because of that it is possible to limit ourselves to the term in η^2) or, expanding $\partial \Phi_0 / \partial x$ in a series:

$$\begin{aligned} \frac{\partial \Phi_0}{\partial x} &= \frac{\partial \Phi_0}{\partial x_0} + (x - x_0) \frac{\partial^2 \Phi_0}{\partial x_0^2} + \dots, \\ - \frac{\partial^2 \Phi_0}{\partial x_0^2} (x - x_0) &= \frac{\partial A}{\partial x} \eta^2. \end{aligned} \quad (9)$$

In the second condition to the same accuracy we put

$$\frac{\partial \Phi_0}{\partial x} \cong \frac{\partial \Phi_0}{\partial x_0}$$

and get

$$\Phi = \Phi_0(x_0) + \frac{\partial \Phi_0}{\partial x_0} (x - x_0).$$

Substituting here the expression for Φ_0 we find

$$A\eta^2 + B\eta^4 = \Phi_0(x_0) - \Phi_0(x) + (x - x_0) \frac{\partial \Phi_0}{\partial x_0}$$

and expanding $\Phi_0(X_0) - \Phi_0(X)$ in a series:

$$A\eta^2 + B\eta^2 = - \frac{(x - x_0)^2}{2} \frac{\partial^2 \Phi_0}{\partial x_0^2}.$$

Further substituting $(x - x_0)$ from equation (9), then

$$A\eta^2 + B\eta^4 = \frac{(x - x_0)}{2} \frac{\partial A}{\partial x} \eta^2,$$

or

$$A + B\eta^2 = \frac{(x - x_0)}{2} \frac{\partial A}{\partial x}. \quad (10)$$

Also remember that one of the conditions for the stability of the state of the body, i.e. the condition that Φ is a minimum, is $\partial \Phi / \partial \eta = 0$ (in that phase where $\eta \neq 0$). From this we get from (7):

$$\eta^3 = - \frac{A}{2B}.$$

Substituting this into (10), we find

$$A - (x - x_0) \frac{\partial A}{\partial x} = 0.$$

Substituting from here $(x - x_0) = A / (\partial A / \partial x)$ and $\eta^2 = - A / 2B$ in equation (9), we find

$$\frac{\frac{\partial^2 \Phi_0}{\partial x_0^2} A}{\frac{\partial A}{\partial x}} = \frac{\partial A}{\partial x} \frac{A}{2B}$$

or

$$B = \frac{\left(\frac{\partial A}{\partial x} \right)^2}{2 \frac{\partial^2 \Phi_0}{\partial x_0^2}}. \quad (11)$$

From this it is obvious that at a 2-point B never becomes zero and that always $B > 0$. The last statement follows from (11) because $\partial^2 \Phi_0 / \partial x_0^2 > 0$

according to the known thermodynamical inequalities for solutions. Further the equation

$$A(x) + (x_0 - x) \frac{\partial A}{\partial x} = 0$$

can be written in the form $A(x_0) = 0$ to the accepted approximation, i.e. the phase transition points for the more symmetric phase satisfy the same equation as the Curie curve.

In this way the neighbourhood of a λ -point has thus for mixtures the form shown in Fig. 4 (plotted along the co-ordinate axes are concentration and temperature). The dotted line is the continuous transition curve, i.e. the Curie curve. I is the more, and II the less symmetric phase. The line 10 goes continuously into the line 03; the line 02 branches away from it. The line 302 is the

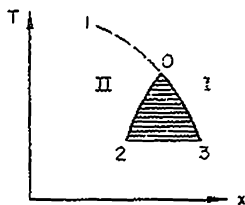


FIG. 4.

phase transition line; the shaded region 302 is the region of separation into two phases I and II, the concentrations of which are determined by the lines 03 and 02.

2. Let the term of third order in the expansion of Φ now not be identically zero. The continuous transition in this case is only possible where terms of the second and third order are equal to zero. The first of these conditions gives again $A(p, T) = 0$. For the second condition to hold it is necessary for only one invariant of the third order to exist, i.e. the terms of third order should only possess one coefficient depending on p and T . Otherwise we would have too many equations which would not be possible to satisfy simultaneously.

Let us again introduce the quantities $\gamma_i = c_i/\eta$.

The term of third order should have the form

$$B(p, T) b(\gamma_i) \eta^3$$

(it is assumed that there is only one invariant of the third order) and the expansion is

$$\Phi = \Phi_0 + A(p, T) \eta^2 + B(p, T) b(\gamma_i) \eta^3 + C(p, T, \gamma) \eta^4 + \dots \quad (12)$$

At a continuous transition point

$$A = B = 0.$$

Consequently the continuous transition points are in this case isolated, i.e. there is no Curie line. Therefore, such points should in some way lie on the

phase transition lines. Accordingly it is necessary to investigate the character of the phase transition line in the vicinity of such points.

In the neighbourhood of a continuous transition point of the type under consideration A and B are close to zero (but $C > 0$). On the equilibrium curves of the more and less symmetric phase their thermodynamic potentials are equal, i.e. $\Phi = \Phi_0$, or

$$A\eta^2 + Bb\eta^3 + C\eta^4 = 0. \quad (13)$$

Besides that $\partial\Phi/\partial\eta$ should be zero, as it should be for all possible equilibrium states, i.e.

$$\eta(2A + 3Bb\eta + 4C\eta^2) = 0. \quad (14)$$

These two equations should have a common solution different from zero (different from zero because the solution $\eta = 0$ would mean that at the transition points $\delta\rho = 0$, i.e. a Curie line would exist and that as has already been mentioned is impossible).

It is easy to see that for this it is necessary that

$$B^2b^2 = 4AC \quad (15)$$

and

$$\eta = -\frac{Bb}{2C}. \quad (16)$$

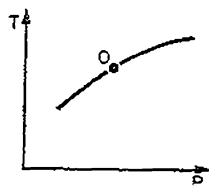


FIG. 5.

It could be thought that the continuous transition points considered simply lie on a phase transition curve like the point 0 in Fig. 5. However that is not so, but instead we shall now show that the point 0 should lie on the intersection of several phase transition curves.

Let us investigate points in the neighbourhood of 0 but not lying on phase transition curves. For them (as in every stable state) $\partial\Phi/\partial\eta = 0$. This equation has solutions $\eta = 0$ and also solutions of the quadratic equation (14).

The solution $\eta = 0$ corresponds to points which represent the state of the more symmetric phase ($\delta\rho = 0$). In the second phase η is determined by equation (14). But quadratic equations have in general two solutions. At the point 0: $A(p, T) = B(p, T) = 0$; in the neighbourhood of the point 0 the equation $B(p, T) = 0$ determines a line. On that line (14) has two solutions with opposite signs

$$\eta = \pm \sqrt{-\frac{A}{2C}}. \quad (17)$$

That means that in the neighbourhood of the point 0 near to the line $B = 0$ (14) has solutions with different signs, almost equal to each other in absolute value (because close to the line $B = 0$, B is small). On one side of the line $B = 0$, B is positive; there the negative solution of (14) corresponds to the stable state, otherwise by changing the sign of η it would be possible to decrease Φ , i.e. Φ would not have a minimum. By the same reason on the other side of the line $B = 0$ (where $B < 0$) the other solution of (14) becomes valid. Consequently the line $B(p, T) = 0$ is also a phase transition line, where η changes sign discontinuously.

In this way the neighbourhood of the point 0 has the appearance shown in Fig. 6, i.e. at the point 0 the other phase transition line ends. The phase I is the more symmetric phase (in it $\eta = 0$, $A > 0$). On the phase transition line AB , $A = 0$. The less symmetric phases II and III (where $A < 0$) have the

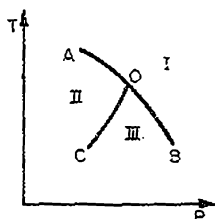


FIG. 6.

same symmetry (in them η differs only in sign, but this does not influence the symmetry of ρ). On the phase line CO , $B(p, T) = 0$. At the point 0 all three phases become identical.

Let us determine the latent heat on the curves CO and AB . For the entropy we have

$$S = - \left(\frac{\partial \Phi}{\partial T} \right)_p = - \left(\frac{\partial \Phi}{\partial T} \right)_{p, \eta} - \left(\frac{\partial \Phi}{\partial \eta} \right)_{p, T} \frac{d\eta}{dT}.$$

But in all stable states $\partial \Phi / \partial \eta = 0$. Therefore

$$S = - \left(\frac{\partial \Phi}{\partial T} \right)_{p, \eta}.$$

Substituting (12), we find in the neighbourhood of the point 0 (i.e. for small η):

$$S = S_0 - \frac{\partial A}{\partial T} \eta^2 \quad (18)$$

$S_0 = -\partial \Phi_0 / \partial T$ is the entropy of the phase I. Terms of higher orders can be neglected because unlike A , $\partial A / \partial T$ does not become zero.

Let us find the latent heat on the curve AB . On it $\eta = -Bb/2c$ (see equation (16)) and the latent heat of transition from the less symmetric to the more symmetric phase is

$$Q = T(S_0 - S) = \frac{\partial A}{\partial T} T \eta^2 = \frac{\partial A}{\partial T} \frac{T b^2}{4 C^2} B^2. \quad (19)$$

Near the point 0 the quantity B is a linear function of the distance along the curve from the point 0 (because at the point 0, $B = 0$). In this way on the curve AB , near the point 0, the latent heat is proportional to the square of the distance from 0.

In order to find Q on the curve OC close to 0, write down the next term in the entropy S :

$$S = - \left(\frac{\partial \Phi}{\partial T} \right)_{p, \eta} = S_0 - \frac{\partial A}{\partial T} \eta^2 - \frac{\partial B}{\partial T} b \eta^3. \quad (20)$$

Since on the curve CO the quantity η is equal in absolute value in both phases, then the difference in entropy between phases II and III is $2b \eta^3 \partial B / \partial T$, where η is determined from (17). The latent heat is

$$Q = 2T \frac{\partial B}{\partial T} b \eta^3. \quad (21)$$

From (21) and (17) it can be seen that Q is proportional to $(-A)^{3/2}$, i.e. proportional to the distance from 0 to the power $3/2$.

Finally, it can be shown that when terms of fourth order have a complex structure new phase transition lines can appear. The neighbourhood of the point 0 then does not look as shown in Fig. 6, but as in Fig. 7.

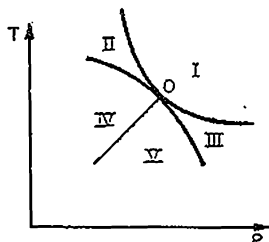


FIG. 7.

Phase I has the highest symmetry. Phases II and III have the same symmetry; the same applies to phases IV and V. At the point 0 all phases become identical, that is indeed the point of continuous transition. At the point 0 two of the phase transition curves have a common tangent and the third ends. Here we have assumed that two curves of phase transitions touch at the point 0. In the general case there may be several of them.

In part II of this paper it will be shown that in the case of transitions between liquids (i.e. isotropic bodies) and crystals terms of the third order are not identically zero. Therefore continuous transitions between liquids and crystals are only possible at isolated points of the type shown in Figs. 6 and 7. In particular Curie lines are impossible.

In the whole of the preceding part of the paper we have assumed that the symmetry properties of crystals are determined by the symmetry of the mean density function ρ . But the moving charges (electrons) in the body can create

in the crystal a mean current density j as well. Then the properties of the crystal will depend not only on the symmetry of the density ρ but also on the symmetry of j . Note that $\int j \, dV$ over the whole volume of the crystal should be equal to zero. Otherwise that current would create a magnetic field and the crystal would possess some magnetic energy. That energy would very rapidly increase with an increase in the dimensions of the crystal and this would be energetically disadvantageous.

In the majority of bodies $j = 0$. In particular $j \neq 0$ in ferromagnetic bodies. In the latter, in addition to this, the magnetic moment is not equal to zero in every part, i.e. $\int [r \wedge j] \, dV \neq 0$ over an elementary cell. However not every body with $j \neq 0$ is ferromagnetic, because although $j \neq 0$, $\int [r \wedge j] \, dV$ can be zero.

If $j = 0$, then the symmetry properties of the crystal are determined by the density ρ . It is known that there exists a limit to the number (230) of possible types of symmetry, i.e. space groups. If besides that $j \neq 0$ then the classification of the types of symmetry follows from the properties of ρ and j ; then it is possible for there to be more than 230 space groups.

The presence of $j \neq 0$ (crystals with $j \neq 0$ we can call magnetic) does not introduce anything essentially new into the preceding discussion about transition points. At transition points the change in symmetry is then determined by $\delta\rho$ and δj . As before only the transition points discussed above are possible.

Let us concentrate for a while on transitions connected with the appearance (or disappearance) of j , i.e. on transitions between magnetic and non-magnetic crystals. Since on one side of these points $j = 0$, then $\delta j = j$. As before we shall consider only the continuous transition points of this type, i.e. points where $\delta j = j = 0$, in the neighbourhood of which (on one side) j is small. Instead of expanding the thermodynamic potential Φ in powers of $\delta\rho$ we shall now have an analogous expansion in powers of j . In view of the symmetry of all the properties of the body in relation to the exchange of the future with the past the potential Φ , in particular, cannot change when the sign of time is reversed. When such a change is made the density ρ does not change, but the current j has its sign reversed. From this it follows that in the expansion of Φ in powers of j all terms with odd powers of j should be identically zero. It means that transitions connected with the appearance of j always belong to the case 1, i.e. Curie points are possible which form Curie lines, and under suitable conditions λ -points also. Such are the Curie points in ferromagnetic bodies. The discontinuities in the specific heats in chlorides of Fe, Cr, Ni at low temperatures are apparently of the same nature, there is also a λ -point in MnO. All these materials have $j \neq 0$ below the transition point, and at the transition point j becomes zero (above that point j remains equal to zero).

Until now we have been talking about transitions with a change in the symmetry of the crystal, but we have not discussed the physical nature of such changes which take place. Atoms in a crystal usually perform small oscillations about their equilibrium positions, i.e. the lattice points. In view of their smallness these oscillations cannot cause changes in the lattice symmetry.

This does not apply, of course, to the jump-like transitions when the atoms start oscillating around new equilibrium positions.

The continuous transitions with a change in the symmetry are always connected with a change in the ordering of the crystals, which follows when the number of places in the lattice where atoms of a given kind can reside is larger than the number of such atoms. There exists one particular distribution of the atoms in the lattice which is energetically most favourable. This is realised at sufficiently low temperatures. At higher temperatures the distribution of atoms deviates from this. As an example, let us consider a crystal formed from two kinds of atoms (binary mixture). The ideal configuration is that in which the atoms of different kinds are placed at lattice points in a definite order one relative to another (this is schematically shown in Fig. 8).

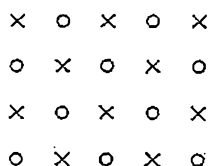


FIG. 8.

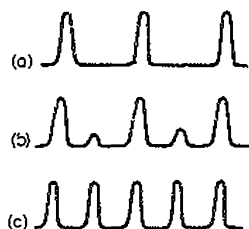


FIG. 9.

Such a crystal is said to be completely ordered. But every atom can in principle be found at any lattice point, i.e. there are more possible places for atoms of a given kind than there are atoms of that kind. Therefore, the crystal can also be incompletely ordered if some atoms are in "foreign" places, i.e. places at which, in the completely ordered crystal, should be atoms of the other kind. The probability, i.e. the density function ρ of finding atoms of one kind at lattice points in the completely ordered crystal can be represented schematically (in one dimension) by the curve in Fig. 9a, where the probability has a sharp maxima at every second lattice point. In the incompletely ordered crystal there appears some probability of finding atoms of a given kind at other (foreign) lattice points (Fig. 9b).

Finally the number of atoms of a given kind in the lattice residing at "foreign" points can be equal to the number of these atoms residing at their "own" places. This means that the probability of finding atoms of a given kind becomes equal at all lattice points (Fig. 9c). The crystal is then called disordered. It is easy to see that at the moment when this disorder appears the symmetry of the crystal changes (namely: the symmetry increases). That can be seen, for instance, in Fig. 9c; the curve c has, in comparison with curves a and b, an extra translational period equal to the distance between two neighbouring lattice points (the curves a and b have only a period equal to twice the distance between lattice points).

A second example is the crystal of NH_4Cl . This crystal has a lattice of the type NaCl, where at the lattice points are Cl and NH_4 . The NH_4 groups have the form of tetrahedra and in the NH_4Cl crystal they can be orientated in

two directions. If all NH_4 groups are pointing in the same direction the crystal is completely ordered; if some of the groups NH_4 are pointing in the opposite direction the crystal is incompletely ordered. Finally if the numbers of NH_4 groups pointing in each direction are equal the crystal is disordered. Its symmetry has then changed, namely: the ordered crystal has the symmetry of a tetrahedron and the disordered the symmetry of a cube.

We can introduce the quantity "degree of order", which would characterise the deviation of the crystal from its ordered state; it is equal to 1 in the completely ordered crystal, decreases as a function of the deviation from the ordered state, and becomes zero in the disordered crystal. In our preceding discussions the transition from the function ϱ to $\varrho + \delta\varrho$ corresponded to the continuous transition from the more to the less symmetric body, i.e. from the disordered crystal to the appearance of the beginning of orderliness. In this way $\delta\varrho$

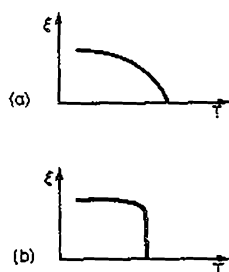


FIG. 10.

just determines how close the crystal is to complete disorder; $\delta\varrho = 0$ in the disordered crystal. But we have seen that $\delta\varrho$ is determined by the quantities c_i which are moreover proportional to η . Obviously η can be chosen as the degree of order. In the above mentioned paper² we used as the degree of order always the positive quantity $\xi = \eta^2$.

At a continuous transition (for instance at a Curie point) ξ as a function of T has the form as shown in Fig. 10a. At the phase transition it becomes zero abruptly (Fig. 10b).

In the case of a binary mixture discussed above, the degree of order can be chosen in the following way. Let N_1 be the number of atoms of a given kind residing at their places, and N_2 at foreign places. In a disordered crystal $N_1 = N_2$. The probability of finding an atom in its place is proportional to $N_1/(N_1 + N_2)$, and in a foreign place $N_2/(N_1 + N_2)$. In a disordered crystal each of those fractions is equal to $1/2$. Therefore the deviations of the probabilities from their values in the disordered crystal are proportional to

$$\frac{N_1}{N_1 + N_2} - \frac{1}{2} = \frac{N_1 - N_2}{2(N_1 + N_2)}, \quad \frac{N_2}{N_1 + N_2} - \frac{1}{2} = \frac{N_2 - N_1}{2(N_1 + N_2)}.$$

In this way $\delta\varrho$ is proportional to the quotient $(N_1 - N_2)/(N_1 + N_2)$, which can indeed be chosen to be η .

In the case of transitions between magnetic and non-magnetic crystals atoms with differently orientated magnetic moments play the role of atoms of different

kinds. To a disordered crystal corresponds the case where the probabilities for an atom to have differently orientated moments are equal for every atom. In the case of the ferromagnetic state these probabilities cease to be equal, since the crystal as a whole has a magnetic moment. Finally in the case of magnetic, but not ferromagnetic, bodies the probabilities for different orientations of the moment for a single atom are also not equal, but in different atoms of the lattice the opposite orientation of the moments are more probable. In that way in this case the mean magnetic moments of different atoms have the opposite directions and the crystal as a whole does not have a magnetic moment.

CONCLUSIONS

1. The transitions between bodies of different symmetry (in particular between a liquid and a crystal) cannot happen continuously, in the same sense as the transition between a liquid and a gas above the critical point; at every moment the body has this or that symmetry.

2. Besides phase transitions the only other possible transitions are those continuous in the sense that at the transition point no abrupt change in the state of the body occurs (in particular there is no latent heat), but the symmetry changes suddenly. Such transitions are inevitably followed by a jump in the specific heat. These transitions are connected with a crystal becoming disordered.

3. The following types of continuous transitions with a change of symmetry are possible: (a) Curie points lying on a curve in the (p , T) diagram. These curves can intersect each other or the phase transition line in points of the kind shown in Figs. 2 and 3. The Curie line can go continuously into a the phase transition line. The point where this happens is a λ -point. At the λ -point of a pure substance the specific heat becomes infinite; if the body is a mixture the specific heat only experiences a finite jump. (b) Isolated continuous transition points. These points lie on the intersections of several phase transition lines (Figs. 6 and 7).

4. Continuous transitions are possible which are connected with the appearance or disappearance of the mean magnetic moments of every atom in the crystal (in particular such is the Curie point of ferromagnetic bodies). For such transitions case a) is appropriate.

PART II

The impossibility of the existence of crystals with a density function which depends only on one or two co-ordinates is proved. The question of transitions between a liquid and a crystal is discussed and it is shown that between them there cannot exist Curie points lying on a curve in the p - T diagram. The question of the nature of liquid crystals is investigated.

In part I† the question of transitions connected with a change in the symmetry of the body has been discussed from a general point of view. In this part we shall investigate the question of the relation between different states of matter from the same point of view.

† Referred to as I in the following.

1. THE IMPOSSIBILITY OF THE EXISTENCE OF CRYSTALS WITH DENSITY ϱ , WHICH DEPENDS ON ONE OR TWO CO-ORDINATES

The density function ϱ of a crystal (see I) is a function of all three co-ordinates x, y, z . The question arises, is the case possible where ϱ is a function of only two, or even one, variable. A crystal with $\varrho = \varrho(x)$ could be considered as consisting of atoms arranged in the form of parallel straight rows, where these rows are all equally orientated along the x -axis, but completely randomly placed with respect to each other. A crystal $\varrho = \varrho(x, y)$ should some how consist of parallel planes. In each of these planes atoms are distributed in a certain order; however the positions of these planes are undetermined.

Let us show that states of matter with density ϱ depending only on one or two co-ordinates are impossible. We shall use a method applied by Peierls¹ to two-dimensional bodies. In particular let us determine the fluctuations in such a body.

Consider some deformation. Such a deformation is characterised by a displacement vector $u(x, y, z)$ with the components $u_i(x, y, z)$ at every point x, y, z of the body. The energy (more specifically the free energy) of the deformed body is determined, as it is known, by the deformation tensor u_{ik}

$$u_{ik} = \frac{i}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right).$$

The change ΔF in the free energy per unit volume element of the body is known to be generally a quadratic function of all components of the tensor u_{ik} . The change in free energy of the whole body is $V \Delta F$, where V is the volume of the body.

The displacement vector u can be expanded in plane waves

$$u_i = \sum_f u_i^{(f)}, \quad u_i^{(f)} = \alpha_i^{(f)} e^{i(f \cdot r)}; \quad (1)$$

obviously the fluctuation is

$$\overline{u_i^2} = \sum_f \overline{u_i^{(f)2}}. \quad (2)$$

The tensor u_{ik} which corresponds to a certain plane wave $u_i^{(f)}$ is obviously proportional to the product of components $u^{(f)}$ with components of the wave vector f [$u_{ik}^{(f)} = i(f_i u_k + u_i f_k)/2$]. The quadratic function ΔF breaks into a sum of terms each of them depending only on $u_i^{(f)}$ of one f .

Let us consider a body with density $\varrho = \varrho(x)$. Then it is easy to see that the free energy of the deformation in such a body does not depend on u_{xy} and u_{xz} . As a matter of fact these deformations exhibit themselves as nothing more than a displacement along the y and z directions. But in these directions

$\varrho = \text{const}$, and therefore such a displacement is not connected with a change in ϱ , consequently neither with a change in F . Let us determine the mean value of $u_x^{(f)2}$. For this determine the part of the free energy which depends on $u_x^{(f)}$. Since $u_{xy}^{(f)}$ and $u_{xz}^{(f)}$ do not enter ΔF then the part of ΔF we are looking for depends only on $u_{xx}^{(f)}$, i.e. on

$$u_{xx}^{(f)} = i f_x u_x^{(f)};$$

consequently ΔF has the form

$$\Delta F = A u_x^{(f)2} f_x^2.$$

Since the probability of fluctuation is to $\exp(-\Delta F \cdot V/kT)$, it is easy to see that

$$\overline{u_x^{(f)2}} \sim \frac{kT}{A V f_x^2}. \quad (3)$$

In order to find the fluctuation of the displacement $\overline{u_x^2}$ it is necessary to sum (3) over all characteristic frequencies. It is known that this summation can be replaced by an integration. For this (3) should be multiplied by the Debye distribution of characteristic oscillations, i.e. by $V df_x df_y df_z$ and integrated over the range zero to the value of f , corresponding to the limiting Debye frequency. In this way

$$\overline{u_x^2} \sim \frac{kT}{A} \int \frac{df_x df_y df_z}{f_x^2}. \quad (4)$$

But this integral diverges like $1/f_x$ when $f_x = 0$. So in this case the fluctuation is infinite.

But the infinite fluctuation results in the fact that the point to which a given value of the function $\varrho(x)$ corresponds can be placed within an arbitrary large distance; in other words the density $\varrho(x)$ "spreads" over the whole body. Saying this differently: no $\varrho(x)$ except $\varrho = \text{const}$ is possible.

If in a crystal $\varrho = \varrho(x, y)$, then in analogy to the above it can be shown that

$$\overline{u_x^2} \sim kT \int \frac{df_x df_y df_z}{\varphi_{II}(f_x, f_y)}, \quad (5)$$

where φ_{II} is a quadratic function of f_x and f_y , and similarly for $\overline{u_y^2}$.

This integral diverges logarithmically when $f_x = 0$, $f_y = 0$. So in this case also the fluctuation is infinite and therefore such crystals cannot exist.

In the case of $\varrho = \varrho(x, y, z)$ it is easy to see that

$$\overline{u_x^2} \sim kT \int \frac{df_x df_y df_z}{\varphi_{III}(f_x, f_y, f_z)} \quad (6)$$

and similarly for $\overline{u_y^2}$ and $\overline{u_z^2}$. This integral is obviously finite.

2. THE TRANSITION BETWEEN A LIQUID AND A CRYSTAL

Let us consider the possibility of transition from a liquid, i.e. an isotropic body, into a crystal, the continuous transition which was discussed in I (without latent heat).

Let ϱ_0 be the density function of the liquid. Since the liquid is isotropic $\varrho_0 = \text{const.}$ At the point of continuous transition ϱ becomes $\varrho = \varrho_0 + \delta\varrho$, where $\delta\varrho$ (and thus also ϱ) have the symmetry of the crystal. Expand $\delta\varrho$ in plane waves

$$\delta\varrho = \sum_f a_f e^{i(f \cdot r)}. \quad (7)$$

Here f are the reciprocal lattice vectors of the crystal. Since ϱ is real we should have

$$a_f = a_{-f}^*, \quad (8)$$

where the symbol $*$ signifies complex conjugate.

The thermodynamic potential φ of the crystal is a functional of ϱ or, what is the same, a functional of $\delta\varrho$ (see I). If for $\delta\varrho$ we substitute expression (7), then φ will be a function of the coefficients a_f . Near the transition point φ can be expanded in powers of a_f . Different terms of this expansion have the form

$$a_{f_1} a_{f_2} a_{f_3} \dots$$

It is easy to see that in the expansion of φ the only terms which can exist are those for which $f_1 + f_2 + \dots = 0$. Actually φ should not change under a translation of the origin of co-ordinates, i.e. under the change of r into $r + R$, where R is an arbitrary constant vector. But under such an exchange a_f is multiplied by $e^{i(f \cdot R)}$, and the expression $a_{f_1} a_{f_2} \dots$ by $e^{i(f_1 + f_2 + \dots \cdot R)}$. This factor is equal to 1 for all values of R only if $f_1 + f_2 + \dots = 0$.

From $\sum f_i = 0$, we have for terms of the first order $f = 0$, i.e. in the expansion of φ there are no terms of first order at all (see also I). Terms of second order should contain only products $a_{-f} a_f$ or according to (8) $|a_f|^2$. The expansion of φ consequently has the form

$$\varphi = \varphi_0 + \sum_f A_f |a_f|^2$$

(φ_0 is the thermodynamic potential of the liquid, the A_f are constants which generally depend on pressure p and temperature T as well as f). Because of the isotropy of the liquid we can conclude that the quantities A_f depend only on the magnitude, but not on the direction, of vector f .

Above the transition point φ has a minimum for all A_f , i.e. all A_f are positive. At a (continuous) transition point the second-order term should become zero for $\delta\varrho$ different from zero (see I). From this it follows that at the transition point one of the A_f should become zero, i.e. the curve $A(f)$ touches the axis of the abscissae at the transition point (Fig. 11).

Touching at two points at the same time is highly improbable therefore at the transition point only one of the coefficients A_f becomes zero. From this it follows that, at the transition a point $\delta\varrho$ arises which corresponds to plane

waves with one definite wavelength, that wavelength which is determined by those values of f which correspond to the vanishing coefficient A_f . All a_f corresponding to other f are equal to zero. Designating that particular coefficient A_f simply by A , we have

$$\varphi = \varphi_0 + A \sum |a_f|^2, \quad (9)$$

where the summation is over f which differ only in their direction.

The terms of the third order have the form

$$\sum_{f_1, f_2, f_3} B_{f_1, f_2, f_3} a_{f_1} a_{f_2} a_{f_3},$$

where in every term $f_1 + f_2 + f_3 = 0$. But as has just been shown, at the transition point there arise periods which all have the same absolute magnitude. Therefore in the third-order terms also only those f_1, f_2, f_3 , which differ only in direction take part. The condition $f_1 + f_2 + f_3 = 0$ means therefore that the vectors f_1, f_2, f_3 should form an equilateral triangle. In all third-order terms these triangles have equal size (because the quantity f is determined by the second order term) and differ only in their orientation in space. Because of the isotropy of the liquid the coefficients B_{f_1, f_2, f_3} can depend only on the size, but not on the orientations, of these triangles. Therefore all B_{f_1, f_2, f_3} in the third-order terms are equal; their common value we shall denote by B . In this way the term of the third order has the form

$$B \sum a_{f_1} a_{f_2} a_{f_3},$$

where the summation is over f_1, f_2, f_3 , which form equal but differently oriented equilateral triangles. Adding this to (9) we have

$$\varphi = \varphi_0 + A(p, T) \sum |a_f|^2 + B(p, T) \sum a_{f_1} a_{f_2} a_{f_3} + \dots \quad (10)$$

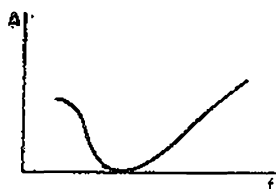


FIG. 11.

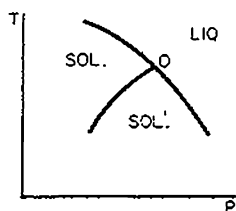


FIG. 12.

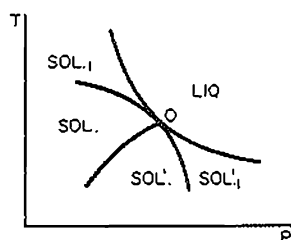


FIG. 13.

We see that the third-order term has only one coefficient $B(p, T)$. In other words we are dealing with the case analysed in I under case II. It means that between liquids and solid crystals there cannot be Curie points forming a line in the p - T diagram, but continuous transitions are possible in isolated points which lie on intersections of ordinary phase transition lines, such as shown in Fig. 12 or in a more complicated case in Fig. 13. In those diagrams the point O is the point of continuous transition; Liq. indicates the liquid phase;

Sol., Sol', Sol.₁, Sol'₁ are different solid phases. Sol., Sol' (in Fig. 12 the phases Sol., Sol', on the one hand and Sol'₁, Sol.₁, on the other hand) have, in the vicinity of the point 0, equal symmetry and differ only in the sign of $\delta\rho$, i.e. they have ρ equal $\rho = \rho_0 + \delta\rho$ (see I). In other words their lattices are such that in those places where in Sol. the probability of finding an atom has a maximum in the lattice Sol' it has a minimum, and vice versa. At the present time it is difficult to say to what extent such continuous transitions exist in nature.

3. LIQUID CRYSTALS

One often finds the opinion that liquid crystals represent bodies in which the molecules are arranged in "chains", orientated in one direction, i.e. bodies in which ρ is a function of one variable. However, it has been shown in section 1, that such bodies cannot exist.

Instead, we can imagine liquid crystals as bodies in which the molecules, or more precisely their centres of mass, are distributed completely randomly, as in ordinary liquids. Anisotropy of the liquid crystal is caused by the equal orientation of its molecules; for instance, if the molecules have an elongated shape, then all of them can be arranged with their axes in one direction.

These ideas about the nature of liquid crystals can be formulated more precisely with the help of the density function.

If the body is isotropic, then $\rho = \text{const}$; however, from $\rho = \text{const}$ it does not follow that the body should necessarily be isotropic. If $\rho = \text{const}$, then this means that all positions of an atom, more precisely its centre of mass, in the body are equally probable. Nevertheless in this case different orientations in the body can be non-equivalent. Namely, when the position of any particular atom No. 1 is given, then the probability of different positions of a neighbouring atom No. 2 is a function of their relative positions (i.e. of the vector r_{12} connecting atom No. 1 and 2). This probability ρ_{12} can depend on the direction of r_{12} . Then the body will be anisotropic regardless of the fact that for every atom $\rho = \text{const}$. On the other hand, when this is so, the body will be a liquid since in it no displacement deformation is possible. If $\rho = \text{const}$ then under any deformation, without a change of volume, ρ does not change, i.e. strictly speaking there is no deformation.

Such bodies ($\rho = \text{const}$, ρ_{12} depends on the orientation) we can consider as liquid crystals. Thus we can talk about the symmetry of liquid crystals as the symmetry of the function ρ_{12} . But ρ_{12} is a function of the vector r_{12} ; when the length of r_{12} is changed without changing its direction, then ρ_{12} does not exhibit any periodicity (when $r_{12} \rightarrow \infty$, ρ_{12} obviously tends to ρ^2). In other words ρ_{12} has no translational symmetry. Therefore the possible symmetry groups of ρ_{12} , i.e. of liquid crystals, are not the 230 space groups, but point groups. Of course the number of these groups is not limited to 32 as in solid crystals; the symmetry of liquid crystals should be classified in the same way as the symmetry of molecules. In particular, symmetry axes of any (and not only of the second, third, fourth and sixth) order are possible. In particular, liquid crystals are possible with total axial symmetry. It is experimentally

known that certain liquid crystals are uniaxial. It would be very interesting to establish whether they possess total axial symmetry or simply have axes of higher than second order.

In principle, liquid crystals with cubic symmetry are possible. Such crystals are impossible to distinguish from ordinary crystals in their optical properties. It is possible that liquid He II is such a crystal. (He II does not exhibit double refraction)³.

If all particles considered are the same then ϱ_{12} obviously has a centre of symmetry. Actually, if on the left of atom No. 1 we have atom No. 2, then standing at the position of atom No. 2 we will have atom No. 1 on the right; in view of the equality of these atoms we conclude that the values of ϱ_{12} should be equal for two anti-parallel but equal r_{12} . If the crystal consists of different atoms then it is possible that ϱ_{12} does not have a centre of symmetry.

Let us consider the possibility of continuous transitions between liquid crystals and liquids (continuous in the sense that there is no discontinuity in the state).

In the case of a continuous transition $\varrho_{12}^{(0)}$ becomes $\varrho_{12}^{(0)} + \delta\varrho_{12}$ in a similar way to that in which ϱ_0 became $\varrho_0 + \delta\varrho$ before. $\varrho_{12}^{(0)}$ relates to the liquid and is therefore isotropic; $\delta\varrho_{12}$ has the symmetry of the liquid crystal.

It was shown in I at a continuous transition point there appear functions $\delta\varrho$, in the present case $\delta\varrho_{12}$, having certain symmetry properties which characterised irreducible representations of the symmetry group $\varrho_{12}^{(0)}$, i.e. in the present case symmetry groups of rotations about the origin of co-ordinates. It is known that functions which characterise irreducible representations of the rotation group either change their sign under inversion (reflection in the origin of co-ordinates) or do not. In the second case the crystal which is being formed has a centre of symmetry, in the first case it has not. The thermodynamic potential Φ of the body should be invariant with respect to every possible transformation, in particular with respect to inversions. Therefore if the crystal, i.e. $\delta\varrho$, has no centre of symmetry then all terms of odd powers in the expansion of Φ in powers of $\delta\varrho_{12}$, in particular the terms of third order, are identically equal to zero. In other words we have the case analysed as the case I, namely; continuous transitions are possible as Curie points lying on a curve in the diagram.

If $\delta\varrho_{12}$ has a centre of symmetry then terms of the third order are not generally speaking identically equal to zero. In this case we have the case II of part I i.e. only isolated points of continuous transition are possible, similar to continuous transitions between a liquid and a solid crystal (section 2).

It is experimentally known that He II becomes an ordinary liquid not at Curie points, i.e. we have the case I. This means, according to what has been said above, that if He II is a liquid crystal it need not have a centre of symmetry. Since, on the other hand, He II consists of identical atoms it should be assumed that the absence of a centre of inversion is caused by the fact that the atoms of He II themselves have asymmetric electron shells. In view of a certain strangeness of such an assumption, the assumption itself about He II being a liquid crystal becomes somehow doubtful.

4. SURFACE PHASES

The surface of separation of two isotropic, i.e. liquid, phases can generally speaking be anisotropic. Apparently in some cases this has been observed. But as it has been shown in section 1, that crystals in which ϱ is a function of only two coordinates are impossible. In particular the existence of two dimensional "solid" crystals, i.e. crystals with anisotropic ϱ , are impossible. Therefore anisotropic surface should be two dimensional "liquid" crystals, i.e. in them the molecules, more precisely their centres of mass, are randomly distributed, but not all of them are equally orientated ($\varrho = \text{const}$, ϱ_{12} is anisotropic).

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30. ON THE THEORY OF SUPERCONDUCTIVITY

In this paper it is shown that at average values of the magnetic field different from zero a superconductor must break up not into two regions, viz. a superconducting and a non-superconducting one, but into a large number of alternating superconducting and non-superconducting layers. This circumstance provides an explanation for the existence of Peierls' intermediate state.

1. THE INTERMEDIATE STATE

Rutgers and Gorter¹ have shown that a number of the properties of superconductors may be explained by supposing that the superconducting and non-superconducting states are two phases, the superconducting phase being characterised by the fact that a magnetic field does not penetrate it, i.e. its magnetic permeability $\mu = 0$. If a superconductor is placed in a magnetic

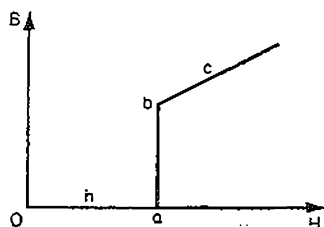


FIG. 1.

field, then on increasing this field the relation between the induction B and the magnetic field H is illustrated by the curve in Fig. 1. While the metal is superconducting $B = 0$; when the magnetic field reaches a critical value, which we shall denote by h , the body goes over to the non-superconducting phase. B then increases with H according to the formula $B = H$ (μ is practically equal to 1). Experiment shows that the section ab of the curve shown also corresponds to a certain actually realisable state of the body. Thus, a superconducting sphere placed in a magnetic field does not divide into two parts—a superconducting and a non-superconducting one—but a uniform magnetic field, which is not, however, equal to the external field, appears within it. With increase in the latter the field within the sphere also increases, approaching the external one. The state corresponding to the indicated section of the B - H curve was analysed by Peierls² and called "intermediate" by him.

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L. Landau, Zur Theorie der Supraleitfähigkeit, *Phys. Z. Sowjet.* 11, 129 (1937).

The task of the present paper consists in establishing the nature of the intermediate state, viz. it can be shown that it follows from thermodynamic considerations that at an average value of the magnetic induction not equal to zero (over the whole body) it is not separation into two parts (superconducting and non-superconducting) which is thermodynamically stable, but separation into a large number of layers, alternately superconducting and non-superconducting. In other words, the intermediate state is not a certain special state; a body in this state consists of a large number of superconducting and non-superconducting layers. As the body approaches the usual non-superconducting state, the non-superconducting layers become thicker at the expense of the superconducting ones until the latter completely disappear.

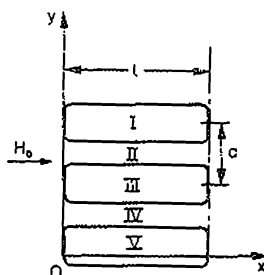


FIG. 2.

In a certain sense this is analogous to ferromagnetic bodies, which, when not magnetised, consist of a series of layers magnetised to saturation, but alternately in opposite directions². We shall now perform the analogous calculation for the intermediate state under consideration.

Let us consider a plane-parallel superconducting plate placed in a magnetic field H_0 perpendicular to its surface. The field H_0 is less than the critical field h . From the boundary conditions it follows that the average value (over the whole body) of the induction B in the plate is equal to the external field H_0 . Thus, inside the plate there exists induction while the external field is less than the critical one; in other words, we are precisely in the intermediate region of states. We shall start from the assumption that such a plate consists of successive layers which are superconducting (I, III, V in Fig. 2) and non-superconducting (II, IV in Fig. 2). The ratio of the quantity of material which has gone over to the non-superconducting state to the total quantity is easily determined from the magnetic induction flow theorem. At large distances from the plate $B = H = H_0$; in the superconducting layers $B = 0$; in the non-superconducting layers at large distance from the edges $B = H = h$. From this it is immediately evident that the non-superconducting layers have a total area of H_0/h . Figure 2 shows a section of the plate. From the requirement that the thermodynamic potential of the plate should be a minimum, we shall determine the distance a between nearest similar layers (Fig. 2), i.e. we shall determine the number of such layers. We shall see that the stable state actually corresponds to a separation into a large number of such layers.

2. THE SHAPE OF THE SUPERCONDUCTING LAYERS

We shall start with a determination of the shape of the layers, i.e. of the shape of the cross-section. It is evident that at large distances from the edges of the plate the layers will be plane-parallel, while near the surface their cross-section will be somewhat more complicated. In a surprising way an exact determination of the shape of the layers proves to be possible.

Let us set up the co-ordinate axes as shown in Fig. 2. In our consideration we shall examine only one surface of the plate, regarding the other as infinitely distant. It is possible to do this, of course, since the thickness of the plate is very large in comparison to the thickness of the region in which the shape of the layers deviates from plane-parallel. One such layer is shown in Fig. 3.

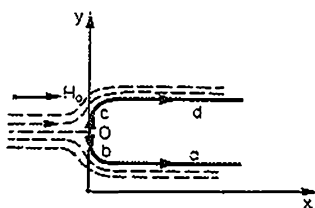


FIG. 3.

At a large distance from the plate the field is equal to H_0 , i.e. at $x = \infty$

$$H_x = H_0, \quad H_y = 0. \quad (1)$$

Since the normal component of magnetic induction must be continuous and in a superconductor $B = 0$, then at the boundary bc it must be the case that

$$H_x = 0. \quad (2)$$

The magnetic field freely penetrates into the region between the superconducting layers; at the boundary cd (or ab) between the superconducting and non-superconducting regions the magnetic field must be parallel to the boundary and equal in magnitude to the critical magnetic field h , i.e.

$$\text{on } cd \quad H^2 = H_x^2 + H_y^2 = h^2. \quad (3)$$

From the equations $\text{div } \mathbf{H} = 0$ and $\text{curl } \mathbf{H} = 0$ it follows that $\mathbf{H} = \text{curl } \mathbf{A}$ or $\mathbf{H} = -\text{grad } \varphi$, where \mathbf{A} and φ are the vector and scalar magnetic potentials. The vector potential \mathbf{A} can always be chosen so that it is directed along the z -axis everywhere. Then

$$H_x = -\frac{\partial \varphi}{\partial x} = \frac{\partial A}{\partial y}, \quad H_y = -\frac{\partial \varphi}{\partial y} = -\frac{\partial A}{\partial x}. \quad (4)$$

Since on the entire boundary $abcd$ of the superconductor the normal component of the magnetic field must be equal to zero, this boundary must consequently be a force line of the field. From symmetry considerations it is evident

that one of the force lines of the field branches into two lines at the point 0, running along the boundary of the layer; all the remaining lines enter the non-superconducting regions between the superconducting layers (in Fig. 3 the lines of force are represented by dotted lines). The determination of the shape of the layer thus amounts to determining the shape of the force line of the field indicated.

On a force line

$$\frac{H_y}{H_x} = \frac{dy}{dx}.$$

Substituting (4) in this, we find that $dA = 0$, i.e. on a force line, in particular at the boundary of the layer, $A = \text{const.}$ Let us suppose that at the boundary of one of the layers $A = 0$. It is easy to prove that at the boundaries of all the other layers A will be an integral multiple of $H_0 a$ (a is the distance between the layers). In fact, at infinite distance it follows from

$$H_x = \frac{\partial A}{\partial y} = H_0 \quad \text{and} \quad H_y = -\frac{\partial A}{\partial x} = 0$$

that $A = H_0 y + \text{const.}$, and along a line of force $A = \text{const.}$ Consequently, the difference between the values of A on two lines of force bordering on two adjacent superconducting layers is equal to $H_0 a$.

Let us introduce the variable

$$x + iy = u \tag{5}$$

and the quantities

$$\psi = \varphi - iA, \tag{6}$$

$$\eta = H_x - iH_y. \tag{7}$$

Equation (4) may then be regarded as the Cauchy-Riemann conditions for ψ and η and we see that ψ and η are functions of the complex variable u . From equation (4) it is further evident that

$$d\psi = -\eta du. \tag{8}$$

The boundary condition (3) may be written in the form

$$\text{on } cd \quad |\eta| = h, \tag{9}$$

and, noting that H_x is the real part of η , the condition (2) may be written in the form

$$\text{on } bc \quad \text{Re}(\eta) = 0. \tag{10}$$

Let us further introduce the quantity

$$\zeta = e^{-\frac{2\pi\psi}{H_0 a}} - 1 \tag{11}$$

and let us regard η as a function of ζ . With the aid of this quantity it is possible to write down boundary conditions which are the same for all the layers. Since, as has been pointed out, A is an integral multiple of $H_0 a$ at the

boundary of the layers, then at all these boundaries

$$\zeta = e^{-\frac{2\pi\varphi}{H_0 a}} - 1, \quad (12)$$

i.e. ζ is real.

Since φ is determined apart from a constant, we may arbitrarily choose a value of φ at one point. Let $\varphi = 0$ at the point 0 in Fig. 3. Then at the point 0, $\zeta = 0$. At an infinite distance from the plate on the line of force under consideration ($x = -\infty$) $\zeta = -1$ (since at infinity $\varphi = \infty$ for our choice of φ). Let us denote by ζ_0 the value of ζ at the point c (or b) where the line of force enters the interior of the plate. The limiting conditions (1), (2), (3) or (1), (9), (10) now have the form:

$$\text{for } \zeta = -1: \quad \eta = H_0, \quad (13)$$

$$\text{for } 0 < \zeta < \zeta_0: \quad \text{Re}(\eta) = 0, \quad (14)$$

$$\text{for } \zeta_0 < \zeta: \quad |\eta| = h. \quad (15)$$

In addition η must be finite for all values of ζ .

The last two conditions may be satisfied by the function

$$\eta = h \frac{\sqrt{-\zeta \zeta_0} - \sqrt{\zeta(\zeta - \zeta_0)}}{\zeta}. \quad (16)$$

This solution is finite (equal to zero) for $\zeta = 0$; for $\zeta < \zeta_0$ both terms are imaginary and therefore $\text{Re}(\eta) = 0$; for $\zeta > \zeta_0$, η is complex and its modulus equal to h . This expression, being correct on a single line, gives a solution everywhere by virtue of the properties of functions of complex variables (i.e. even where ζ is not real). Here the signs of the roots are taken for the region $\zeta < 0$. For $0 < \zeta < \zeta_0$ both roots are imaginary, since it follows from the theory of functions that their signs are opposite as before. For $\zeta > \zeta_0$ the second root changes its sign.

As regards ζ_0 , it is determined from the condition (13). For $\zeta = -1$

$$\sqrt{1 + \zeta_0} - \sqrt{\zeta_0} = \frac{H_0}{h},$$

and hence

$$\zeta_0 = \left(\frac{h^2 - H_0^2}{2hH_0} \right)^2. \quad (17)$$

Substituting (16) and (11) in $du = -d\varphi/\eta$ (8) and integrating for real values of ζ , we can find the shape of the layer. This gives

$$u = - \int \frac{d\varphi}{\eta} = \frac{H_0 a}{2\pi} \int \frac{d\zeta}{\eta(\zeta + 1)} = - \frac{H_0 a}{2\pi h} \int \frac{d\zeta [\sqrt{-\zeta \zeta_0} + \sqrt{\zeta(\zeta + \zeta_0)}]}{\zeta(\zeta + 1)}. \quad (18)$$

I shall not reproduce the result of this integration here.

3. THE NUMBER OF LAYERS

The thermodynamic potential Φ of a body in the conditions under consideration (more accurately, that part of it which originates from the presence of the layers) consists of two parts. One part, Φ_1 , is due to the surface tension at the boundary between the superconducting and non-superconducting regions. Let α be the coefficient of surface tension. If l_1 and l_2 are the lengths of the plate along the y and z axes, then there are in total l_1/a superconducting layers in the plate. Their total surface of contact with the non-superconducting layers is equal to $2l_1 l_2 l/a$ (deviations of the shape of the layers from planes near the surface of the plate are not important here). There is consequently $2l/a$ layer surface per unit area of the plate surface. The thermodynamic potential Φ_1 per unit area of the plate surface is consequently equal to

$$\Phi_1 = \frac{2l}{a} \alpha.$$

The other part Φ_2 of the thermodynamic potential is due to phenomena occurring near the surface of the plate³. To find it we must find the energy of the superconducting layer and calculate from it the energy the same layer would have if its section was rectilinear right up to the surface (i.e. if this section was rectangular).

The energy of unit volume of the superconductor differs from the energy per unit volume of the same material in the non-superconducting state by the quantity $-\hbar^2/8\pi$. At each of the four corners of the section of the layer (as in the neighbourhood of the points c and b in Fig. 3) with respect to a rectangular section (dotted line in Fig. 3) the area

$$\int_0^\infty (Y - y) dx, \quad (19)$$

is missing, where x and y are the co-ordinates of the boundary line of the section of the layer and Y is the co-ordinate y of this line at a large distance from the surface, i.e. where the section is bounded by straight lines. We can integrate to ∞ since at a large distance from the surface $y - Y$ is in any case $= 0$. The "missing" volume of the layers per unit plate surface is consequently equal to

$$\frac{4}{a} \int_0^\infty (Y - y) dx, \quad (20)$$

and the excess energy of the corners to

$$\frac{\hbar^2}{2\pi a} \int_0^\infty (Y - y) dx. \quad (21)$$

However, this is not the total energy. At the boundary of the superconducting layer the tangential component of the field undergoes a jump. In other words, at this boundary there are surface currents and, consequently, a magnetic moment. It is easy to show that per unit length of the z axis there is a magnetic moment for every boundary surface of

$$-\int \frac{|\eta|}{8\pi} y \, ds,$$

where $ds = \sqrt{dx^2 + dy^2}$ and the integration is carried out along the line bounding the section of the layer. If it were not for the perturbing influence of the surface, the layer would be plane-parallel and $|\eta|$ would be equal to h everywhere. The excess moment for each corner is equal to

$$-\int \frac{|\eta|}{8\pi} y \, ds + \int \frac{h}{8\pi} Y \, dx.$$

In total there is an additional moment

$$\frac{4}{a} \left\{ \int \frac{h}{8\pi} Y \, dx - \int \frac{|\eta|}{8\pi} y \, ds \right\}$$

per unit surface of the plate. The corresponding energy is obviously equal to

$$-\frac{4}{a} \left\{ \int \frac{h}{8\pi} Y \, dx - \int \frac{|\eta|}{8\pi} y \, ds \right\} \cdot H_0. \quad (22)$$

Let us remember that the point 0 (Fig. 3) corresponds to $\zeta = 0$, the point c to $\zeta = \zeta_0$ and at infinity $\zeta = \infty$. Therefore the energy (21) can be written in the form

$$\frac{h^2}{2\pi a} \int_{\zeta=\zeta_0}^{\infty} (Y - y) \, dx, \quad (21a)$$

and the energy (22) in the form

$$\frac{H_0}{2\pi a} \int_{\zeta=0}^{\zeta_0} H_y y \, dy + \frac{H_0 h}{2\pi a} \int_{\zeta=\zeta_0}^{\infty} (y \, ds - Y \, dx) \quad (22a)$$

[remembering that for $0 < \zeta < \zeta_0$, $H_x = 0$ and for $\zeta > \zeta_0$, $|\eta| = h$ (see equations (14) and (15))].

Φ_2 is equal to the sum of these two energies. Regrouping and replacing $H_y \, dy$ by $-d\varphi$ in the integral in (22a):

$$\Phi_2 = \frac{h(h - H_0)}{2\pi a} \int_{\zeta=\zeta_0}^{\infty} (Y - y) \, dx + \frac{h H_0}{2\pi a} \int_{\zeta=\zeta_0}^{\infty} y \, (ds - dx) - \frac{H_0}{2\pi a} \int_{\zeta=0}^{\zeta_0} y \, d\varphi. \quad (23)$$

Here we must substitute the equation of the curve bounding the section of the layer according to (18). In the general case this leads to quite complicated

expressions. However, one can easily reveal important properties of Φ_2 without carrying out the substitution. All three terms in Φ_2 are obviously positive (it is always the case that $Y > y$, $ds > dx$ and $d\varphi$ is less than zero along the positive direction of the line of force) so that $\Phi_2 > 0$. It is further evident from (18) that x and y , expressed in terms of ζ , are always proportional to a , so that all the integrals in Φ_2 are proportional to a^2 and Φ_2 is proportional to a . This result is quite evident in itself since the depth at which the surface of the plate exhibits a distorting influence must be of the order of the thickness of the layer. Since according to (19) Φ_1 is inversely proportional to a , the sum of Φ_1 and Φ_2 has a minimum at a certain value of a . Since Φ_1 is proportional to the thickness of the plate l and Φ_2 does not depend on l , it can easily be seen that the distance between the layers is inversely proportional to \sqrt{l} †.

Let us determine a for the case when the external field H_0 is small ($H_0 \ll h$). Investigation shows that only the third term in expression (23) is then important for Φ_2 and at the same time only the region near the point b in Fig. 3, i.e. the region near $\zeta = \zeta_0$, is important in this integral. At small fields the body is almost entirely superconducting, i.e. the gaps between the superconducting layers are very small. Therefore y is approximately equal to $a/2$.

From (23) we then have

$$\Phi_2 = -\frac{H_0}{4\pi} \varphi_0,$$

where φ_0 is the value of φ at $\zeta = \zeta_0$. From (12) we have

$$-\varphi_0 = \frac{H_0 a}{2\pi} \ln(1 + \zeta_0),$$

while from (17) we have for $h \gg H_0$

$$\zeta_0 = \left(\frac{h}{2H_0} \right)^2.$$

Since $\zeta_0 \gg 1$, then

$$-\varphi_0 = \frac{aH_0}{\pi} \ln \frac{h}{2H_0}$$

and therefore

$$\Phi_2 = \frac{H_0^2 a}{4\pi^2} \ln \frac{h}{2H_0}.$$

From the condition that the sum $\Phi_1 + \Phi_2$ should be a minimum we find that

$$a = \frac{2\pi}{H_0} \sqrt{\frac{2\alpha l}{\ln \frac{h}{2H_0}}}. \quad (24)$$

† An analogous situation occurs for the thickness of the layers in ferromagnetics, see L. Landau and E. Lifshitz³.

The thickness b of the non-superconducting layer will be determined from the magnetic field flow theorem:

$$b h = a H_0,$$

from which in the present case

$$b = \frac{2\pi}{h} \sqrt{\frac{2\alpha l}{\ln \frac{h}{2H_0}}}. \quad (25)$$

It is evident from (25) that the thickness of the non-superconducting layers gradually becomes zero as $H_0 \rightarrow 0$.

It can be shown that when H_0 increases the distance a at first falls, reaches a minimum, after which it begins to increase. For H_0 near to h , i.e. when the body is almost completely non-superconducting, it turns out that

$$a = \frac{2\pi}{h - H_0} \sqrt{\alpha l}, \quad (26)$$

and that the thickness of the superconducting layers is

$$a - b = a \left(1 - \frac{H_0}{h} \right) = \frac{2\pi}{h} \sqrt{\alpha l}.$$

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31. ON THE STATISTICAL THEORY OF NUCLEI

It is shown that the formulae for the distribution of nuclear energy levels which Bethe has obtained by considering the nucleus as an ideal gas, can also result from quite general considerations. A formula is derived for determining the width of the nuclear energy levels which are associated with scattering. Both inelastic scattering of neutrons, and emission of γ -rays are considered.

1. DISTRIBUTION OF NUCLEAR LEVELS

Niels Bohr has shown in his fundamental work that the nuclei of elements of sufficiently large atomic weight can be treated by statistical methods.

Developing these ideas, Bethe¹ has investigated the distribution of nuclear energy levels as function of the energy. In his treatment he has made assumptions which are equivalent to considering the nucleus as an ideal gas. In reality the interaction of the particles within the nucleus is definitely very strong, so that the nucleus should not be treated as a gas. It turns out, however, that Bethe's results do not depend on his assumptions.

If we allow for the interaction of the particles in the nucleus, there is naturally very little reason to consider the nucleus as a "solid body". That is to say we should imagine it as a "liquid drop" of protons and neutrons, and not as a "crystal". In contrast to normal liquids quantum effects are extremely important in this liquid. The uncertainty of the co-ordinates of the particles is distinctly larger than their mutual separation. Notwithstanding that we have no methods available for the theoretical treatment of "quantum liquids", we can still derive a few properties of the nuclei.

We set the energy of the nucleus in the ground state equal to zero. In this state the "temperature" of the nucleus is zero (T will be measured in units of energy). Since the excitation energy of the nucleus in the case which are of interest to us is small by comparison with the total nuclear binding energy, we can also assume the temperature of the excited nucleus to be small. Thus the free energy of the excited nucleus can be expressed in powers of T . Taking only the first term, we have:

$$F = - \frac{\alpha T^2}{2}, \quad (1)$$

where α is a constant. The linear term is equal to zero in accordance with the Nernst theorem. That the expansion for liquids begins with the quadratic term can be seen either from the analogous conditions for gases, or directly

Л. Ландау, К статистической теории ядер, *Журнал Экспериментальной и Теоретической Физики* 7, 819 (1937).

L. Landau, Zur statistischen Theorie der Kerne, *Phys. Z. Sowjet.* 11, 556 (1937).

from the measurements of Keesom, which show that the specific heat of an "electron liquid"† in metals is proportional to T .

From (1) we find for the energy E of the nucleus, and the entropy S :

$$S = - \frac{\partial F}{\partial T} = \alpha T, \quad E = F + TS = \frac{\alpha T^2}{2}. \quad (2)$$

Expressing S as a function of E , we have

$$S = \sqrt{2\alpha E}. \quad (3)$$

It is well known that we can write the number, N , of states, i.e. of energy levels, with an energy less than E as:

$$N = e^{S(E)} = e^{\sqrt{2\alpha E}}. \quad (4)$$

The number of levels per unit energy interval is:

$$\frac{dN}{dE} = e^S \frac{dS}{dE} = \frac{1}{T} e^S = \frac{1}{T} e^{\sqrt{2\alpha E}}. \quad (5)$$

Equation (5) gives the distribution of the energy levels, where all states, i.e. all possible angular momenta are included.

We now consider states with a specific total angular momentum j . Since the nucleus may be considered as a spherical drop, the energy of the nucleus, "rotating" with angular momentum j , is given by

$$\frac{\hbar^2}{2J} j(j+1),$$

where J is the moment of inertia of the nucleus. The corresponding number of states for this energy we know from the theory of the spherical top to be $(2j+1)^2$ ($a(2j+1)$ -fold degeneracy is due to the $(2j+1)$ different orientations of the vector j in space, and a further $(2j+1)$ due to the different possible orientations of the same vector with respect to the top.)

We can now write down the number of states for given j :

$$\frac{dN_j}{dE} = A(2j+1)^2 \exp \left[- \frac{\hbar^2}{2JT} j(j+1) \right],$$

where A is a quantity which does not depend on j . A is defined by the relation:

$$\sum_j \frac{dN_j}{dE} = A \sum_j (2j+1)^2 \exp \left[- \frac{\hbar^2}{2JT} j(j+1) \right] = \frac{dN}{dE}.$$

The exponential expression decreases rapidly from $j = 7$ to 8 onwards. Thus we may consider j as large compared with unity, and replace the summation

† The term "electron gas" is purely conventional, the interactions between electrons being very large.

by integration:

$$\frac{dN}{dE} = A \int_0^{\infty} e^{-\frac{\hbar^2}{2JT} j^2} 4j^2 dj = A \left(\frac{2JT}{\hbar^2} \right)^{3/2} \sqrt{\pi}.$$

Comparison with (5) gives the value of A , from which:

$$\frac{dN_j}{dE} = \frac{1}{\sqrt{\pi}} \left(\frac{\hbar^2}{2J} \right)^{3/2} \frac{1}{T^{5/2}} (2j+1)^2 \exp \left[-\frac{\hbar^2}{2JT} j(j+1) \right] e^S, \quad (6)$$

or, for the number of energy levels with angular momentum j :

$$\frac{1}{2j+1} \frac{dN_j}{dE} = \frac{1}{\sqrt{\pi}} \left(\frac{\hbar^2}{2J} \right)^{3/2} \frac{1}{T^{5/2}} (2j+1) \exp \left[S - \frac{\hbar^2}{2JT} j(j+1) \right]. \quad (7)$$

When a nucleus with angular momentum j_0 , comes into collision with a neutron, the resonance levels correspond to the states of the compound nucleus, i.e. the system, nucleus + neutron, with an angular momentum $j_0 \pm \frac{1}{2}$. If we sum the expression (7) for $j = j_0 \pm \frac{1}{2}$, we find for the number, $(2j_0+1)^{-1} \times dN_0/dE$:

$$\frac{1}{2j_0+1} \frac{dN_0}{dE} = \frac{1}{\sqrt{2\pi}} \left(\frac{\hbar^2}{J} \right)^{3/2} \frac{1}{T^{5/2}} (2j_0+1) e^S \quad (8)$$

(since the value of j_0 is small, we can neglect the second term in the exponential.) As this expression is in any case not very accurate, we may put $j_0 = 0$ in what follows. The reciprocal of dN/dE is the spacing D between the levels. From (8) we obtain for the spacing of the resonance levels:

$$D = \sqrt{2\pi} \left(\frac{J}{\hbar^2} \right)^{3/2} T^{5/2} e^{-S}. \quad (9)$$

From (2) and (3) we have $T = 2E/S$. Putting this in (9) gives:

$$S^{5/2} e^S = \frac{\sqrt{2\pi}}{D} \left(\frac{J}{\hbar^2} \right)^{3/2} (2E)^{5/2}. \quad (10)$$

Using the experimentally known values of E and D , we can calculate S from this formula. For heavy nuclei, $E \sim 8$ MeV. For the spacing of the energy levels, we put $D \sim 5$ eV. The moment of inertia, J , of a sphere of mass, m , and radius r , is $J = 2mr^2/5$. For a nuclear radius $r = 7 \times 10^{-13}$ cm and an atomic weight of 100, we find $J = 3 \times 10^{-47}$ gcm². Finally we obtain from (10):

$$S = 18,$$

and thus for the temperature, $T (= 2E/S)$:

$$T \sim 1 \text{ MeV}.$$

Let us now determine the order of magnitude of the energies of the lowest nuclear energy levels. We are thus specifying that the number, N (equation 4), of states belonging to the corresponding energy is of the order one. From (4) follows for $N \sim 1$ that S is also ~ 1 . Since from (3) the energy is proportional to the square of the entropy, we can write:

$$E_{\min} \sim \frac{E}{[S(E)]^2}. \quad (11)$$

We put $S \sim 20$ and obtain for E_{\min} a value of the order 2×10^4 eV.

We should note that whereas the formula (5) is valid for an energy of a few ten thousand eV onwards, formula (9) is only applicable for a range with a lower limit of about 2 MeV. For smaller energies the condition $E \gg D$ is not satisfied.

Since E is roughly equal for the various elements, we can deduce from the relation $S = \sqrt{2\alpha E}$ that S is proportional to $\sqrt{\alpha}$. Since the free energies are additive, we can say further that α increases with the number of nuclear particles, that is it is proportional to \sqrt{A} (A is the atomic weight). Since the number of particles in the nucleus is usually not very large, we do not expect this formula to be particularly useful.

2. THE WIDTH OF RESONANCE PEAKS FOR NEUTRONS

The neutron can be captured on collision by the nucleus, forming the so-called compound nucleus of neutron + atomic nucleus. The neutron may subsequently be expelled from the nucleus, with or without loss of energy (inelastic or elastic scattering, respectively). Alternatively, it may be absorbed finally by the nucleus, and the compound nucleus will change to its ground state with the emission of light quanta (capture of neutrons).

Let us consider the elastic scattering. Let $2\Gamma_e/\hbar$ be the probability per unit time that the neutron, having been captured into a level, E_n , of the compound nucleus, is then re-emitted Γ_e (for brevity the index n is omitted) is the width of the energy peak for this process.

In order to determine the approximate magnitude of Γ_e we treat the problem as follows: we consider the nucleus in thermodynamic equilibrium with the neutron gas. The probability that the nucleus will be in the compound state with energy E_n , is $\exp[(\mu - E_n)/T]$ (μ is the chemical potential). The nucleus can only capture neutrons with angular momentum, $l = 0$, (neutrons with $l = 1, 2 \dots$ pass by at greater distances). The number of such neutrons whose momentum lies in the range dp and whose distance from the nucleus lies in the range dr , is:†

$$\frac{dp dr}{2\pi \hbar} e^{(\mu - E)/T},$$

† Both the spin of the neutron and the angular momentum of the nucleus have been ignored, since the results are anyway only approximate.

where E is the energy of the neutron. Instead of dp we can put dE/v where v is the speed of the neutron. Then the number of neutrons which strike the nucleus per sec is:

$$e^{(\mu - E)/T} \frac{dE}{2\pi\hbar}.$$

Now let us consider a small energy interval which is nonetheless large enough to contain many energy levels of the compound nucleus. We use \bar{w} for the mean probability that a compound nucleus is formed on the collision of the neutron with the nucleus. Then per sec there are:

$$e^{(\mu - E)/T} \frac{\Delta E \bar{w}}{2\pi\hbar} \quad (12)$$

compound nuclei formed in some state or other in the energy interval ΔE . This number must be equal to the number of neutrons which leave the compound nucleus on its decay. Since the number of compound nuclei with energy E is $\exp[(\mu - E)/T]$, the mean probability of decay, $2\bar{\Gamma}_e/\hbar$, and the number of levels in the energy interval ΔE is $\Delta E/D$, where D is the mean spacing between levels, then the number of nuclei which decay per second is:

$$e^{(\mu - E)/T} \frac{\Delta E}{D} \frac{2\bar{\Gamma}_e}{\hbar}. \quad (13)$$

From (12) and (13) we have:

$$\bar{\Gamma}_e = \frac{D\bar{w}}{4\pi}. \quad (14)$$

Since the normalised wave function of the neutron in the region of the nucleus is proportional to the square root of the speed, then $\bar{\Gamma}_e$ and \bar{w} are proportional to the speed itself, i.e. to the square root of the energy, and in particular:

$$\bar{w} = a\sqrt{E}.$$

When we consider the nucleus as a liquid drop, on the other hand, we can afford to disregard the spacing of the energy levels, just as for a real liquid. We can now argue that if the energy of the neutron were of the same order as the binding energy E_0 of the nucleus, then the probability of neutron capture would be of the order one. We can thus write:

$$\bar{w} = \sqrt{\frac{E}{E_0}}. \quad (15)$$

Comparing (14) and (15) gives:

$$\frac{\bar{\Gamma}_e}{D} = \frac{1}{4\pi} \sqrt{\frac{E}{E_0}}. \quad (16)$$

Putting, for example, $E \sim 1$ eV, $E_0 \sim 8$ MeV, $D \sim 5$ eV we find:

$$\bar{\Gamma}_e \cong 2 \times 10^{-4} \text{ eV}$$

which is apparently in agreement with experimental data.

Formula (14) remains obviously valid when a proton or an α -particle is emitted on decay of the nucleus instead of a neutron. In place of (15) we now have:

$$\bar{w} = e^{-\gamma},$$

where γ is the Gamow penetrability of the potential barrier for a proton or an α -particle. We then have for the decay constant of a radioactive nucleus:

$$\lambda = \frac{2\bar{\Gamma}_e}{\hbar} = \frac{D}{2\pi\hbar} e^{-\gamma}. \quad (17)$$

For D we must here substitute the difference between the first excited nuclear states. According to the above calculations, D is $\sim 2 \times 10^4$ eV. In place of this Bethe² has taken $1 \text{ eV} \cdot e^{-\gamma}$ for $\bar{\Gamma}_e$. Thus he has obtained the value of 13×10^{-13} cm for the radii of the heavy nuclei, which is too large. By means of (17) we obtain the value of 10.5×10^{-13} cm for the radius of the ThA nucleus.

3. INELASTIC SCATTERING

In the case of inelastic scattering the neutron leaves the compound nucleus with a reduced energy, and the nucleus remains in an excited state. We denote the probability of such a decay of the compound nucleus by $2\bar{\Gamma}_i/\hbar$; $\bar{\Gamma}_i$ is the corresponding part of the level width. Since there is no fundamental difference between the ground state and the excited states, it is apparent that a formula for $\bar{\Gamma}_i$ is valid, analogous to formula (16) for elastic scattering:

$$\bar{\Gamma}_i = \frac{D}{4\pi} \sqrt{\frac{\varepsilon}{E_0}}. \quad (18)$$

Here ε is the energy of the emitted neutron. Since ε is smaller than the energy E of the incoming neutron, we can see by comparing (16) and (18):

$$\bar{\Gamma}_i < \bar{\Gamma}_e.$$

At the same time the number of energy levels increases very sharply with increasing energy, and the nucleus can remain in any one of the lower excited states for inelastic scattering. For this reason the width of the highly excited nuclear levels is defined principally by the inelastic scattering. It should not be concluded from this that the scattering will be principally inelastic since, as Bethe and Placzek^{3†} have shown, the cross section for elastic scattering has an extra term, $4\pi r^2$ (r is the nuclear radius), which corresponds to elastic

† The author would like to express his thanks to G. Placzek for communicating the results of this work prior to publication.

scattering without the formation of a compound nucleus. In fact the resonance at highly excited levels will be almost entirely due to inelastic scattering.

Owing to the rapid broadening of the levels for inelastic scattering, the tendency for resonance must decrease sharply at sufficiently high energy. We should expect that at an energy of the order of 1 MeV it should vanish.

When the energy of the incident neutron is sufficiently large to enable us to speak of the temperature of the nucleus left in an excited state, then the mean energy of the emitted neutron will clearly be of the order:

$$\varepsilon \sim T, \quad (19)$$

where T is the temperature of the excited nucleus. A more exact calculation shows that the mean value of ε is about $2T$. From (19) and (2) it can be seen that the energy of the neutron on leaving the nucleus varies with the square root of its incident energy. Thus when the incident energy is large, the neutron leaves with appreciably less energy. We can estimate, for example, that when a neutron of energy 10 MeV is captured by the nucleus, the latter retains about 8 MeV in its excited state.

4. NUCLEAR RADIATION

Taking a statistical standpoint, we can consider the emission from the highly excited nucleus as thermal radiation. In particular, we can apply Kirchhoff's law. Unfortunately, the calculation of the absorption coefficient is very difficult. It can be assumed, however, that whatever form the absorption coefficient may take, its dependence on frequency will be basically defined from Planck's formula; i.e. principally quanta of energy,

$$\hbar \omega \sim T$$

will be emitted, where T is the temperature of the nucleus. We should thus expect that the nucleus will radiate its energy not in one, but in several stages. In the particular case of neutron capture, γ -quanta with energy 1-3 MeV will be emitted, rather than quanta of 8 MeV.

Finally, I should like to thank G. Placzek for the very interesting discussions of this problem.

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32. X-RAY SCATTERING BY CRYSTALS IN THE NEIGHBOURHOOD OF THE CURIE POINT

Formulae are derived for coherent and incoherent scattering of X-rays from crystals in the region of the Curie point. The temperature dependence of the intensity of the scattering was determined. With the help of these formulae the intensity distribution in the spots of the Laue diagram and the rings of the Debye-Scherrer diagram was determined.

LINES are observed in the scattering of X-rays from ordered crystals, which occur in addition to those lines seen from the disordered crystal, and which are specified precisely by the existence of order; these are the superstructure lines. These lines vanish as the crystal changes into the disordered state. In this paper the properties are discussed of the scattering in the region of that point where the crystal changes from the ordered to the disordered state (the Curie point).

We shall describe the state of the crystal by the "degree of order" η , as in I¹. At the Curie point η becomes zero. In the region of the Curie point the thermodynamic potential, Φ , per unit volume of the body, takes the form (see I, instead of A and B , we put here $A/2$ and $B/4$):

$$\Phi = \Phi_0 + \frac{1}{2} A \eta^2 + \frac{1}{4} B \eta^4. \quad (1)$$

Here Φ_0 , A , and B are functions of the pressure, p , and the temperature T . Φ_0 is the thermodynamic potential of the disordered crystal. In the ordered crystal we obtain from the equilibrium condition, $\partial \Phi / \partial \eta = 0$, in the region of the Curie point (see I)

$$\eta = \sqrt{-A/B}. \quad (2)$$

The coefficient B is positive. The coefficient A is negative below the Curie point (in the ordered state), and positive above it (in the disordered state). At the Curie point itself,

$$A_c = 0. \quad (3)$$

In the region of the Curie point we can expand A in powers $\theta - T$, where θ is the Curie temperature:

$$A = A_c + \frac{\partial A}{\partial T} (T - \theta) + \dots \cong \frac{\partial A}{\partial T} (T - \theta).$$

Л. Д. Ландау, Рассеяние рентгеновых лучей кристаллами вблизи точки кюри, *Журнал Экспериментальной и Теоретической Физики* 7, 1232 (1937).

L. Landau, Streuung von Röntgenstrahlen an Kristallen in der Nähe des Curie-Punktes, *Phys. Z. Sowjet.* 12, 123 (1937).

If we substitute this in (2) and write $\partial A/\partial T = \beta$, we obtain:

$$\eta = \sqrt{\frac{\beta(\theta - T)}{B}}. \quad (4)$$

Consider first the coherent scattering. The amplitudes of coherent scattering from atoms which are in normal and abnormal conditions are clearly equal and opposite. (By normal, we mean the position which an atom would take in the fully ordered crystal; a "strange" position is then abnormal). In the disordered crystal ($\eta = 0$) the number of atoms in normal and abnormal positions is the same, and thus the amplitude of the scattering is zero. Clearly in the region of the Curie point the amplitude of the coherent scattering is proportional to η , and the intensity, J_1 , to η^2 . From (4) we see that

$$J_1 \sim \theta - T, \quad (5)$$

i.e. the intensity of the coherent scattering is proportional to the distance from the Curie point.

In the region of the Curie point the incoherent scattering, which is due to the fluctuation of order, becomes very large, similar to the scattering of light near the critical point. This is connected with the fact, which we will show presently, that at the Curie point the fluctuation of order becomes infinite.

We calculate the mean square variation $(\Delta\eta)^2$. We emphasise that a fluctuation, and hence also incoherent scattering, naturally occurs on both sides of the Curie point: in the ordered as well as the disordered state. The probability of a variation for const p and T , we know to be $\propto e^{-V\Delta\Phi/kT}$, where $\Delta\Phi$ is the change per unit volume of the thermodynamic potential with the variation, V is the volume of the body, and k is Boltzmann's constant. Above the Curie point $\eta = 0$ at equilibrium, so that the deviation $\Delta\eta = \eta$. From (1) we obtain in second order:

$$\Delta\Phi = \Phi - \Phi_0 = \frac{1}{2} A \eta^2 = \frac{1}{2} A (\Delta\eta)^2.$$

The probability of the fluctuation is thus proportional to

$$e^{-\frac{1}{2} A (\Delta\eta)^2 V / kT}.$$

From this, we have for the mean value $\overline{(\Delta\eta)^2}$:

$$\overline{(\Delta\eta)^2} = \frac{kT}{AV}. \quad (6)$$

At the Curie point, however, $A = 0$ (3), thus here $\overline{(\Delta\eta)^2} = \infty$.

This expression naturally deals with fluctuations in every small region of the body. The variation from order is not in general homogeneous over the whole body. This leads to the appearance of the hazy superstructure in the scattering.

Let the wave vectors of the light before and after scattering be k_1 and k_2 ; k_1 and k_2 are different only in direction. The amplitude of the scattering, as is known, is proportional to the integral

$$\int \rho e^{i(k_1 - k_2) \cdot r} dV,$$

where ρ is the charge density in the crystal, and the integral is taken over the entire volume of the crystal. In the region of the Curie point, ρ has the form

$$\rho = \rho_0 + \eta \rho_1,$$

where ρ_0 is the density function in the disordered crystal. It has the symmetry of the disordered crystal, as against ρ_1 which has the lower symmetry of the disordered crystal. To follow this further see I (in I we wrote $\rho = \rho_0 + \delta\rho$ where $\delta\rho$ becomes zero at the Curie point). The superstructure lines are due to the factor, so that the required amplitude, a , of the corresponding scattering is proportional to

$$a \sim \int \eta \rho_1 e^{i(k_1 - k_2) \cdot r} dV, \quad (7)$$

ρ_1 can be represented in the form

$$\rho_1 = \sum \lambda_i e^{i(K_i \cdot r)}$$

where the K_i are the periods of that reciprocal lattice which possesses the symmetry of ρ_1 . Amongst the K_i are also periods which do not occur in the expression for ρ_0 . It is precisely these periods which are of interest to us.

Let us consider the scattering from a plane of the superstructure lattice, i.e. from one K_i , which we shall denote simply by K . From (7) we obtain for the amplitude of the scattering

$$a \sim \int \eta e^{i(k_1 - k_2 + K) \cdot r} dV = \int \eta e^{i(q \cdot r)} dV, \quad (8)$$

where $q = k_1 - k_2 + K$. The intensity of the scattered ray is given by $|a|^2$. The degree of order, η , can be written in the form $\eta = \bar{\eta} + \Delta\eta$, where $\bar{\eta}$ is the constant mean value of η and $\Delta\eta$ is the fluctuation. The intensity, J , becomes proportional to:

$$J \sim \left| \int \eta e^{i(q \cdot r)} dV \right|^2.$$

If we substitute $\eta = \bar{\eta} + \Delta\eta$ here, allowing that $\overline{\Delta\eta} = 0$, we find that

$$J \sim \left| \int \bar{\eta} e^{i(q \cdot r)} dV \right|^2 + \left| \int \Delta\eta e^{i(q \cdot r)} dV \right|^2. \quad (9)$$

We note that in the mean amplitude,

$$\bar{a} \sim \int \bar{\eta} e^{i(q \cdot r)} dV$$

there is no term with $\Delta\eta$. Thus the first term in (9) (we denote it by J_1) describes the coherent scattering. The second term (J_2), which does not appear in the amplitude, specifies the incoherent scattering. $\Delta\eta$ can change along the crystal. The scattering for given q is determined according to (8) by the part of $\Delta\eta$

which has the form $\text{const} \times e^{-i(\mathbf{q} \cdot \mathbf{r})}$. Since $\Delta \eta$ is naturally a real quantity, the part which is of interest to us may be written:

$$\Delta \eta = c e^{-i(\mathbf{q} \cdot \mathbf{r})} + c^* e^{i(\mathbf{q} \cdot \mathbf{r})}. \quad (10)$$

The asterisk denotes the complex conjugate.

Since the term, $A \eta^{3/2}$, in the expansion (1) is small near the Curie point, and as we are considering the fluctuation as a function of the co-ordinates, we must bear in mind the dependence of Φ not only on η , but also on the gradient, $\nabla \eta$. We must therefore also expand in powers of $\nabla \eta$. Since Φ is a scalar, the first term in the expansion will contain $(\nabla \eta)^2$. Hence we obtain for Φ the expression

$$\Phi = \Phi_0 + \frac{1}{2} A \eta^2 + \frac{1}{4} B \eta^4 + \frac{1}{2} \alpha (\nabla \eta)^2. \quad (11)$$

We shall limit ourselves to considering cubic crystals. All results can easily be generalised to other classes of crystal: one must merely substitute the expression

$$\frac{1}{2} \sum_{i,k=1}^3 \alpha_{ik} \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_k}$$

instead of $\frac{1}{2} \alpha (\nabla \eta)^2$, where α_{ik} is a tensor.

Let us consider first the scattering above the Curie point (in the disordered state). Since $\bar{\eta} = 0$, then $\Delta \eta = \eta$. The thermodynamic potential of the whole crystal is $\int \Phi dV$. We substitute (11), ignoring higher order terms, and obtain

$$\int (\Phi - \Phi_0) dV = \left(\frac{1}{2} A \eta^2 + \frac{1}{2} \alpha (\nabla \eta)^2 \right) dV.$$

We substitute (10) and, noting that $\int e^{i(\mathbf{f} \cdot \mathbf{r})} dV = 0$ if $\mathbf{f} \neq 0$, and equals V when $\mathbf{f} = 0$, we obtain $\int (\Phi - \Phi_0) dV = (A + \alpha q^2) c c^* V$.

As we have already remarked above, the probability of the fluctuation is proportional to

$$\exp \left[- \frac{1}{\kappa T} \int (\Phi - \Phi_0) dV \right]$$

where $\int (\Phi - \Phi_0) dV$ is the change in the thermodynamic potential with the fluctuation. In the given case this probability is proportional to

$$\exp \left(- \frac{1}{\kappa T} V (A + \alpha q^2) (c_1^2 + c_2^2) \right).$$

(The complex constant, c , is written in the form $c = c_1 + i c_2$.)

From this we get

$$\overline{c_1^2} = \overline{c_2^2} = \frac{\kappa T}{2V(A + \alpha q^2)}$$

or

$$\overline{c c^*} = \frac{\kappa T}{V(A + \alpha q^2)}. \quad (12)$$

Above the Curie point the first term in (9) is zero, since $\bar{\eta} = 0$. Thus there is no coherent scattering. With the use of (10) and (12), the second term gives for the intensity of the incoherent scattering,

$$J \sim \overline{c c^*} V^2 = \frac{V \kappa T}{A + \alpha q^2}. \quad (13)$$

Let us now consider the scattering below the Curie point. Here $\eta = \bar{\eta} + \Delta\eta$, where from (2) and (4) we have

$$\bar{\eta} = \sqrt{-\frac{A}{B}} = \sqrt{\frac{\beta}{B}(\theta - T)}.$$

If we substitute this in (11), then ignoring terms higher than the second order in $\Delta\eta$, we have

$$\int (\Phi - \Phi(\bar{\eta})) dV = \int \left\{ \beta(\theta - T)(\Delta\eta)^2 + \frac{\alpha(\nabla\eta)^2}{2} \right\} dV$$

$\Phi(\bar{\eta})$ denotes the value of Φ when $\Delta\eta = 0$. We substitute (10) and find

$$\int (\Phi - \Phi(\bar{\eta})) dV = (2\beta(\theta - T) + \alpha q^2) V c c^*.$$

Analogous to (12), we have

$$\overline{c c^*} = \frac{\kappa T}{(2\beta(\theta - T) + \alpha q^2) V}. \quad (14)$$

The first term in (9), the coherent scattering, now becomes

$$J_1 \sim \bar{\eta}^2 \left| \int e^{i(\mathbf{q} \cdot \mathbf{r})} dV \right|^2 = \frac{\beta}{B} (\theta - T) \left| \int e^{i(\mathbf{q} \cdot \mathbf{r})} dV \right|^2. \quad (15)$$

We now transform this expression by writing the square of the modulus of the integral as a double integral:

$$\left| \int e^{i(\mathbf{q} \cdot \mathbf{r})} dV \right|^2 = \int \int e^{i(\mathbf{q} \cdot \mathbf{r}_1 - \mathbf{r}_2)} dV_1 dV_2.$$

If we change to the variables $\mathbf{r}' = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ this becomes

$$\int \int e^{i(\mathbf{q} \cdot \mathbf{r})} dV dV'.$$

The integral over dV' simply gives the volume of the crystal. From the well known formula for the δ -function,

$$\int_{-\infty}^{+\infty} e^{i\alpha x} dx = 2\pi \delta(\alpha)$$

follows

$$\int e^{i(\mathbf{q} \cdot \mathbf{r})} dV = 8\pi^3 \delta(\mathbf{q}),$$

where $\delta(\mathbf{q})$ stands for the product $\delta(q_x) \delta(q_y) \delta(q_z)$.

If we substitute this in (15), we find

$$J_1 \sim 8\pi^3 \frac{\beta}{B} (\theta - T) \delta(q) V. \quad (16)$$

From (14) we obtain for J_2

$$J_2 \sim c c^* V^2 = \frac{\kappa T V}{2\beta(\theta - T) + \alpha q^2}. \quad (17)$$

Thus for the scattering intensity $J = J_1 + J_2$ we obtain for $T < \theta$

$$J \sim \frac{\theta - T}{\theta} \delta(q) + \frac{\kappa}{16\pi^3 \Delta C_p} \cdot \frac{1}{2 \frac{\theta - T}{\theta} + \alpha q^2}. \quad (18)$$

Here we have divided the sum of (16) and (17) by the constant $8\pi^3 (\beta/B) \theta V$, since (18) in any case only gives the intensity without a constant factor. Also the notations $\alpha = (\alpha/\beta) \theta$ and $\Delta C_p = (2\beta/B^2) \theta$ have been introduced. The latter quantity is the jump in the specific heat, C_p , at the Curie point (see formula (8) in I). Since we are considering the scattering near the Curie point, $\theta - T$ is small, and we have written θ for T wherever possible throughout.

We can write the formula (13) for the scattering above the Curie point in a form analogous to (18): for $T > \theta$

$$J \sim \frac{\kappa}{16\pi^3 \Delta C_p} \cdot \frac{1}{\frac{T - \theta}{\theta} + \alpha q^2}. \quad (19)$$

For $T < \theta$ the superstructure lines are comprised of two parts: (1) a sharp line due to coherent scattering (the first term in (18)) with $q = 0$, i.e. corresponding to the Bragg scattering condition. (2) a hazy line from the incoherent scattering (the second term in (18)); the intensity maximum occurs, of course at $q = 0$. The first line vanishes at the Curie point. The second, by contrast, has a maximum at this point; above the Curie point it is determined by (19).

Suppose we specify the form of the superstructure lines (i.e. the spots) in a Laue pattern. In order to obtain a Laue pattern the primary radiation must be parallel but not monochromatic. In other words the wave vector, k_1 , of the incident radiation has the same direction for all rays, but takes all possible absolute magnitudes. The crystal also has a fixed orientation, i.e. the vector K of the lattice period is constant. Thus to obtain the distribution of the Laue spots, we must integrate (18) and (19) over all frequencies of the incident radiation.

The centre of the spot corresponds to scattering at the Bragg angle, i.e. the condition $q = 0$ is fulfilled, or:

$$k_1 + K = k_2.$$

Here K is a known vector, k_1 has a particular direction, and k_1 and k_2 have the same absolute value, $k_1 = k_2 = k$; the condition, $k_1 + K = k_2$, can therefore only be satisfied for certain frequencies of the incident light. We denote the value of k for which this condition is satisfied, by k_0 , and the corresponding vectors, k_1 and k_2 , by k_{10} and k_{20} . The vectors, K , k_{10} and k_{20} clearly form an isosceles triangle. We also introduce the unit vector n_2 in the direction of scattering, i.e., in the direction k_2 ; for $k_2 = k_{20}$, it shall be written n_{20} . We write k and n_2 in the form:

$$k = k_0(1 + \varepsilon), \quad n_2 = n_{20} + v. \quad (20)$$

Since the spot is in general small, ε and v are also small.

We substitute (20) in $q = k_1 - k_2 + K$, and ignoring powers of ε and v higher than the first order, we find

$$q = -k_0 v - K \varepsilon. \quad (21)$$

On the other hand, if we square $k_{20} - K = k_{10}$, we have

$$k_0 = \frac{K^2}{2(K \cdot n_{20})}.$$

We choose the z -axis of the co-ordinate system along n_{20} , and the x -axis in the plane of the vectors k_{10} , k_{20} , K (Fig. 1). We can then write k_0 in the form $k_0 = K^2/2K_z$ and (21) becomes

$$q = -\frac{K^2}{2K_z} v - K \varepsilon. \quad (22)$$

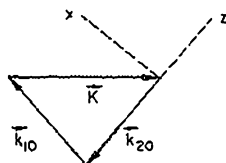


FIG. 1.

It is easily seen that q^2 can be written as

$$q^2 = K^2 \left(\varepsilon + \frac{(K \cdot v)}{2K_z} \right)^2 + \frac{K^2 [K \wedge v]^2}{4K_z^2} \quad (23)$$

Since K lies in the xz -plane, and v is a small change in the vector n_{20} and perpendicular to it, i.e. it has no component in the z -direction, we obtain for the vector product:

$$[K \wedge v]^2 = K_z^2 v_x^2 + K^2 v_y^2$$

v_x and v_y are clearly the sines of the scattering angles measured from the scattering direction to the centre of the spots; because of their small size, these angles may be equated to their sines.

We now consider the integration of (18) for all frequencies of the incident rays, i.e. over all k . This integration can be transformed into an integration over ε . From (22) we can write

$$\int \delta(q) d\varepsilon = \int \delta(-\varepsilon K_z) \delta\left(-\varepsilon K_x - \frac{K^2 v_x}{2K_z}\right) \delta\left(-\varepsilon K_y - \frac{K^2 v_y}{2K_z}\right) d\varepsilon.$$

From

$$\delta(x) = \delta(-x)$$

and

$$\delta(ax) = \frac{1}{a} \delta(x).$$

follows

$$\delta(-\varepsilon K_z) = \frac{1}{K_z} \delta(\varepsilon).$$

Because of the presence of $\delta(\varepsilon)$ in the integrand, we must put $\varepsilon = 0$ in the two other δ -functions for the integration in ε , i.e.

$$\int \delta(q) d\varepsilon = \frac{1}{K_z} \delta\left(\frac{K^2 v_x}{2K_z}\right) \delta\left(\frac{K^2 v_y}{2K_z}\right) = \frac{4K_z}{K^4} \delta(v), \quad (24)$$

where $\delta(v) = \delta(v_x) \delta(v_y)$.

For the integral in the second term of (18), we obtain after substituting (23),

$$\int \frac{d\varepsilon}{2 \frac{\theta - T}{\theta} + a q^2} = \int \frac{d\varepsilon}{2 \frac{\theta - T}{\theta} + a K^2 \left(\varepsilon + \frac{(K \cdot v)}{2K_z} \right)^2 + a \frac{K^2 [K \wedge v]^2}{4K_z^2}}.$$

Since the integral converges rapidly, the limits may be extended to $\pm \infty$, and we have

$$\int_{-\infty}^{+\infty} \frac{d\varepsilon}{2 \frac{\theta - T}{\theta} + a q^2} = \frac{\pi}{K \sqrt{a \left(2 \frac{\theta - T}{\theta} + a \frac{K^2}{4K_z^2} [K \wedge v]^2 \right)}}. \quad (25)$$

With the help of (24) and (25), we find the distribution of intensities in the Laue spot from (18), and for $T < \theta$ in the form

$$I \sim \frac{\theta - T}{\theta} \delta(v) + \frac{\kappa K^2}{32\pi^2 \Delta C_p a \sqrt{K_z^2 v_x^2 + K^2 v_y^2 + \frac{8K_z^2}{a K^2} \cdot \frac{\theta - T}{\theta}}}. \quad (26)$$

We have again divided by a constant in order to have simply $(\theta - T)/T$ as the coefficient of $\delta(v)$.

Similarly we obtain above the Curie point, for $T > \theta$:

$$I \sim \frac{\kappa K^2}{32\pi^2 \Delta C_p a \sqrt{K_z^2 v_x^2 + K^2 v_y^2 + \frac{4K_z^2}{a K^2} \cdot \frac{T - \theta}{\theta}}}. \quad (27)$$

Thus in the region of the Curie point, in order to obtain a very sharp maximum at the centre of the spot (the first term in (26); for $T > \theta$ this does not occur) we have also a blurred spot (second term) due to the incoherent scattering. This blurred spot also has an intensity maximum at the centre (when $v = 0$), but nevertheless has a somewhat diffuse character. This can be seen since the integral of the second term in (26) or of (27) diverges for all v_x and v_y .

To estimate the order of magnitude of the incoherent scattering we note that $a = \alpha/\beta\theta$ is of the order $a \cong K^2$. That this is so can be seen as follows: when η changes over the length of one lattice constant, then for temperatures which are not too near the Curie point, the second and fourth terms are of the same order of magnitude (i.e. we have $\Delta\eta \cong \eta K$ and $\beta\theta\eta^2 \cong \alpha(\Delta\eta)^2$). Furthermore, the jump in the specific heat per elementary cell is of the order κ , thus ΔC_p per unit volume is

$$\Delta C_p \cong \frac{\kappa}{\left(\frac{2\pi}{K}\right)^3} = \frac{\kappa K^3}{8\pi^3}.$$

The coefficient cancels out roughly, and we find that the second term in (26) is of the order

$$\sqrt{\frac{\theta}{\theta - T}}. \quad (28)$$

By comparison, we can point out that the total intensity corresponding to the first term in (26) is equal to

$$\frac{\theta - T}{\theta}. \quad (29)$$

Comparing finally the two terms under the square root in (26) or (27), we find that the mean "width", \bar{v} , of the blurred spot is of the order

$$\bar{v} = \sqrt{\frac{\theta - T}{\theta}}. \quad (30)$$

We now specify the distribution of the intensities of the superstructure lines (i.e. of the ring) in a Debye-Scherrer pattern. The incident rays now have a particular direction and frequency; i.e. k_1 has now a given absolute magnitude. The crystals take all possible orientations in the powder, so that the vector K has a specified magnitude and all possible directions. Thus to obtain the intensity distribution in the Debye-Scherrer ring, (18) and (19) must be averaged for all directions of the vector K .

Let us start with the first term in (18), averaging it over all directions of K , for a given absolute value, K . To do this we multiply by

$$\frac{\delta(|K| - K) dK_x dK_y dK_z}{4\pi K^2}$$

and integrate over $dK_x dK_y dK_z$. This gives

$$\int \delta(\mathbf{k}_1 + \mathbf{K} - \mathbf{k}_2) \delta(|\mathbf{K}| - K) dK_x dK_y dK_z = \delta(|\mathbf{k}_2 - \mathbf{k}_1| - K).$$

We introduce the absolute value $k = k_1 = k_2$ of the vectors, \mathbf{k}_1 and \mathbf{k}_2 , and also the angle, φ , between them (i.e., the scattering angle) and obtain

$$\delta(|\mathbf{k}_2 - \mathbf{k}_1| - K) = \delta\left(2k \sin \frac{\varphi}{2} - K\right).$$

The reflection at the Bragg angle corresponds to the scattering angle, $\varphi = \varphi_0$, given by

$$K = 2k \sin \frac{\varphi_0}{2}.$$

Since the angle $\varphi - \varphi_0$ in the δ -function is always small, we can write

$$\begin{aligned} \delta\left(2k \sin \frac{\varphi}{2} - K\right) &= \delta\left(K \frac{\sin \frac{\varphi}{2}}{\sin \frac{\varphi_0}{2}} - K\right) = \delta\left(\frac{K}{2} \cot \frac{\varphi_0}{2} (\varphi - \varphi_0)\right) \\ &= \delta[K(\chi - \chi_0)] = \frac{1}{K} \delta(\chi - \chi_0), \end{aligned}$$

where

$$\chi = \ln \sin \frac{\varphi}{2}. \quad (31)$$

With the aid of these formulae we obtain as the result of the averaging of the first term of (18),

$$J_1 \sim \frac{1}{4\pi K^3} \frac{\theta - T}{\theta} \delta(\chi - \chi_0). \quad (32)$$

To average the second term of (18), we denote the angle between the vectors \mathbf{K} and $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{f}$ by ϑ , multiply the expression to be averaged by $2\pi \sin \vartheta d\vartheta/4\pi$, and integrate over $d\vartheta$ from 0 to π . Thus we obtain for J_2 ,

$$\begin{aligned} J_2 &\sim \frac{\kappa}{16\pi^3 \Delta C_p} \frac{1}{2} \int_{\pi}^0 \frac{\sin \vartheta d\vartheta}{2 \frac{\theta - T}{\theta} + a f^2 + a K^2 - 2 a f K \cos \vartheta} \\ &= - \frac{\kappa}{64\pi^3 \Delta C_p a f K} \ln \frac{2 \frac{\theta - T}{\theta} + a(f - k)^2}{2 \frac{\theta - T}{\theta} + a(f + k)^2}. \end{aligned}$$

However,

$$f = |k_2 - k_1| = 2k \sin \frac{\varphi}{2} = K \frac{\sin \frac{\varphi}{2}}{\sin \frac{\varphi_0}{2}};$$

therefore since $\varphi - \varphi_0$ is small,

$$f - K \cong \frac{K}{2} \cot \frac{\varphi_0}{2} (\varphi - \varphi_0) = K(\chi - \chi_0), \quad f + K \cong 2K.$$

Because of the smallness of the difference, $\theta - T$, the term $(\theta - T)/\theta$ in the denominator of the expression for J_2 under the logarithm can be ignored. We then obtain

$$J_2 \sim -\frac{\kappa}{64\pi^3 \Delta C_p a K^2} \ln \left[\frac{(\chi - \chi_0)^2}{4} + \frac{\theta - T}{2a K^2 \theta} \right]. \quad (33)$$

We add (32) and (33), divide the sum by the constant $1/4\pi K^2$ and we have finally for $T < \theta$:

$$J \sim \frac{\theta - T}{\theta} \delta(\chi - \chi_0) - \frac{\kappa}{16\pi^2 \Delta C_p a K^2} \ln \left[\frac{(\chi - \chi_0)^2}{4} + \frac{\theta - T}{2a K^2 \theta} \right] \quad (34)$$

Similarly above the Curie point, for $T > \theta$:

$$J \sim -\frac{\kappa}{16\pi^2 \Delta C_p a K^2} \ln \left[\frac{(\chi - \chi_0)^2}{4} + \frac{T - \theta}{4a K^2 \theta} \right]. \quad (35)$$

We see then that the intensity of the coherent scattering increases slowly with decreasing $\theta - T$. The second term in (34) is then of the order of magnitude

$$\ln \frac{\theta}{\theta - T}. \quad (36)$$

REFERENCE

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33. THE SCATTERING OF X-RAYS BY CRYSTALS WITH VARIABLE LAMELLAR STRUCTURE

* A formula is derived for the distribution of intensities in X-ray lines on scattering from crystals with variable lamellar structure. The shape of the spots in a Laue-diagram and of the lines in a rotational pattern are examined.

A NUMBER of crystals are known to possess "variable structure", i.e. to consist of separate regions of regular structure with lattices shifted relatively one to another^{1,2}. In some cases these regions seem to be plane-parallel layers², so that the crystal can be regarded as consisting of a series of layers with regular structure shifted with respect to one another. This is the case we shall deal with in this paper.

The scattering intensity I of a ray with a wave vector k_1 , the corresponding wave vector of the scattered ray being k_2 , is proportional to the expression

$$I \sim \left| \int \varrho e^{i(k_1 - k_2) \cdot r} dV \right|^2, \quad (1)$$

the integration being extended over the whole volume of the crystal, and ϱ being the density of charge in the crystal. As is known, ϱ can be represented in the form

$$\varrho = \sum a_i e^{i(K_i \cdot r)},$$

where K_i are periods of the inverse lattice. The scattering from one of the faces of the lattice is determined by a corresponding term of this expansion.

Let us consider the scattering from one of the faces of the lattice, i.e. the scattering near one of the Bragg lines. Let K be the corresponding period of the inverse lattice; then we must consider in ϱ only the term proportional to $e^{i(K \cdot r)}$. If the crystal is a single crystal, then inserting $\varrho = \text{const} \cdot e^{i(K \cdot r)}$ in equation (1) and introducing the denotation

$$q = k_1 - k_2 + K, \quad (2)$$

we find that

$$I \sim 8\pi^3 \delta(q) V, \quad (3)$$

where V is the volume of the crystal and $\delta(q)$ is the product of three δ -functions: $\delta(q) = \delta(q_x) \delta(q_y) \delta(q_z)$.

Now let us assume that the crystal has a variable structure and is divided by parallel planes (parallel to the $X Y$ plane) into a series of layers, each pos-

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sessing a regular lattice, identical for all the layers. The lattice of each layer is shifted with respect to that of the preceding layer by a constant vector \mathbf{p} .

We shall assume that \mathbf{p} is the same for all the pairs of layers (an analogous derivation is possible also when \mathbf{p} can have several different values). If the considered part of the density ρ is equal to $C e^{i(\mathbf{K} \cdot \mathbf{r})}$ in one layer, then in the next layer it will evidently be equal to $C e^{i(\mathbf{K} \cdot \mathbf{r} + \mathbf{p})}$.

The integral in (1) breaks into several integrals over each layer,

$$\int \rho e^{i(\mathbf{k}_1 - \mathbf{k}_2 \cdot \mathbf{r})} dV = \sum_{n=0}^{N-1} \int \rho_n e^{i(\mathbf{k}_1 - \mathbf{k}_2 \cdot \mathbf{r})} dV_n,$$

the layers being numbered from 0 to N . Substituting $\rho_n \sim e^{i\mathbf{n}(\mathbf{K} \cdot \mathbf{p})} e^{i(\mathbf{K} \cdot \mathbf{r})}$ for the n th layer and introducing the notation

$$(\mathbf{K} \cdot \mathbf{p}) = \chi,$$

we obtain

$$I \sim \left| \int \rho e^{i(\mathbf{k}_1 - \mathbf{k}_2 \cdot \mathbf{r})} dV \right|^2 \sim \left| \sum_{n=0}^{N-1} e^{i\mathbf{n}\chi} \int e^{i(\mathbf{q} \cdot \mathbf{r})} dV_n \right|^2.$$

The integration with respect to x and y is extended over the entire area S of the cross-section of the crystal and yields, analogously to (3), $4\pi^2 S \delta(q_x) \delta(q_y)$. The integration with respect to z in the n th layer is carried out within the limits z_n and z_{n+1} , z_n being the coordinate of the n th boundary plane between the layers. We finally obtain

$$I \sim \frac{4\pi^2 S \delta(q_x) \delta(q_y)}{q_z^2} \left| \sum_{n=0}^{N-1} e^{i\mathbf{n}\chi} (e^{iq_z z_{n+1}} - e^{iq_z z_n}) \right|^2$$

N being very large, this may be written in the form

$$I \sim \frac{4\pi^2 S \delta(q_x) \delta(q_y)}{q_z^2} |e^{i\chi} - 1|^2 \left| \sum_{n=0}^{N-1} e^{i\mathbf{n}\chi} e^{iq_z z_n} \right|^2. \quad (4)$$

The square of the modulus of the sum can be represented as a double sum:

$$\left| \sum_{n=0}^{N-1} e^{i\mathbf{n}\chi} e^{iq_z z_n} \right|^2 = \sum_{n=0}^{N-1} \sum_{l=-n}^{N-1-n} e^{i\mathbf{l}\chi} e^{iq_z (z_n - z_{n-l})}. \quad (5)$$

As N is large, this may be replaced by

$$N \sum_{l=-\infty}^{+\infty} e^{i\mathbf{l}\chi} \overline{e^{iq_z (z_n - z_{n-l})}},$$

where the bar indicates the mean value, the averaging being taken over the different n .

Let us introduce the thicknesses d_n of the layers

$$d_n = z_{n+1} - z_n. \quad (6)$$

Then, with $l > 0$,

$$z_n - z_{n-l} = d_{n-l} + d_{n-l+1} + \dots + d_{n-1}$$

and with $l < 0$:

$$z_n - z_{n-l} = -d_n - d_{n+1} - \dots - d_{n-l-1}.$$

Let us assume that no correlation exists between the thicknesses of different layers, i.e. the probability of one layer having some thickness or other does not depend on the probabilities for the other layers. Let us denote by ξ the mean value

$$\xi = \overline{e^{iq_z d}}, \quad (7)$$

where d is the thickness of one of the layers. There being no correlation between the thicknesses of the layers we can evidently write, if $l > 0$,

$$\overline{e^{iq_z(z_n - z_{n-l})}} = \overline{(e^{iq_z d})^l} = \xi^l$$

and, if $l < 0$,

$$\overline{e^{iq_z(z_n - z_{n-l})}} = \overline{(e^{-iq_z d})^l} = \xi^{*l},$$

ξ^* being the complex conjugate of ξ .

We then obtain approximately

$$\begin{aligned} \left| \sum_0^{N-1} e^{in\chi} e^{iq_z z_n} \right|^2 &\cong N \left(1 + \sum_1^\infty \xi^l e^{il\chi} + \sum_1^\infty \xi^{*l} e^{-il\chi} \right) \\ &= N \left(1 + \frac{\xi e^{i\chi}}{1 - \xi e^{i\chi}} + \frac{\xi^* e^{-i\chi}}{1 - \xi^* e^{-i\chi}} \right) = N \frac{1 - \xi \xi^*}{(1 - \xi e^{i\chi})(1 - \xi^* e^{-i\chi})} \end{aligned}$$

(obviously, $|\xi| < 1$ and therefore the progressions converge). Substituting this in equation (4), we find

$$I \sim \frac{16\pi^2 S N}{q_z^2} \delta(q_x) \delta(q_y) \frac{\sin^2 \frac{\chi}{2} (1 - |\xi|^2)}{|1 - \xi e^{i\chi}|^2}. \quad (8)$$

Since the absence of correlation has been already assumed it is natural to proceed to determine the probabilities $w(d)$ of the thicknesses of the layers, assuming that the probability for a layer to increase in thickness on its formation by a given value is independent of the thickness it already possesses. This leads in a well known manner to a distribution of probabilities of the form $w(d) \sim e^{-\text{const} \cdot d}$. The normalisation gives

$$w(d) = \frac{e^{-d/\bar{d}}}{\bar{d}}, \quad (9)$$

where \bar{d} is the mean thickness. Using (9) we find that

$$\xi = \frac{1}{1 - i q_z \bar{d}}$$

and (8) is transformed to

$$I \sim \frac{16\pi^2 S}{q_z^2} \delta(q_x) \delta(q_y) N \frac{q_z^2 \bar{d}^2 \sin^2 \frac{\chi}{2}}{\left[4 \sin^2 \frac{\chi}{2} + 2 \sin \chi \cdot q_z \bar{d} + q_z^2 \bar{d}^2 \right]}. \quad (10)$$

Let us choose the coefficient in this expression so that with the same normalisation the intensity I for single crystals with regular structure shall have the simple form

$$I \sim \delta(q).$$

As appears from (3) this will be attained by dividing (10) by $8\pi^3 V = 8\pi^2 N \bar{d} S$. We finally obtain

$$I \sim \frac{2\bar{d} \sin^2 \frac{\chi}{2}}{\pi} \frac{\delta(q_x) \delta(q_y)}{(q_z \bar{d} + \sin \chi)^2 + (1 - \cos \chi)^2}. \quad (11)$$

This expression has with respect to q_z the so-called dispersion form with its maximum shifted relatively to the Bragg line $q_z = 0$ by

$$\Delta q_z = -\frac{\sin \chi}{\bar{d}}. \quad (12)$$

The width δq_z of the line is

$$\delta q_z = \frac{1 - \cos \chi}{\bar{d}} \quad (13)$$

q has a component along the z axis only, as appears from (11) containing $\delta(q_x) \delta(q_y)$. We can therefore write

$$k_1 - k_2 + K = q_z z, \quad (14)$$

where z is a unit vector along the z axis. k_2 and k_1 have the same absolute magnitude (the frequency being unaltered on scattering) $|k_1| = |k_2| = k$. We shall denote the unit vectors along the directions k_2 and k_1 by n_2 and n_1 respectively.

Let us use (11) for determining the shape of the spot in the Laue pattern. The crystal is here immobile, i.e. $K = \text{const}$. The incident beam has a constant direction ($n_1 = \text{const}$) but all possible frequencies, i.e. k . Let k_{10} and $k_{20} = k_0 n_{20}$ be wave vectors of the incident and the scattered ray satisfying the Bragg condition

$$k_{20} - k_{10} = k_0(n_{20} - n_1) = K \quad (15)$$

and δk_1 and δk_2 the deviations of k_1 and k_2 from these values. Inserting $k_1 = k_{10} + \delta k_1$, $k_2 = k_{20} + \delta k_2$ in (14) we obtain

$$\delta k_1 - \delta k_2 = -\delta k(n_{20} - n_1) - k \delta n_2 = q_z z$$

or, inserting (15),

$$\frac{K}{k_0} \delta k + k_0 \delta n_2 = -q_z z \quad (16)$$

δn_2 being a small variation of the unit vector n_2 , it must be perpendicular to n_{20} ; moreover, as appears from (16), it lies in the plane of the vectors z , K . We can therefore write

$$\delta n_2 = \alpha [n_{20} \wedge [z \wedge K]].$$

where α is a certain constant. Inserting this in (16) and multiplying both sides vectorially by \mathbf{K} we find

$$q_z = -\alpha k_0 (\mathbf{K} \cdot \mathbf{n}_{20}).$$

Thus

$$\delta \mathbf{n}_2 = \frac{q_z [\mathbf{z} \wedge \mathbf{K}] \wedge \mathbf{n}_{20}}{k_0 (\mathbf{K} \cdot \mathbf{n}_{20})}. \quad (17)$$

Thus the spot widens in the direction determined by the vector $[[\mathbf{z} \wedge \mathbf{K}] \wedge \mathbf{n}_{20}]$. The absolute magnitude of $\delta \mathbf{n}_2$ is evidently represented by the angle φ between the direction \mathbf{n}_2 of the scattered ray and its direction \mathbf{n}_{20} when fulfilling the Bragg condition.

From (12), (13) and (17) we find that the line has the dispersion form, the displacement $\Delta \varphi$ of the centre of the spot with regard to its Bragg position is

$$\Delta \varphi = -\frac{\sin \chi}{\bar{d}} \frac{|[[\mathbf{z} \wedge \mathbf{K}] \wedge \mathbf{n}_{20}]|}{k_0 (\mathbf{K} \cdot \mathbf{n}_{20})}$$

and the width of the spot is

$$\delta \varphi = \frac{1 - \cos \chi}{\bar{d}} \frac{|[[\mathbf{z} \wedge \mathbf{K}] \wedge \mathbf{n}_{20}]|}{k_0 (\mathbf{K} \cdot \mathbf{n}_{20})}. \quad (18)$$

Let us now use (12) and (13) to determine the form of the line in the rotational diagram. There the incident beam has a definite direction and frequency $k_1 = \text{const}$, i.e. $\mathbf{n}_1 = \text{const}$, $k = \text{const}$ and the crystal rotates, i.e. \mathbf{K} has a variable direction. With the Bragg condition satisfied let again $\mathbf{K} = \mathbf{K}_0$, $k_2 = k_{20}$,

$$k_{20} - k_1 = K_0.$$

The substitution $k_2 = k_{20} + \delta k_2$, $\mathbf{K} = \mathbf{K}_0 + \delta \mathbf{K}$ now gives

$$\delta \mathbf{K} - \delta k_2 = \delta \mathbf{K} - k \delta \mathbf{n}_2 = q_z \mathbf{z}_0 \quad (19)$$

(for \mathbf{z} also we must now write $\mathbf{z} = \mathbf{z}_0 + \delta \mathbf{z}$; q_z being small in itself, the term $q_z \delta \mathbf{z}$ may be neglected). Keeping in mind that $\delta \mathbf{K}$, being a small variation of the vector \mathbf{K} without altering its absolute magnitude, is perpendicular to \mathbf{K}_0 , we find analogously to the foregoing:

$$\delta \mathbf{n}_2 = \alpha [\mathbf{n}_{20} \wedge [\mathbf{z}_0 \wedge \delta \mathbf{K}]],$$

$$q_z = -\alpha k (\mathbf{n}_{20} \cdot \delta \mathbf{K}),$$

whence

$$\delta \mathbf{n}_2 = \frac{q_z [[\mathbf{z}_0 \wedge \delta \mathbf{K}] \wedge \mathbf{n}_{20}]}{k (\mathbf{n}_{20} \cdot \delta \mathbf{K})}. \quad (20)$$

We must here substitute for $\delta \mathbf{K}$ the well known expressions for the variation of the vector on turning.

$$\delta \mathbf{K} = [\delta \Omega \wedge \mathbf{K}],$$

where $\delta \Omega$ is a vector equal in magnitude to the angle of rotation and directed along the axis about which the rotation takes place. Of course the absolute magnitude of $\delta \Omega$ will cancel in (20). Analogously to formulae (18) we now have:

$$\left. \begin{aligned} \Delta \varphi &= -\frac{\sin \chi}{\bar{d}} \frac{|[z_0 \wedge \delta K] \wedge n_{20}|}{k(n_{20} \cdot \delta K)}, \\ \delta \varphi &= \frac{1 - \cos \chi}{\bar{d}} \frac{|[z_0 \wedge \delta K] \wedge n_{20}|}{k(n_{20} \cdot \delta K)}. \end{aligned} \right\} \quad (21)$$

Usually χ is near either to zero or π . Of course, a noticeable expansion occurs only in the second case.

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34. PRODUCTION OF SHOWERS BY HEAVY PARTICLES

CARLSON and Oppenheimer¹ and Bhabha and Heitler² have proposed a very ingenious explanation of showers by a multiplicative production of photons, electrons and positrons. The greater part of the showers observed can in this way be regarded as originated by electrons, positrons or photons penetrating from the atmosphere. Such showers are absorbed if the layer of lead is sufficiently thick. It is known, however, that at sea-level and especially under earth, the frequency of showers with increasing thickness of lead layer does not diminish but tends to a limit. We wish to point out that this phenomenon does not require a new concept of showers, but can be explained by irradiation with heavy particles, which easily penetrate through such layers of lead.

A more rigorous treatment of showers³ shows that a shower has a certain range given by the formula

$$L = 0.25 \frac{\hbar c}{e^2} \left(\frac{m c^2}{e^2} \right)^2 \frac{1}{\bar{Z} N} \ln \frac{E}{\varepsilon},$$

where \hbar , c , e have the usual meaning; m is the electronic mass; N is the number of electrons per cm^3 ; $\bar{Z} = \sum N_z Z(Z+1) / \sum N_z Z$ (N_z is number of atoms with charge Ze); E is the energy of the incident particle and ε a critical energy of the order of magnitude $750/\bar{Z}$ MeV.

If we combine this formula with the well-known formula for radiation probability for a heavy particle, we get for the probability that a shower produced by a photon emitted by a heavy particle gets out of the layer of sufficient thickness:

$$W = 0.8 \left(\frac{m}{M} \right)^2 \left(\ln \frac{E}{\varepsilon} \right)^2 \ln \frac{E}{(M c^2)^{3/4} \varepsilon^{1/4}};$$

where M is the mass of the heavy particle and E its energy, which is assumed $\gg M c^2$. If we take for M the value of some tens of electronic mass proposed by Anderson and Neddermeyer⁴, we get a fair agreement with the observed order of magnitude.

Details of the calculations will be published elsewhere.

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L. Landau and G. Rumer, Production of showers by heavy particles, *Nature*, **140**, 682 (1937).

† *Editor's note*: This paper was apparently never published; see, however, Collected Papers No. 36, p. 252.

35. STABILITY OF NEON AND CARBON WITH RESPECT TO α -PARTICLE DISINTEGRATION

KALCKAR, Oppenheimer and Serber in a recent article¹ have proposed as an explanation for the experimentally observed stability of compound nuclei ^{20}Ne , ^{12}C against α -decay a small l - s coupling. The smallness of coupling would mean that the energy of coupling is small compared with the separation of energy levels, but it is well known that the distance between energy levels in heavy nuclei is very small and in the above elements is surely much less than the energy of l - s coupling. Therefore such an explanation of the anomalous stability seems untenable.

I should like to point out that there is no need to look for special explanations, because the observed phenomenon can be explained by the fundamental conservation laws which, as is well known in wave mechanics, include also the conservation of parity. As already pointed out by Fermi for atoms it can sometimes occur that a decay which is energetically possible cannot take place because of this conservation law. Let us assume for instance that the state in question is an odd state with angular momentum zero and the states of the disintegration particles are even states with momenta also zero. Then it is easy to see that no states of relative motion can satisfy both the laws of conservation of angular momentum and of parity.

REFERENCE

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36. THE CASCADE THEORY OF ELECTRONIC SHOWERS

1. Starting from the physical idea proposed by Bhabha and Heitler, and Carlson and Oppenheimer, on the cascade production of showers, a more rigorous treatment of the problem is given (§§ 1-5).

2. A numerical table permits the calculation of the number of shower particles as a function of depth of penetration for any given initial energy of the creating particle, and also the energy distribution for shower particles at a given depth (§ 6).

3. The showers possess a definite range (§ 7).

4. The behaviour of a shower passing from air to a medium with a different atomic number is studied (§ 9).

5. The consequences arising from the hypothesis of the new "semi-heavy" particles are discussed.

1. Bhabha and Heitler¹ and also Carlson and Oppenheimer² have recently proposed a very ingenious theory of showers which explains nearly all the observed phenomena; both these treatments, however, contain approximations the validity of which cannot always be justified. The calculations of the authors make use of the method of successive approximations whose exactness is difficult to estimate, while the second authors replace one of the fundamental equations by a different one, and it is impossible to see how this replacement may distort the final result. Consequently, we have thought it useful to make the calculation in a more rigorous way; the fundamental physical ideas, however, are exactly the same as those of the authors mentioned above.

2. Consider a fast electron penetrating into a layer of matter; as long as its energy is below a certain order of magnitude, the loss of energy of the electron will be determined mainly by ionization; for higher energies, however, loss of energy by radiation (Bremsstrahlung) becomes more important. This critical energy lies in the region of about $\varepsilon = 750/\bar{Z}$ MeV, where \bar{Z} denotes a certain average nuclear charge (there may be several nuclei of different kinds present). Since the collision radiation, which is determined by the interaction between the electron and the nuclear field, increases as Z^2 , while the ionization is approximately proportional to the number of electrons in the material, the most suitable definition of \bar{Z} is

$$\bar{Z} = \sum_i n_i Z_i (Z_i + 1) / N, \quad (1)$$

where n_i is the number per cm³ of nuclei having the charge Z_i , and N is the total number of electrons per cm³. In the numerator we have put $Z_i(Z_i + 1)$

¹L. Landau and G. Rumer, The cascade theory of electronic showers, *Proc. Roy. Soc. A* 166, 213 (1938).

instead of Z_1^2 , since in the case of collision radiation we have to consider not only interactions with the nuclei but also with the electrons; it is important to remember this in the case of hydrogen. This same critical energy ϵ is also that for which the absorption of photons by pair production rather than by the Compton effect begins to become important.

3. Consider a particle (electron, positron) or photon moving normally to the boundary of a sufficiently thick layer of matter. If the energy of the particle inside this layer is high enough, both photons (produced by collision radiation) and new "material particles" (produced by pair production) will appear—in other words a shower will be generated. Let us consider how the number of particles $\Pi(E) dE$ and the number of photons $\Gamma(E) dE$, with energies in the range dE and at distance x from the surface of the layer, will change in consequence of pair production and collision radiation (we assume the energy E to be so high that the Compton effect and ionization can be neglected).

Let $\gamma(E, E') dE'$ be the probability per unit path length of the production of an electron of energy between E' and $E' + dE'$ by a photon of energy E , and let $\pi(E, E') dE'$ be the probability of the radiation of a photon of energy between E' and $E' + dE'$ by an electron of energy E . The rate of change of $\Pi(E)$ caused by pair production will consequently be

$$\left(\frac{d\Pi(E)}{dx} \right)_P = 2 \int_E^\infty \Gamma(u) \gamma(u, E) du. \quad (2)$$

The factor 2 occurs because there are always two particles (electron and positron) created with equal probabilities. Similarly the rate of change of $\Pi(E)$ caused by radiation will be

$$\left(\frac{d\Pi(E)}{dx} \right)_R = \int_E^\infty \Pi(u) \pi(u, u-E) du - \int_0^E \Pi(E) \pi(E, E-u) du. \quad (3)$$

The first term gives the number of particles of energy greater than E , which have lost energy by radiating a photon, thus bringing them into the energy range between E and $E + dE$; the second term gives the number of particles which leaves this interval by radiating a photon. For the rate of change of the number of photons $\Gamma(E)$ we have in a similar way

$$\left(\frac{d\Gamma(E)}{dx} \right)_P = - \int_0^E \Gamma(E) \gamma(E, u) du, \quad (4)$$

$$\left(\frac{d\Gamma(E)}{dx} \right)_R = \int_E^\infty \Pi(u) \pi(u, E) du. \quad (5)$$

Thus for the total value of the change of the numbers of material particles and photons, we shall have:

$$\frac{d\Pi(E)}{dx} = 2 \int_E^\infty \Gamma(u) \gamma(u, E) du + \int_E^\infty \Pi(u) \pi(u, u-E) du - \int_0^E \Pi(E) \pi(E, E-u) du, \quad (6)$$

$$\frac{d\Gamma(E)}{dx} = \int_E^\infty \Pi(u) \pi(u, E) du - \int_0^E \Gamma(E) \gamma(E, u) du. \quad (7)$$

This is the basis of the cascade theory of showers.

4. The expressions for $\gamma(E, E')$ and $\pi(E, E')$ have been given by Bethe and Heitler². For the region of energies which concern us here, the screening can be considered as complete and we have

$$\gamma(E, E') = A \frac{E'^2 + (E - E')^2 + \frac{2}{3} E' (E - E')}{E^3}, \quad (8)$$

$$\pi(E, E') = A \frac{E^2 + (E - E')^2 - \frac{2}{3} E (E - E')}{E^2 E'}, \quad (9)$$

where

$$A = \frac{4}{137} \left(\frac{e^2}{m c^2} \right)^2 N \bar{Z} \ln 183 \bar{Z}^{-1/3} \text{ cm}^{-1}. \quad (10)$$

Putting in numerical values we find

$$A = 1.19 \times 10^{-26} N \bar{Z} (1 - 0.067 \ln \bar{Z}). \quad (10a)$$

It is convenient to follow Carlson and Oppenheimer³ and measure lengths x in terms of the dimensionless unit, so that the length is defined by

$$x = \frac{t}{A}. \quad (11)$$

We obtain the lengths in cm which correspond to unit value of t for various materials, shown in Table 1, together with the values of \bar{Z} and ε .

TABLE 1

	\bar{Z}	ε (eV)	Equivalent thickness in cm corresponding to $t = 1$
Al	14	50×10^6	9.2
Pb	83	9×10^6	0.52
H ₂ O	7.6	100×10^6	38.0
Air	8.3	90×10^6	39.0 (height of water barometer)

The height of the atmosphere is $t = 26$. When this is done, all the material constants disappear from the fundamental equations, which then assume the form

$$\frac{d\Pi(E)}{dt} = \int_E^\infty \Pi(u) \frac{u^2 + E^2 - \frac{2}{3}uE}{u^2(u-E)} du - \int_0^E \Pi(E) \frac{E^2 + u^2 - \frac{2}{3}uE}{E^2(E-u)} du + \int_E^\infty \Gamma(u) \frac{2u^2 - \frac{8}{3}uE + \frac{8}{3}E^2}{u^3} du, \quad (6a)$$

$$\frac{d\Gamma(E)}{dt} = \int_E^\infty \Gamma(u) \frac{\frac{4}{3}u^2 - \frac{4}{3}uE + E^2}{u^2E} du - \int_0^E \Gamma(E) \frac{\frac{4}{3}u^2 - \frac{4}{3}uE + E^2}{E^3} du. \quad (7a)$$

For the discussion of these equations we introduce a new independent variable s in the place of E , in the following way: if $f(E)$ is any function of E , and f_s is defined by

$$f_s = \int_0^\infty f(E) E^s dE \quad (0 \leq s < \infty), \quad (12)$$

then f_s is defined uniquely in terms of $f(E)$ and the converse may also be proved. Thus to any equation valid in a complete range of E there corresponds an equation in s , and vice versa. We now transform each term of equations (6a) and (7a) to the new variables s , and also write $\xi = E/u$

$$\begin{aligned} \int_0^\infty E^s dE \int_0^\infty \Pi(u) \frac{u^2 + E^2 - \frac{2}{3}uE}{u^2(u-E)} du - \int_0^\infty \Pi(E) E^s dE \int_0^E \frac{E^2 + u^2 - \frac{2}{3}uE}{E^2(E-u)} du \\ = - \int_0^\infty \Pi(u) u^s du \int_0^1 \frac{(1-\xi^s)}{1-\xi} \left(1 - \frac{2}{3}\xi + \xi^2\right) d\xi, \\ \int_0^\infty E^s dE \int_E^\infty \Gamma(u) \frac{2u^2 - \frac{8}{3}uE + \frac{8}{3}E^2}{u^3} du = \int_0^\infty \Gamma(u) u^s \int_0^1 \xi^s \left(\frac{8}{3}\xi^2 - \frac{8}{3}\xi + 2\right) d\xi, \\ \int_0^\infty E^s dE \int_E^\infty \Pi(u) \frac{\frac{4}{3}u^2 - \frac{4}{3}uE + E^2}{u^2E} du \\ = \int_0^\infty \Pi(u) u^s du \int_0^1 \xi^{s-1} \left(\xi^2 + \frac{4}{3}\xi - \frac{4}{3}\right) d\xi, \\ \int_0^\infty \Gamma(E) E^s dE \int_0^E \frac{\frac{4}{3}u^2 - \frac{4}{3}uE + E^2}{E^3} du \\ = \int_0^\infty \Gamma(E) E^s dE \int_0^1 \left(\frac{4}{3}\frac{1}{\xi^2} - \frac{4}{3}\frac{1}{\xi} + 1\right) d\frac{1}{\xi}. \end{aligned}$$

The definite integrals with respect to ξ occurring in these expressions, we shall denote by $A(s)$, $B(s)$, $C(s)$ and D , i.e.

$$A(s) = \int_0^1 (1 - \xi^s)(1 - \xi)^{-1} \left(1 - \frac{2}{3}\xi + \xi^2\right) d\xi = \frac{4}{3} \{\psi(s) + \gamma\} - \frac{s(5s+7)}{6(s+1)(s+2)}, \quad (13)$$

$$B(s) = \int_0^1 \xi^s \left(2 - \frac{8}{3}\xi + \frac{8}{3}\xi^2\right) d\xi = \frac{2}{1+s} - \frac{\frac{8}{3}}{2+s} + \frac{\frac{8}{3}}{3+s}, \quad (14)$$

$$C(s) = \int_0^1 \xi^{s-1} \left(\xi^2 + \frac{4}{3}\xi - \frac{4}{3}\right) d\xi = \frac{\frac{4}{3}}{s} - \frac{\frac{4}{3}}{s+1} + \frac{1}{s+2}, \quad (15)$$

$$D = \int_0^1 \left(\frac{4}{3}\xi^2 - \frac{4}{3}\xi + 1\right) d\xi = \frac{7}{9}, \quad (16)$$

where $\psi(s)$ is defined by $\psi(s) = (d/ds) \ln s!$, and γ is Euler's constant.

Thus in terms of the new variable s , the system of fundamental equations assumes the simple form

$$\frac{d\Pi_s}{dt} = -A(s)\Pi_s + B(s)\Gamma_s, \quad \frac{d\Gamma_s}{dt} = C(s)\Pi_s - D\Gamma_s. \quad (17)$$

5. As is well known, the solutions of (17) are given by

$$\left. \begin{aligned} \Pi_s(t) &= a_s \exp(-\lambda_s t) + b_s \exp(-\mu_s t), \\ \Gamma_s(t) &= c_s \exp(-\lambda_s t) + d_s \exp(-\mu_s t), \end{aligned} \right\} \quad (18)$$

where λ_s and μ_s are the roots of the secular equations

$$\begin{vmatrix} -A(s) + x & B(s) \\ C(s) & -D + x \end{vmatrix} = 0, \quad (19)$$

so that

$$\begin{aligned} \lambda(s) = \lambda_s &= \frac{1}{2} \{A(s) + D\} - \frac{1}{2} \sqrt{\{A(s) - D\}^2 + 4B(s)C(s)} \\ \mu(s) = \mu_s &= \frac{1}{2} \{A(s) + D\} + \frac{1}{2} \sqrt{\{A(s) - D\}^2 + 4B(s)C(s)}, \end{aligned}$$

and a_s , b_s , c_s , d_s are constants determined by the initial conditions ($t = 0$). The case which interests us is when the rate of change of the number of particles is large, i.e. when the exponents are large, and consequently we consider all quantities which do not vary exponentially as being of order of magnitude

unity.† With this approximation we can neglect the solutions corresponding to μ_s , and approximately satisfy the initial conditions by using only the λ_s -solutions.

An examination of the secular equation (19) shows that for change of s in the interval $0 \leq s \leq \infty$, the function $\lambda(s)$ increases from $-\infty$ to the limiting value $7/9$.

6. If only a single photon passes through the layer with energy E_0 , the initial conditions are

$$\Pi(E) = 0, \quad I(E) = \delta(E - E_0);$$

thus to our approximation we have

$$\int_0^\infty \{\Pi(E) + \Gamma(E)\} E^s dE = E_0^s \exp(-\lambda_s t). \quad (20)$$

Following Bhabha and Heitler¹ we measure the energy by means of the dimensionless quantity y , so that

$$y = \ln \frac{E_0}{E} \quad \text{and} \quad \eta = \ln \frac{E_0}{\varepsilon}, \quad (21)$$

and η will then be the quantity corresponding to the critical energy. Let $P(y, t) = \exp \varphi(y, t)$ be the distribution function with respect to y so that‡

$$\{\Gamma(E) + \Pi(E)\} dE = -P dy. \quad (22)$$

The expression (20) then becomes

$$\int_{-\infty}^\infty \exp\{\varphi(y, t) - sy\} dy = \exp(-\lambda_s t). \quad (23)$$

The integrand has a sharp maximum for a certain value of y , say \bar{y} ; then \bar{y} will be the root of the equation

$$\frac{\partial \varphi(y, t)}{\partial y} = s. \quad (24)$$

Neglecting the non-exponential terms, we obtain from (23)

$$\varphi(\bar{y}, t) - s\bar{y} = -\lambda(s)t. \quad (25)$$

From equations (24) and (25) knowing the function $\varphi(y, t)$, we could determine the function $\lambda(s)$; since, however, it is $\lambda(s)$ which is given, and we wish

† The assumption that the exponentials are respectively large or small implies that the total number of secondary particles is large. This assumption, absolutely necessary, if radiation and creation of pairs are treated separately from ionization and Compton effect, is of course made (together with other assumptions which we have avoided) by Bhabha and Heitler and by Carlson and Oppenheimer.

‡ When in the following we refer to the number of particles we mean both charged particles and light quanta.

to find the function $\varphi(y, t) = \ln P(y, t)$, we differentiate equation (25) with respect to s , which gives

$$\frac{\partial \varphi}{\partial \bar{y}} \cdot \frac{d\bar{y}}{ds} - y - s \frac{d\bar{y}}{ds} = -\lambda'(s)t.$$

On account of (24) this simplifies to

$$\bar{y} = \lambda'(s)t. \quad (26)$$

We substitute this value \bar{y} in equation (25), and write simply y instead of \bar{y} in what follows.

The equations

$$y = t\lambda'(s), \quad \varphi = t\{s\lambda'(s) - \lambda(s)\}, \quad (27)$$

give a parametric representation of φ as a function of y and t , and in Fig. 1

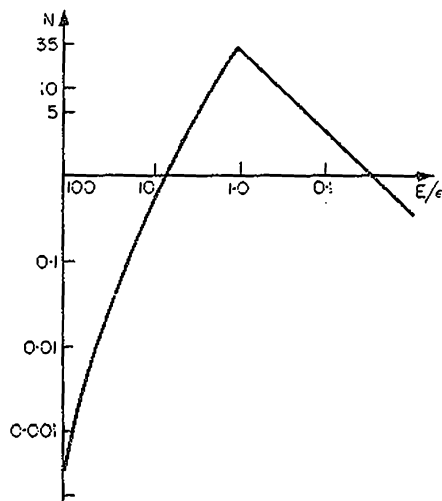


FIG. 1

we show the dependence of φ on y for the particular values $t = 10$ and $E_0/\epsilon = 100$ (for other values the scale of Fig. 1 must be correspondingly altered). Since it follows from (24) that

$$\frac{\partial \varphi}{\partial y} = s > 0,$$

we see then that the function increases monotonically, or in other words, the number of created particles observed in a given thickness t of the layer increases with y , i.e. with diminishing energy of these created particles.

The expression for φ ceases to be valid as soon as the energy of the created particles falls to the order of magnitude of the critical energy, since the ioniza-

tion and Compton effect which we could neglect for higher energies, then become increasingly important. As can be seen from equations (8) and (9) photons of small energy are formed in much larger quantities, by collision radiation, than are material particles of small energy by pair production. The number of created photons with energies in the range E' , $E' + dE'$ is proportional to $1/E'$. If $E' \ll \epsilon$ then the absorption of these photons is determined mainly by the Compton effect and not by pair production: consequently in this energy region the absorption coefficient is nearly proportional to $1/E'$ (as follows from the theory of the Compton effect), i.e. the numbers of photons $\Gamma(E)$ for $E < \epsilon$ is independent of the energy.

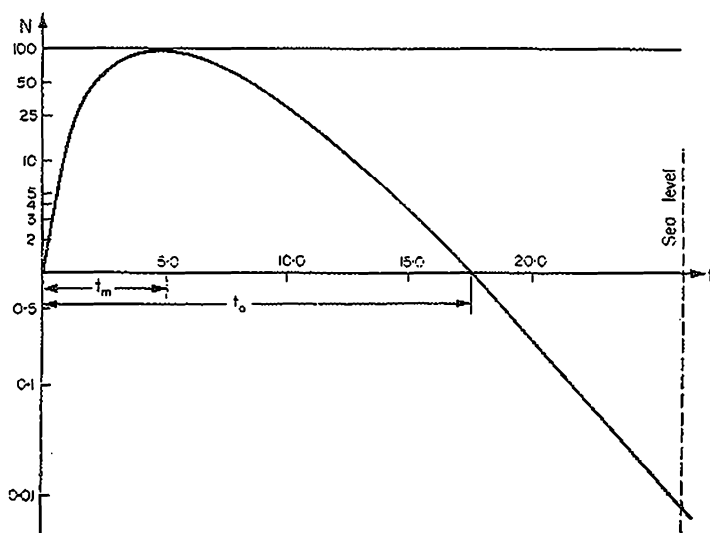


FIG. 2

As can be shown by a more detailed analysis, the same is true of the number of particles $\Pi(E)$ for $E < \epsilon'$. If we go over from E to the logarithmic measure y we find in consequence of (21) that the distribution function $P(y, t)$ for $y > \eta$ has the form

$$P(y, t) = \exp \{ \text{const} - y \}.$$

We see then that the number of particles for given penetration depth t increases with diminution of energy until the critical region is reached, when further diminution of the energy diminishes the number of particles. This means that for given penetration depth the majority of particles in a shower will have the energy ϵ .

The distribution of particles along the path of penetration into the layer of matter is given by the function which is shown graphically in Fig. 2 for $\eta = \ln 100$. For other values of η the scale has to be correspondingly changed. We give in Table 2 the corresponding values of s , t/y , ϕ/y and λ . The curve

has a maximum at the point $t = t_m$, for which the derivative $\partial \varphi / \partial t$ vanishes. From (25) we have

$$\delta \varphi - \eta ds - s d\eta = -\lambda dt - t d\lambda.$$

Since $d\eta = 0$ and, on account of (26), $\eta ds = t d\lambda$, we obtain

$$\frac{\partial \varphi}{\partial t} = -\lambda(s). \quad (28)$$

From the law of conservation of energy it follows that

$$\frac{d}{dt} \{I_1 + II_1\} = 0;$$

hence $\lambda(1) = 0$; this can of course also be proved by a direct calculation.

7. Equation (26) gives for the position of the maximum

$$t_m = \frac{\eta}{\lambda'(1)}, \quad \varphi_{\max} = \eta,$$

TABLE 2

t/y	φ/y	λ	s
0.04	0.25	-3.76	0.1
0.11	0.44	-2.26	0.2
0.19	0.59	-1.56	0.3
0.28	0.71	-1.12	0.4
0.37	0.80	-1.80	0.5
0.48	0.88	-0.57	0.6
0.60	0.93	-0.38	0.7
0.72	0.97	-0.23	0.8
0.86	0.99	-0.11	0.9
1.01	1.00	0.00	1.0
1.18	0.99	0.09	1.1
1.37	0.97	0.17	1.2
1.58	0.92	0.24	1.3
1.81	0.86	0.30	1.4
2.07	0.78	0.35	1.5
2.36	0.67	0.39	1.6
2.69	0.53	0.44	1.7
3.07	0.36	0.47	1.8
3.49	0.16	0.50	1.9
3.96	-0.08	0.53	2.0
4.49	-0.37	0.55	2.1
5.08	-0.70	0.57	2.2
5.73	-1.08	0.59	2.3
6.46	-1.52	0.61	2.4
7.29	-2.03	0.62	2.5
8.17	-2.67	0.63	2.6
...
∞	$-\infty$	0.78	∞

i.e. the maximum number of particles will be observed at a depth

$$t_m = 1.01 \ln \frac{E}{\varepsilon},$$

and will be equal to E_0/ε particles. The length of the shower track will be, expressed in t -units,

$$t_0 = 3.8 \ln \frac{E}{\varepsilon},$$

which gives for air $1.48 \ln E_0/100$ m of water barometer.

Let us calculate the fraction of the number of particles which, having penetrated the atmosphere with an initial energy of 10^{10} eV, can be observed at sea-level; we have $\eta = \ln 100 = 4.6$ and $t/\eta = 26/4.6 = 5.65$, and from Table 2 we find the corresponding value of $\varphi/\eta = -1.03$ so that $\varphi = -4.74$. Consequently the required fraction is $\exp(-4.74)$, or slightly less than 1 per cent. As regards the particles penetrating the atmosphere with energies of 3×10^9 eV, only 1 in 1000 can reach sea level.

In the third column of Table 2 we have written down the values of λ . Now equation (28) shows that λ is the coefficient of absorption and we see that the limiting value of λ (for large t) is $7/9$, which gives 2 per metre of water barometer in the case of air. If we estimate the energy of the particles entering the atmosphere as of order of magnitude $3 \times 10^9 - 10^{10}$ eV, we obtain 0.6 as the order of magnitude of λ , and consequently an absorption coefficient 1.5 per metre of water barometer. This then is the size of absorption coefficient we should expect to find for light particles at sea-level.

The total energy of the shower is given by

$$E(t) = \int_0^\infty \{\Gamma(E) + \Pi(E)\} E \, dE = E_0 \int_{-\infty}^\infty \exp\{\varphi(y, t) - y\} \, dy. \quad (29)$$

The integrand has a sharp maximum at the point $y = \bar{y}$, where the derivative of the exponent vanishes, i.e.

$$\frac{\partial \varphi}{\partial y} = 1 = s \left(\frac{t}{y} \right) \quad (30)$$

As can be seen from Table 2 $\bar{y} \doteq t$; if $\bar{y} < \eta$, $\varphi(\bar{y}, t) = \bar{y}$ and the integral becomes of order unity, and we find that the total energy of the shower is of order E_0 . For $\bar{y} > \eta$ the maximum of the integrand corresponds to the point $y = \eta$ and we have

$$E(t) = E_0, \quad t < \eta$$

$$E(t) = E_0 \exp\{\varphi(\eta, t) - \eta\}, \quad t > \eta.$$

The graph of this function is shown in Fig. 3.

8. In principle it is possible to deduce the spectral distribution of the particles entering the atmosphere and producing showers along their path from the experimentally observable number of particles $N(t)$ per unit length of

path. Let $\exp \xi(\eta)$ be the number of particles entering the atmosphere with energy E_0 ; each of these will produce a shower, so that the total number of particles $N(t)$ per unit length observed at depth t is

$$N(t) = \int \exp \{ \xi(\eta) + \varphi(\eta, t) \} d\eta. \quad (31)$$

The integrand has a maximum at the point $\eta = \bar{\eta}$, where the derivative $\xi'(\eta) + \partial \varphi(\eta, t) / \partial \eta$ vanishes, and $\bar{\eta}$ of course depends on t . The majority of particles observed at different heights are formed by particles of different energies. In our approximation we can write

$$N(t) = \exp \{ \xi(\bar{\eta}) + \varphi(\bar{\eta}, t) \},$$

and we have

$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{\partial \varphi}{\partial t} + \left\{ \xi'(\bar{\eta}) + \frac{\partial \varphi(\bar{\eta}, t)}{\partial \bar{\eta}} \right\} \frac{d\bar{\eta}}{dt} = -\lambda \left(\frac{t}{\bar{\eta}} \right). \quad (32)$$

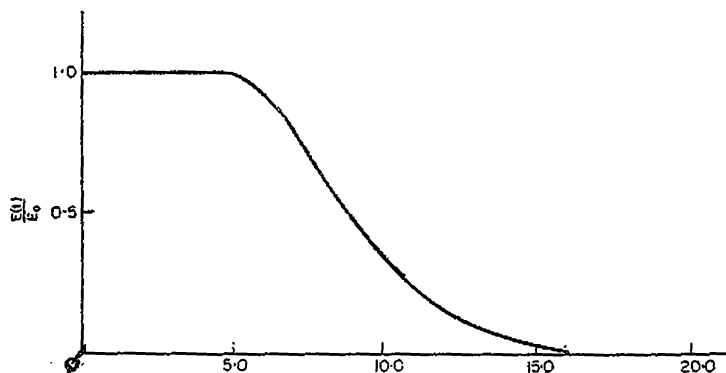


FIG. 3

The left-hand side of this equation can be obtained from experiment, and thus knowing $\lambda(t/\bar{\eta})$ we can determine for given depth t the corresponding value of the energy of the shower producing particles. In the same column of Table 2 we find the corresponding value of $\varphi(t/\bar{\eta})$, and finally obtain the required functions as

$$\exp \xi(\eta) = N(t) \exp (-\varphi(\bar{\eta}, t)) \quad (33)$$

$$t = t(\bar{\eta}).$$

9. We shall now consider the behaviour of cosmic particles in passing from air to a medium with a different atomic number, and first of all determine the distribution of particles with respect to energy at given height t . Let $\eta = \ln(E/\varepsilon)$, where E is the energy of a particle at height t , and let $\eta_0 = \ln(E_0/\varepsilon)$, where E_0 is the energy of a particle at the boundary of the atmosphere.

The number of particles with a given value of η will then be given by

$$\tilde{N}(t, \eta) = \int \exp \{ \xi(\eta_0) + \varphi(\eta_0 - \eta, t) \} d\eta_0, \quad (34)$$

or to our approximation

$$N(t, \eta) = \exp \{ \xi(\bar{\eta}_0) + \varphi(\bar{\eta}_0 - \eta, t) \}, \quad (34a)$$

where η_0 is the root of the equation

$$\xi'(\bar{\eta}_0) + \frac{\partial \varphi(\bar{\eta}_0 - \eta, t)}{\partial \eta_0} = 0. \quad (35)$$

If we are not concerned with particles with very high energies, we can assume that η is small compared with η_0 , and retain only the linear term in the expansion of the expression in the exponent in powers of η ; we then have

$$\left\{ \frac{\partial}{\partial \eta} \ln N(t, \eta)_{\eta=0} = \{ \xi'(\bar{\eta}_0) + \varphi(\bar{\eta}_0 - \eta, t) \} \frac{\partial \bar{\eta}_0}{\partial \eta} - \frac{\partial \varphi(\bar{\eta}_0, t)}{\partial \bar{\eta}_0} \right\}$$

or,

$$\ln N(t, \eta) = \text{const} - s \left(\frac{t}{\bar{\eta}_0} \right) \eta, \quad (36)$$

where the constant of course depends on t but not on η . The quantity s in this formula is connected directly with the absorption coefficient λ at the given height, and can be calculated from it by means of the table. We obtain finally

$$N(t, \eta) = N_0 \exp(-s\eta), \quad (37)$$

and going over to the ordinary scale of energy, we have evidently

$$dN = \text{const} \frac{dE}{E^{s+1}} \quad (38)$$

(this formula applies of course only for $E \gg \varepsilon$). For example, for $t = 10$, $E_0 = 10^{10}$ eV, we obtain from the table $s \sim 1.5$, for $t = 20$, $E_0 = 10^{10}$ eV we find $s \sim 2.0$.

We now determine the number of showers produced by a particle passing from air through a not too thin layer of some material; let

$$\ln \left(\frac{\varepsilon}{\varepsilon'} \right) = \ln \left(\frac{\bar{Z}'}{\bar{Z}} \right) = \zeta,$$

where ε and \bar{Z} refer to air and ε' and \bar{Z}' to the layer, then we can write $\eta' = \eta + \zeta$. As was shown in section 7, the showers will emerge from the layer only if $\eta' > t/3.8$ or $\eta > t/3.8 - \zeta$.

Since the number of particles diminishes with energy, for negative values of ζ (i.e. $\bar{Z}' < \bar{Z}$) we can simply write

$$\eta = \frac{t}{3.8} + |\zeta| \quad \text{and} \quad N = N_0 \exp \left\{ -s \left(\frac{t}{3.8} + |\zeta| \right) \right\} \quad (39)$$

We see from this that with increase of the layer thickness the number of showers falls off with an absorption coefficient $s/3.8$. For positive ζ we have to

distinguish between two cases: (1) $t < 3.8\zeta$ and (2) $t > 3.8\zeta$; in the first case the lower limit given by the formula for η is negative, and we have simply to put $\eta = 0$, from which $N = N_0$; in the second case we obtain as before

$$N = N_0 \exp \left\{ -s \left(\frac{t}{3.8} - \zeta \right) \right\}. \quad (40)$$

To calculate the number of particles emerging from the given layer, we have

$$N = N_0 \int \exp(-s\eta + \varphi(\eta + \zeta, t)) d\eta. \quad (41)$$

For negative ζ the integrand always has a maximum for a value of η different from zero, and in our approximation we can write

$$N_0 = N_0 \exp \{ -s\bar{\eta} + \varphi(\bar{\eta} + \zeta, t) \}, \quad (42)$$

where $\bar{\eta}$ is the root of the equation

$$s = \frac{\partial \varphi(\eta + \zeta, t)}{\partial \eta} = s \left(\frac{t}{\eta + \zeta} \right). \quad (43)$$

Now we have (see equation (25))

$$\varphi(\bar{\eta} + \zeta, t) - (\bar{\eta} + \zeta)s = -t\lambda(s),$$

hence

$$N = N_0 \exp \{ s\zeta - t\lambda(s) \}. \quad (44)$$

In other words the number of particles continues to diminish in the material with the same absorption coefficient λ as in air. For $t = 0$,

$$N = N_0 \exp \{ s\zeta \}, \quad (45)$$

i.e. since ζ is negative for a transition from air to a medium of smaller atomic number, the number of particles falls discontinuously (actually in a thin layer $t \ll 1$) by a factor $\exp s|\zeta|$. For a transition from air to lithium we have $Z' = 4$, $\zeta \sim -0.76$, which for $s \sim 1.5$ gives a diminution of the number of particles by a factor 3.

Passing to a medium with larger Z , we have two cases according as to whether equation (43) has or has not a positive solution for $\bar{\eta}$. In the former case we evidently obtain the same result as for negative ζ , but in the latter which occurs when $s(t/\zeta) < s$ we must write

$$N = N_0 \exp \varphi(\zeta, t). \quad (46)$$

This expression has a maximum for $t \cong \zeta$, as was shown, this maximum being

$$N = N_0 \exp \zeta = N_0 \left(\frac{Z'}{Z} \right). \quad (47)$$

We see therefore that for passage through a medium with Z larger than in air the number of particles at first increases, 10 times for lead, for instance, and then falls, reaching after a certain point the absorption coefficient for air.

10. The showers observed at great heights should evidently be ascribed to the so-called soft component of the penetrating radiation, i.e. particles produced by penetration of photons or electrons. Correspondingly the number of showers leaving a lead plate has, as we know, a maximum for a certain thickness of lead, falling off with further increase of the thickness corresponding to the finite length of path of the showers. The number of showers at and below the Earth's surface tends to a certain limit with increase of the thickness of the lead plate. As had already been pointed out, such showers can be related to secondary particles created during the passage of the hard component through matter and such particles may evidently be either electrons or photons. From the relativistic scattering formula, we can obtain for the number of fast electrons produced per unit path of the "semi-heavy" particle the following expression (we assume the velocity of the particle to be close to that of light)

$$dI = \frac{2\pi}{m} N \frac{e^4 dE}{c^2 E^2}. \quad (48)$$

Since the number of particles increases rapidly with decrease of their energies, the number of observed showers should be very sensitive to the minimum energy of the shower which the apparatus can detect. In order to obtain a very rough estimate of the order of magnitude, we assume arbitrarily that this lower limit is $E \doteq \varepsilon$ and the length of path of the shower is $t = 1$. It is seen from the calculation that the atomic number of the medium drops out of the final expression, so that the number of showers is independent of the nature of the medium, and we obtain 3 or 4 as the order of magnitude of the number of showers per 100 semi-heavy particles. Thus if the particle producing the shower is an electron we should expect that the number of showers should be independent of the mass and energy of the penetrating particle, and independent of the properties of the medium in which the shower is created, but that it should be strongly dependent on the particular experimental arrangement used for detecting the showers.

On the other hand, if it is a proton which produces the shower, calculation shows that the number of showers should depend chiefly on the mass of the penetrating particle, and also to some extent on its energy and the atomic number of the medium, but much less on the particular experimental arrangement.

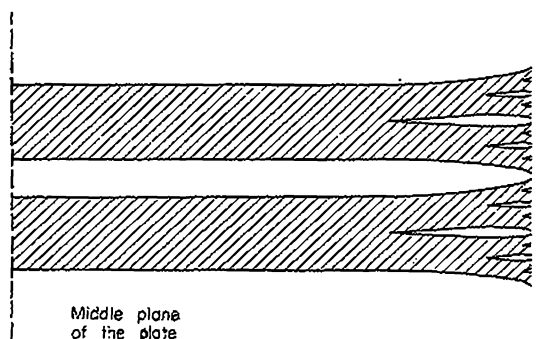
The relative importance of these two possibilities depends very much on what we assume about the mass of the initial particle. Since experimentally it is found that the number of showers is independent of the medium, we may conclude that it is the first possibility which is correct.

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37. THE INTERMEDIATE STATE OF SUPRACONDUCTORS

In a previous paper¹, I have shown that a body in the so-called "intermediate" state consists of alternating layers of the supraconducting and normal phases. It can be shown, however, that the plane-parallel form of these layers discussed in this paper does not correspond to a minimum of the free energy†. A detailed investigation leads to the result that the thickness of the layers does not remain constant throughout the body, but is greatest in the inner part, and becomes very small near the surface of the body. The accompanying figure shows a section of a plate in a transverse magnetic field. The normal layers (shaded in the figure) branch in both directions towards the surface of the plate. A similar branching along the lines of force occurs in the intermediate state of a body of any other shape.



Calculation shows that the thickness of the layers in the middle of the body is of the order of magnitude $(l_0 l^2)^{1/3}$, l being the total length of the line of force inside the body and l_0 the depth of penetration of the magnetic field in a superconductor (probably of the order 10^{-5} cm).

The branching of the layers towards the surface continues until the thickness of the layers becomes of the order of magnitude of l_0 , when it is no longer really possible to speak of the normal and supraconducting states as distinct phases. Hence, a region of the width l_0 near the surface of the body is not in the intermediate state, that is, does not consist of layers; but is in some different state which may be called the "mixed" state.

L. Landau, The intermediate state of superconductors, *Nature*, 141, 688 (1938).

† I am indebted to Professor R. Peierls for directing my attention to this point.

The conception of the alternating layers leads to the conclusion that the intermediate state must be supraconducting in the direction of the magnetic field, but has resistance in perpendicular directions. It might seem that the body should be supraconducting also in the direction which is parallel to the layers but perpendicular to the direction of the field; however, it can easily be shown that, if the current is flowing in this direction, the layers turn so as to be perpendicular to the current.

For the resistance of the intermediate state perpendicular to the magnetic field, we should have $R = R_0 B/H_c$ (B , magnetic induction; H_c , critical field; R_0 , resistance of the normal state). However, experiment shows² that the resistance of a wire in a transverse field tends to this value only if the current is comparatively large, the resistance for very small currents being much smaller. This indicates that an appreciable part of the current flows not in the middle of the wire through the intermediate state, but in the mixed state near the surface. More detailed experiments could in this way supply information about the properties of the mixed state, about which at present very little can be said theoretically.

The detailed calculations will appear elsewhere³.

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38. ON THE DE HAAS—VAN ALPHEN EFFECT

To obtain the formula for the magnetic susceptibility at low temperatures for the isotropic case, we start from equations (17) and (18) of Blackman¹, which in our notation† become:

$$\frac{F}{V} = - \int_{-\infty}^{\infty} \frac{Z(E, H) e^{-(E-E_0)/kT} dE}{(1 + e^{-(E-E_0)/kT})} + \frac{NE_0}{V}, \quad (\text{A } 1)$$

$$Z(E, H) = \frac{\beta H \sqrt{2m^3}}{\pi^2 \hbar^3} \sum_l \left(E - \left(l + \frac{1}{2} \right) \beta H \right)^{1/2}. \quad (\text{A } 2)$$

We assume E_0 constant, in accordance with Blackman's method (p. 12) (for the weak fields of the experiments the condition $N = \text{const}$ gives practically the same result as $E_0 = \text{const}$). Introducing the abbreviations

$$\frac{E}{\beta H} = \varepsilon, \quad \frac{kT}{\beta H} = \theta, \quad \frac{(2m)^{3/2}}{3\pi^2 \hbar^3} = \alpha, \quad (\text{A } 3)$$

and integrating by parts, we have

$$\frac{F - NE_0}{V} = \alpha (\beta H)^{3/2} \int_{-\infty}^{\infty} \varphi(\varepsilon) \frac{d}{d\varepsilon} g\left(\frac{\varepsilon - \varepsilon_0}{\theta}\right) d\varepsilon, \quad (\text{A } 4)$$

where

$$\varphi(\varepsilon) = \sum_l \left(\varepsilon - l - \frac{1}{2} \right)^{3/2}, \quad (\text{A } 5)$$

the summation being over all positive integers l which make the radicand positive, and where

$$g(x) = \frac{1}{1 + e^x}. \quad (\text{A } 6)$$

Using Poisson's summation formula, we have

$$\varphi(\varepsilon) = \sum_{p=-\infty}^{\infty} (-1)^p \int_0^{\varepsilon} (\varepsilon - x)^{3/2} e^{2\pi i p x} dx. \quad (\text{A } 7)$$

L. Landau, Appendix to a paper by D. Shoenberg, *Proc. Roy. Soc. A* 170, 363 (1939).

† For the isotropic case, $m_1 = m_2 = m_3 = m$, and $m_4 = 0$, so the β 's defined in equations (4) and (6) of the text all become equal to $\beta = e\hbar/mc$.

The term in $p = 0$ can be immediately evaluated; for $p \neq 0$, integrating twice by parts, the integral can be expressed in terms of Fresnel's integrals for the argument $2\pi p \varepsilon$. It can be seen, however, from (A 4) that the value of φ need be known only in the neighbourhood of $\varepsilon = \varepsilon_0$, since the derivative of $g((\varepsilon - \varepsilon_0)/\theta)$ is appreciable only there, provided $E_0 \gg kT$. Since ε_0 is a large number, provided $E_0 \gg \beta H$, the Fresnel integrals may be replaced by their values for infinite argument. This transforms (A 7) into

$$\begin{aligned} \varphi(\varepsilon) &= \frac{2}{5} \varepsilon_0^{5/2} + \sum_{p=1}^{\infty} (-1)^p \frac{3}{4\pi^2 p^2} \varepsilon^{1/2} - \sum_{p=1}^{\infty} (-1)^p \frac{3}{8\pi^2 \sqrt{2} p^{5/2}} \cos\left(2\pi p \varepsilon - \frac{\pi}{4}\right) \\ &= \frac{2}{5} \varepsilon_0^{5/2} - \frac{1}{16} \varepsilon^{1/2} - \sum_{p=1}^{\infty} (-1)^p \frac{\cos\left(2\pi p \varepsilon - \frac{\pi}{4}\right)}{8\pi^2 \sqrt{2} p^{5/2}}. \end{aligned} \quad (\text{A } 8)$$

Inserting this into (A 4), we notice that the first two terms of (A 8) vary very slowly in the range for which $dg((\varepsilon - \varepsilon_0)/\theta)/d\varepsilon$ is appreciable, and we may therefore use their values at ε_0 . The last term, however, varies rapidly. Hence

$$\begin{aligned} \frac{F - N E_0}{V} &= -\alpha (\beta H)^{5/2} \left\{ \frac{2}{5} \varepsilon_0^{5/2} - \frac{1}{16} \varepsilon_0^{1/2} \right. \\ &\quad \left. + \sum_{p=1}^{\infty} \frac{(-1)^p 3}{8\pi^2 \sqrt{2} p^{5/2}} \mathcal{R} \left[e^{(2\pi p \varepsilon - 1/4\pi)i} \int_{-\infty}^{\infty} e^{2\pi i p (\varepsilon - \varepsilon_0)} \frac{d}{d\varepsilon} g\left(\frac{\varepsilon - \varepsilon_0}{\theta}\right) d\varepsilon \right] \right\}, \end{aligned} \quad (\text{A } 9)$$

where \mathcal{R} stands for "the real part of".

The integral in (A 9) can be evaluated, and gives

$$\int_{-\infty}^{\infty} e^{2\pi i p \varepsilon} \frac{d}{d\varepsilon} g\left(\frac{\varepsilon}{\theta}\right) d\varepsilon = -\frac{2\pi^2 p \theta}{\sinh 2\pi^2 p \theta}. \quad (\text{A } 10)$$

So, substituting for α , ε_0 , and θ from (A 3), and omitting the term which does not depend on H , we have

$$\frac{F}{V} = \frac{(2m)^{3/2}}{3\pi^2 \hbar^3} \left\{ \frac{1}{16} (\beta^2 H^2 E_0^{1/2}) + \sum_{p=1}^{\infty} (-1)^p \frac{3kT(\beta H)^{3/2}}{4\sqrt{2} p^{3/2}} \frac{\cos\left(2\pi p \varepsilon_0 - \frac{\pi}{4}\right)}{\sinh 2\pi^2 p \theta} \right\},$$

and hence the susceptibility, $\frac{I}{H} = -\frac{1}{H} \frac{\partial}{\partial H} \left(\frac{F}{V} \right)$, is given by

$$\begin{aligned} \frac{I}{H} &= \frac{\sqrt{2} m^{3/2} \beta^2 E_0^{1/2}}{12\pi^2 \hbar^3} + \frac{m^{3/2} kT E_0 (\beta H)^{1/2}}{\pi \hbar^3 H^2} \sum_{p=1}^{\infty} (-1)^p \frac{\sin\left(\frac{2\pi p E_0}{\beta H} - \frac{\pi}{4}\right)}{\sqrt{p} \sinh\left(\frac{2\pi^2 p kT}{\beta H}\right)} \\ &= m A' f(\beta H), \end{aligned} \quad (\text{A } 11)$$

where f is the function defined by Shoenberg² and A' is

$$A' = \frac{\sqrt{2} e^2 E_0}{2\pi^4 c^2 \hbar \sqrt{k} m^{3/2}} \quad (\text{A } 12)$$

The transformation to the non-isotropic case is carried out just as in section 1 of Blackman's paper, except that the coefficient m_4 is taken into account, and the formulae quoted by Shoenberg² (equations (3-9)) are obtained.

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39. ON THE POLARISATION OF ELECTRONS BY SCATTERING

It is well known that experiments fail to observe any appreciable polarisation by scattering¹ which is sometimes regarded as a failure of the theory. Bethe and Rose² have shown in a recent article that this fact cannot be explained by depolarisation effects. I wish to point out that the negative result of experiments seems to follow from a much simpler reason, namely from the fact that the scattered electrons observed have got their deflection not in a single act of scattering but as a result of multiple scattering.

The angular width of the beam originated by multiple scattering can be easily calculated. The mean square angle of deflection resulting from a multiple scattering is³

$$\langle \theta^2 \rangle_{av} = 2\pi N \left(\frac{Z e^2}{E} \right)^2 l \ln \left[\frac{\langle \theta^2 \rangle_{av}^{1/2}}{\theta_0} \right],$$

where N is the number of atoms in a unit volume, Z the atomic number, E the energy of the electrons (more exactly $E = m v^2 / 2 [1 - (v^2/c^2)]^{1/2}$), θ_0 the angle for which effects of screening become important, and l the path which the electrons have travelled. Inserting the values for gold we get for $E = 100$ keV

$$\langle \theta^2 \rangle_{av} = 2.5 \times 10^3 l \ln(5 \times 10^4 l).$$

Even for $l = 7 \times 10^{-6}$ cm (the thinnest foil used by Dymond) we get

$$\langle \theta \rangle_{av}^{1/2} = 0.23.$$

But this means that most of the deflected electrons observed in experiment were scattered many times on small angles. As the polarisation formula given by Mott⁴ shows that the polarisation falls very rapidly for small angles of scattering, this seems to explain the experimental results.

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40. ON THE NATURE OF THE NUCLEAR FORCES

It was shown recently by one of us¹ that the properties of the wave equation for a particle with spin 1 in a Coulomb field are essentially different from those of the wave equation for particles with spin 0 or $\frac{1}{2}$. It turns out that the solutions of this equation corresponding to the total momenta $j \geq 1$ and orbital momenta $l = j + 1$ have an essential singularity in the co-ordinate origin. The wave equations are in the vicinity of the origin of the form

$$\psi'' + f(r)\psi = 0 \quad (1)$$

with $f \sim (e^2/\mu c^2)(1/r^3)$ (μ is the mass of the particle).

However, it is necessary to hold in mind that the applicability of the Proca equations is limited by distances of the order $r_0 = e^2/\mu c^2$, as was shown by one of us². If we note that (1) is of the form analogous to that of the Schrödinger equation the qualitative picture of the motion can be formally described as the motion of a fictitious non-relativistic particle with mass 1 in a potential hole with the radius r_0 and of the depth $U \sim \hbar^2/r_0^2$. It is known that the levels corresponding to the motion of the particle inside the hole exist if $U > \pi^2 \hbar^2/8r_0^2$. As in the present case U is itself of the order \hbar^2/r_0^2 , it is *a priori* impossible to decide if there are states which correspond to the motion limited by the distances of the order r_0 . In particular such states can prove to be possible only for certain values of the momentum j .

If we identify, as is commonly done, mesotrons with particles with spin 1, and assume that the neutron consists of a proton and a negative mesotron, then we can conclude from the foregoing considerations that there is no specific interaction between the proton and the mesotron and the formation of the neutron is the result of the ordinary electrical interaction. If such a point of view is adopted we must assume that the system proton + negative mesotron possesses for the total momentum of the system equal to $\frac{1}{2}$ and the total momentum of the mesotron $j = 1$ a level which corresponds to the relative motion of both particles at the distances of the order r_0 .

So we are led to the view that the nuclear forces are to be regarded as originating from the electrical interaction of the nuclear particles by means of the mesotrons, and are connected with the peculiar properties of the equations of motion of the mesotrons.

It could appear that such a point of view is inconsistent with the symmetry in the properties of the neutrons and the protons, which is substantially supported by experiment. However, it follows from the mere fact that the

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L. Landau and I. Tamm, On the nature of the nuclear forces, *C. R. Acad. Sci. USSR*, 29, 556 (1940).

system proton + negative mesotron possesses a level with the binding energy $\sim \mu c^2$, that the system neutron + positive electron forms a proton with the liberation of nearly the same energy μc^2 (the possibility of the mesotron pair creation must be held in mind).

As the radius $e^2/\mu c^2$ of the proton and of the neutron is, in this theory, small as compared with that of the nucleus, the possibility arises of calculating various effects (the scattering of the mesotrons on the nuclei, the mesotron photo-effect, etc.) without going into the detailed consideration of the unknown interaction inside the protons and the neutrons themselves (as is done in the theory of the deuteron)³.

A detailed discussion of these problems will be published in the *Journal of Physics* (edited by the Academy of Sciences of the U.S.S.R.)[†].

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[†] *Editor's note:* This paper appears not to have been published.

41. ON THE "RADIUS" OF THE ELEMENTARY PARTICLES

A method is developed to find the limit beyond which the electrodynamical conceptions in quantum mechanics lose their validity. The method is applied to electrons and to particles with spin 1.

It is well known that experiment does not detect any limit in the applicability of the Dirac theory for large energies, i.e. small wavelengths (in Compton effect, Bremsstrahlung, etc.), whereas from classical electrodynamics we get e^2/mc^2 as the limit of applicability of its fundamental principles.

This limit e^2/mc^2 can be deduced not only from the considerations connected with the self-energy of the electron, but most convincingly from the condition that the reaction of the field of the electron must be small¹. This means that in the equations of motion of the electron in the electromagnetic field

$$m \ddot{\mathbf{v}} = e \mathbf{E} + \frac{2e^2}{3c^3} \ddot{\mathbf{v}} \quad (1)$$

the last term, the Lorentz damping force, must be small compared with the first one. Hence we easily get for the frequency ω of the field $\omega \ll mc^2/e^2$, i.e. $\lambda \gg e^2/mc^2$ for its wavelength.

The real limit of the classical electrodynamics is, as is well known, $\lambda \gg \hbar/mc$, which lies much higher than $\lambda \sim e^2/mc^2$. As for the limits of the applicability of the electrodynamical conceptions in quantum theory, there are no reasons to assume that they are given by e^2/mc^2 , and some other way must be found to determine the real "radius" of elementary particles.

The previous discussions of this problem have always included some additional physical assumptions[†] which make the result quite unconvincing. It is our purpose here to show that in quantum theory a simple method can be developed to find the limit beyond which the theory comes into contradiction with itself due to the neglect of the reaction of the field.

The neglect of the reaction of the field means that we assume the particle to move under the influence of the external field without taking into account the variation of this field by the electron itself. In the mathematical treatment this is expressed by calculating the matrix elements of the interaction of the

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L. Landau, On the "radius" of the elementary particles, *Phys. Rev.* 58, 1006 (1940).

† Cf. Heisenberg² and also a recent paper by Weisskopf³.

charged particle and a photon with the help of free particle wave functions. Hence the smallness of the reaction force corresponds in the quantum theory to the applicability of the perturbation theory to the collision of a charged particle and a photon.

The effective cross-section σ_l for the scattering of particles with an angular momentum $l\hbar$ in the frame of reference moving with the centre of inertia is

$$\sigma_l = 4(2l + 1) \pi \lambda^2 \sin^2 \delta_l, \quad (2)$$

λ being the wavelength (divided by 2π), δ_l —the phase of the wave function at infinity. The applicability of the perturbation theory requires $\delta_l \ll 1$. Hence we get the condition:

$$\sigma_l \ll 4(2l + 1) \pi \lambda^2. \quad (3)$$

The condition (3) means that the applicability of the perturbation theory requires the number of the scattered particles with any angular momentum to be small as compared with the number of the incident particles with the same momentum. It is to be noted that the condition

$$\sigma_l < 4(2l + 1) \pi \lambda^2,$$

must be in any case fulfilled, quite independently of the applicability of the perturbation theory. The opposite would mean that the number of scattered particles exceeds the number of corresponding incident particles, which is impossible.

As the scattering decreases with increasing l , it suffices to require the condition (3) to be fulfilled for $l \sim 1$; the corresponding σ_l we denote as σ_0 . Thus, we come finally to the condition

$$\sigma_0 \ll \lambda^2. \quad (4)$$

It is easy to see, that in the classical theory this condition leads to the correct result. As in the classical theory all the scattering is dipole scattering, we have simply to insert for σ_0 the full Thomson scattering cross-section, thus leading to

$$\left(\frac{e^2}{m c^2} \right)^2 \ll \lambda^2.$$

We now apply (4) to the quantum theory of the electron. We must first of all transform the Klein-Nishina formula to the proper frame of reference and then take the part of the scattering corresponding to the angular momentum of the order 1. This gives

$$\sigma_0 \sim \left(\frac{e^2}{m c^2} \right)^2 \frac{m c^2}{E}$$

(E being the energy of the photon in the frame of reference where the electron is at rest). σ_0 differs from the total effective cross-section by the absence of

† As all the considerations are qualitative there is no necessity to consider the spin of the particles.

the factor $\ln E/mc^2$ which arises from scattering with large angular momenta. The energy ε of the photon in the frame of reference moving with the centre of inertia is given by $2\varepsilon^2 = mc^2 E$ (when $E \gg mc^2$) so that $\lambda \sim \hbar c/\varepsilon \sim \hbar c/\sqrt{mc^2 E}$.

The condition (4) gives correspondingly

$$\sigma_0 \sim \frac{e^4}{mc^2 E} \ll \lambda^2 \sim \frac{\hbar^2 c^2}{mc^2 E}. \quad (5)$$

The energy E disappears from this inequality and we get $e^2 \ll \hbar c$ which is always fulfilled. This means, that in the quantum theory of the electron there are no limits of its applicability, arising from itself. The "radius" of the electron in quantum electrodynamics is in some sense equal to zero.

One gets the same result for particles with zero spin, if the corresponding expression for the Compton effect⁴ is used.

The Compton effect for particles with spin 1 (mass μ) was recently calculated by Booth and Wilson⁵, and by J. Smorodinsky⁶. Their result is

$$\sigma_0 \sim \left(\frac{e^2}{\mu c^2} \right)^2 \frac{E}{\mu c^2}.$$

Hence the condition (4) gives

$$E \ll \mu c^2 \frac{\hbar c}{e^2}. \quad (6)$$

The "radius" of the particle is equal to the corresponding wavelength in the frame of reference in which the particle is at rest, i.e.

$$r \sim \frac{\hbar c}{E} \sim \frac{e^2}{\mu c^2}. \quad (7)$$

Hence the radius of the particle with spin 1 is equal to its classical value.

When considering the scattering of particles with spin 1 ("mesotrons") by a charged particle, we must determine the condition when the neglect of radiation effects is justified. For this purpose we shall write (6) in the frame of reference moving with the centre of inertia:

$$\varepsilon \ll \mu c^2 \sqrt{\frac{\hbar c}{e^2}}. \quad (8)$$

From the other side a finite "radius" of the particle means the breakdown of the theory for the processes connected with the transfer of large momenta. The change of momentum in scattering at a given angle depends only on ε and does not depend on the nature of the scattering particle. Hence the condition (8) is a criterion of the applicability of the theory to the collisions with any particles. The energy E of the incident mesotron is connected with the energy ε in the frame of reference moving with the centre of inertia by means of $E \sim \varepsilon^2/mc^2$, m being the mass of the particle with which the mesotron

collides. Now for E (8) gives the condition:

$$E \ll \mu c^2 \frac{\mu}{m} \frac{\hbar c}{e^2}. \quad (9)$$

If we consider the formulae derived by Massey and Corben⁷ for the scattering of mesotrons and apply the condition (4) now treated as a general condition for the applicability of the perturbation theory, we get:

$$E \ll \mu c^2 \frac{\mu}{m} \left(\frac{\hbar c}{e^2} \right)^2. \quad (10)$$

We see that (9) is violated earlier than (10). Hence the theory of the scattering of mesotrons breaks down because of radiation effects much earlier than the perturbation theory will become inapplicable.

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42. ON THE SCATTERING OF MESOTRONS BY "NUCLEAR FORCES"

It is shown that correct expressions for the scattering cross-sections of mesotrons can be deduced in a simple manner from the diverging expressions obtained by means of perturbation theory.

THE effective cross-sections, calculated¹ for the scattering of particles with spin 1 ("mesotrons") by the so-called "nuclear forces", increase rapidly with the energy. However it is easy to see that these expressions cannot be correct for energies just exceeding μc^2 (μ being the mass of a mesotron). Indeed, this scattering has a central symmetry, i.e. only particles with zero angular momentum take part in the scattering. The corresponding cross section cannot, therefore, exceed $4\pi \lambda^2$, λ being the wave length (divided by 2π) of the scattered particle in the frame of reference moving with the centre of inertia (cf. e.g. this issue p. 485; the accompanying paper³).

It is the aim of the present work to show, that a correct result can be obtained in that case also where perturbation theory cannot be applied.

The assumptions made for the interaction of a "mesotron" with a heavy particle are formally identical with those made in deriving the Breit-Wigner formula for the scattering of neutrons by nuclei. Moreover in the present case there is only a single "compound state" of the mesotron and the heavy particle. The Breit-Wigner formula for the elastic scattering is²:

$$\sigma_e = \pi \lambda^2 \frac{\Gamma_e^2}{(E - E_0)^2 + \frac{\Gamma^2}{4}} = \pi \lambda^2 \frac{\gamma_e^2}{1 + \frac{\gamma^2}{4}}, \quad (1)$$

where

$$\gamma_e = \frac{\Gamma_e}{E - E_0}, \quad \gamma = \frac{\Gamma}{E - E_0}. \quad (2)$$

For some inelastic scattering the corresponding formula is

$$\sigma_i = \pi \lambda^2 \frac{\Gamma_i \Gamma_e}{(E - E_0)^2 + \frac{\Gamma^2}{4}} = \pi \lambda^2 \frac{\gamma_i \gamma_e}{1 + \frac{\gamma^2}{4}}. \quad (3)$$

Л. Ландау, О рассеянии мезотронов ядерными силами, *Журнал Экспериментальной и Теоретической Физики*, 10, 721 (1940).

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The total "width" is $\Gamma = \Gamma_e + \sum \Gamma_i$ and $\gamma = \gamma_e + \sum \gamma_i$ (the summation being extended over all possible kinds of inelastic scattering).

The ordinary perturbation theory corresponds to neglecting γ^2 in the denominator of (1) and (3). The cross-section calculated in this way are hence

$$\sigma'_i = \pi \lambda^2 \gamma'_i \gamma_e, \quad \sigma'_e = \pi \lambda^2 \gamma_e^2. \quad (4)$$

It is seen from formulae (1-4) that if one knows σ'_e and σ'_i calculated by means of the perturbation theory, it is always possible to obtain correct expressions for σ_e and σ_i ; these correct expressions, of course, always satisfy the condition $\sigma < 4\pi \lambda^2$.

In the scattering of mesotrons by heavy particles, the inelastic scattering corresponds to a creation of several mesotrons (the radiation effects are comparatively improbable). For sufficiently large energies γ_e becomes large as compared with unity and simultaneously the probability of the inelastic scattering becomes large as compared with that of elastic scattering. The order of the magnitude of the probability of the scattering with a creation of n particles is apparently given by the Poisson formula:

$$(\gamma_i)_n \sim \frac{\gamma_e^n}{n!} \quad (5)$$

and correspondingly $\gamma \sim e^{\gamma_e}$, i.e. increases exponentially with increasing γ_e and hence with the energy. From formulae (1) and (3) it is now clear that the scattering does not increase for large energies, but on the contrary decreases swiftly.

It is of interest to examine to what extent the divergence of some expressions, obtained by means of "nuclear forces" (the self-energy, the magnetic moment of the heavy particles, etc.) depends on the application of perturbation theory and if it would not be possible to deduce converging expressions by means of an exact theory.

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43. THE ANGULAR DISTRIBUTION OF THE SHOWER PARTICLES

A precise expression (3) for the number of the charged particles in a shower at the point of maximum is obtained. The mean angles of the deflection of the shower particles (16-19) and also the width of the shower (22) are calculated.

1. THE NUMBER OF CHARGED PARTICLES IN A SHOWER AT THE MAXIMUM POINT

Starting from the theory of Bhabha and Heitler¹ and Carlson and Oppenheimer², G. Rumer and the author³† have calculated the number of particles in a shower as a function of their energy E and penetration depth. It was shown that this number N is of the form

$$N = a e^{\varphi(\eta, t)}, \quad (1)$$

with $\eta = \ln(E_0/\varepsilon)$ (E_0 —the initial energy of the primary particle, ε —the critical energy value which determines the ionisation energy loss). The exponent φ was calculated in I and the non-exponential factor a was there omitted, as being less essential. Later on we shall denote by N the number of the charged particles in a shower; the total number of all particles—electrons and photons—is equal to the same exponential factor with another non-exponential coefficient.

The number N has a maximum at a definite depth $t = t_m$. We shall now show that it is possible to calculate this maximum value N_m more accurately, i.e. it is possible to calculate the value of the non-exponential factor a in (1) for $t = t_m$.

Suppose that along the whole length the primary particles (electrons or photons) with the energy E_0 are uniformly (one per unit path) generated. (This assumption is, of course, purely formal.) From the energy conservation we conclude that the energy E_0 generated on a unit length must be equal (in a stationary state) to the energy loss of the shower particles which are in equilibrium with a primary particle. This energy loss is due only to ionisation, because the Compton effect only transfers the energy from photons to the

Л. Д. Ландау, Угловое распределение частиц в ливнях, *Журнал Экспериментальной и Теоретической Физики* 10, 1007 (1940).

L. Landau, The angular distribution of the shower particles, *J. Phys. U.S.S.R.* 3, 237 (1940).

† We refer in the following to this paper as to I; the notation in the present article is the same as in I.

charged shower particles. If the number of particles which are in equilibrium with a one primary is n we have, consequently, $n\varepsilon = E_0$, i.e.

$$n = \frac{E_0}{\varepsilon}.$$

On the other hand, we can calculate n as the total number of shower particles at a given distance from the surface of the layer which were created above this level. As the number of particles on the level t created by one primary particles is $a e^{\varphi(t, \eta)}$ we have:

$$n = \int_0^{\infty} a e^{\varphi(t, \eta)} dt.$$

From the two expressions for n we get:

$$\int_0^{\infty} a e^{\varphi(t, \eta)} dt = \frac{E_0}{\varepsilon}. \quad (2)$$

As $\varphi(t, \eta)$ has a maximum at $t = t_m$, we can expand $\varphi(t, \eta)$ under the integral into a series:

$$\varphi = \varphi_m + \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial t^2} \right)_{t=t_m} (t - t_m)^2,$$

and the non-exponential factor a can be taken simply at the point $t = t_m$. Thus, we obtain:

$$a_m e^{\varphi_m} \int \sqrt{\frac{2\pi}{-\frac{\partial^2 \varphi}{\partial t^2}}} = N_m \int \sqrt{\frac{2\pi}{-\frac{\partial^2 \varphi}{\partial t^2}}} = \frac{E_0}{\varepsilon}.$$

According to formulae deduced in I, at the maximum point

$$\left| \frac{\partial \varphi}{\partial t} \right|_{t=t_m} = -\lambda(s).$$

Hence,

$$N_m = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\partial \lambda}{\partial t} \frac{E_0}{\varepsilon}}.$$

The derivative $\partial \lambda / \partial t$ must be taken at constant energy, i.e. at constant y or, according to (27) in I, constant product $t\lambda'(s)$ (' denotes differentiation with respect to the parameter s). We write the derivative in the form of a Jacobian and get:

$$\left(\frac{\partial \lambda}{\partial t} \right)_{t\lambda' = \text{const}} = \frac{\partial(\lambda, t\lambda')}{\partial(t, t\lambda')} = \frac{\frac{\partial(\lambda, t\lambda')}{\partial(t, s)}}{\frac{\partial(t, t\lambda')}{\partial(t, s)}} = -\frac{\lambda'^2}{t\lambda''}.$$

The maximum point corresponds to $s = 1$, $t = t_m = \eta/\lambda'(1)$ (cf. I). Hence

$$N_m = \sqrt{-\frac{1}{2\pi} \frac{\lambda'^3(1)}{\lambda''(1) \ln \frac{E_0}{\varepsilon}} \frac{E_0}{\varepsilon}}.$$

The calculation of $\lambda'(1)$ and $\lambda''(1)$ leads finally to the following expression for the number of charged particles in a shower at the maximum point:

$$N_m = 0.3 \frac{E_0}{\varepsilon \sqrt{\ln \frac{E_0}{\varepsilon}}} \quad (3)$$

(without taking into account the non-exponential factor we would get simply $N_m = E_0/\varepsilon$). For the energies $E_0 = 100\varepsilon$ and $E_0 = 1000\varepsilon$ this formula gives correspondingly $N_m = 15$ and $N_m = 120$. For ε in (3) it is necessary to use correct values of ε taking into account the correction introduced by Fermi⁴. These values in MeV are for Pb-8, Al-40, H₂O-75, air-100.

2. DISTRIBUTION OF THE PARTICLES IN A SHOWER OVER THE DIRECTIONS

The deflection of the particles in a shower from their primary directions is due mainly to the Rutherford scattering on the heavy nuclei. It is known that in this scattering the particles are deflected chiefly over small angles. This fact can be used for the determination of additional terms, which must be added to the kinetic equations (6), (7) in I, if we wish to take into account the scattering of the particles over the different directions.

The kinetic equation for the distribution of particles which are scattered over small angles was discussed several times (cf. e.g. (5)). For the sake of convenience we shall derive here these equations in a form, suitable for the present case. Let $\Pi(E, x, \theta)$ be the distribution function of the charged particles over their energies E , co-ordinates x and the angles θ (in I Π was a function only of E and x). It is convenient to consider the (small) angle of the deflection as a vector θ with two components $\theta_x = \theta \cos \varphi$, $\theta_y = \theta \sin \varphi$, θ and φ being the polar angle and the azimuth in a spherical co-ordinate system with a polar axis along the primary direction of the shower. We calculate the variation $(\partial \Pi / \partial x)_{sc}$ of the function Π along the unit path which is due to the scattering. We have:

$$\left(\frac{\partial \Pi}{\partial x} \right)_{sc} = \int \{ \Pi(\theta + \chi) - \Pi(\theta) \} d\sigma, \quad (4)$$

$d\sigma$ being the effective cross-section for scattering on an angle χ . The first term in the integral is as usual the number of particles, which obtain the angle θ after being scattered; the second term represents the number of particles which are scattered out of the given solid angle element. Expanding

$\Pi(\theta + \chi)$ in a series with respect to the powers of χ we have:

$$\Pi(\theta + \chi) - \Pi(\theta) = \chi_i \frac{\partial \Pi}{\partial \theta_i} + \frac{1}{2} \chi_i \chi_k \frac{\partial^2 \Pi}{\partial \theta_i \partial \theta_k}$$

(the summation over the indices which occur twice is implied). The integral of the first term vanishes (when integrated over the angle φ), and averaging over φ in the second term we get:

$$\int \chi_i \chi_k d\sigma = \frac{1}{2} \delta_{ik} \int \chi^2 d\sigma,$$

δ_{ik} being the two-dimensional unit tensor. Expression (4) becomes finally

$$\left(\frac{\partial \Pi}{\partial x} \right)_{sc} = \frac{1}{4} \int \chi^2 d\sigma \cdot \Delta_\theta \Pi. \quad (5)$$

According to the relativistic Rutherford formula the cross-section for the scattering of the particles with energy E is

$$d\sigma = 8\pi \left(\frac{Z e^2}{E} \right)^2 \frac{N}{Z} \frac{d\chi}{\chi^3}$$

(N —the number of electrons in 1 cm³ of the matter, N/Z —the number of the nuclei). The integral $\int \chi^2 d\sigma$ diverges logarithmically, and in calculating this integral we must take as an upper limit the mean angle of the deflection, which will prove to be in our case of the order of unity. For the lower limit we must take the angle which correspond to the momentum variation \hbar/a , a being the screening radius of the Coulomb field of the nucleus. In other words,

$$\chi_{\min} \sim \frac{\hbar c}{a E} \sim \frac{m c}{E} \frac{\hbar c}{a}.$$

The ratio $a/(\hbar/mc)$ enters under the logarithm in the expressions for the Bremsstrahlung and for the pair creation by photons and was calculated several times⁶. Thus, we find:

$$\int \chi^2 d\sigma = 8\pi \left(\frac{Z e^2}{E} \right)^2 \frac{N}{Z} \ln \frac{183 Z^{-1/3} E}{m c^2}.$$

Inserting this into (5) and introducing $t = x/A$ (equation (10) in I) instead of x we get finally:

$$\left(\frac{\partial \Pi}{\partial t} \right)_{sc} = \delta \Delta_\theta \Pi, \quad (6)$$

with

$$\delta = \frac{2\pi \hbar c}{e^2} \left(\frac{m c^2}{E} \right)^2 \left[1 + \frac{\ln \frac{E}{m c^2}}{\ln 183 Z^{-1/3}} \right]. \quad (7)$$

The expression thus obtained must be added to the right-hand side of the equation (6a) in I in which Π and Γ are now functions not only of E and t , but also of θ . We denote the integral operators in equations (6a), (7a) in I as A, B, C, D (after being multiplied by E^s and integrated they give the quantities $A(s), B(s), C(s), D$) (cf. (13-17) in I), and obtain:

$$\frac{\partial \Pi}{\partial t} = -A\Pi + B\Gamma + \delta \Delta_\theta \Pi, \quad (8)$$

$$\frac{\partial \Gamma}{\partial t} = C\Pi - D\Gamma. \quad (9)$$

We multiply these equations by θ^2 and integrate over the angles ($d\sigma_0$ is the element of the solid angle). This leads to

$$\frac{\partial}{\partial t} \int \Pi \theta^2 d\sigma = -A \int \Pi \theta^2 d\sigma + B \int \Gamma \theta^2 d\sigma + 4\delta \int \Pi d\sigma, \quad (10)$$

$$\frac{\partial}{\partial t} \int \Gamma \theta^2 d\sigma = C \int \Pi \theta^2 d\sigma - D \int \Gamma \theta^2 d\sigma. \quad (11)$$

As in I we introduce $y = \ln(E_0/E)$ instead of E . The total number $\int \Pi d\sigma$ of the particles in a given energy interval is of the form $a e^{\varphi(y, t)}$. The quantity δ is inversely proportional to the square of the energy, i.e. is of the form $\text{const} \cdot e^{2y}$. In other words, the free term in the equation (10) contains the exponential expression $e^{\varphi(y, t) + 2y}$. We seek correspondingly the solutions of (10) and (11) in the form proportional to

$$e^{\varphi(y, t) + 2y}, \quad (12)$$

the coefficient being a slowly varying function of y and t . It follows from the results of I that the operators A, B, C, D when multiplied by an exponential function of y give:

$$A e^{\varphi(y)} = A \left(\frac{\partial \varphi}{\partial y} \right) e^{\varphi(y)},$$

etc. The expression $A(\partial \varphi / \partial y)$ here corresponds to $A(s)$ in I. In our case

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial y} + 2 = s + 2.$$

On the other hand,

$$\frac{\partial}{\partial t} e^{\varphi + 2y} = \frac{\partial \varphi}{\partial t} e^{\varphi + 2y} = -\lambda(s) e^{\varphi + 2y}.$$

Hence the equations (10), (11) become (we write A_s, B_s, \dots for $A(s), B(s), \dots$):

$$\left. \begin{aligned} (A_{s+2} - \lambda_s) \int \Pi \theta^2 d\sigma - B_{s+2} \int \Gamma \theta^2 d\sigma &= 4\delta \int \Pi d\sigma, \\ -C_{s+2} \int \Pi \theta^2 d\sigma + (D - \lambda_s) \int \Gamma \theta^2 d\sigma &= 0, \end{aligned} \right\} \quad (13)$$

A_s, B_s, C_s, D being given by (13-16) in I.

It is our aim here to calculate the mean angles of the deflection of the shower particles, i.e. for electrons and photons respectively the quantities:

$$\overline{\theta_H^2} = \frac{\int \Gamma \theta^2 d\phi}{\int \Gamma d\phi}, \quad \overline{\theta_T^2} = \frac{\int \Pi \theta^2 d\phi}{\int \Pi d\phi}. \quad (14)$$

If we divide equations (13) by $\int \Pi d\phi$, we see that for calculating $\overline{\theta_T^2}$ it is necessary to calculate the ratio $\int \Gamma d\phi / \int \Pi d\phi$. For this sake we integrate equations (8), (9) with respect to $d\phi$; the third term on the right-hand side of (8) disappears, and we get:

$$\begin{aligned} \frac{\partial}{\partial t} \int \Pi d\phi &= -A \int \Pi d\phi + B \int \Gamma d\phi, \\ \frac{\partial}{\partial t} \int \Gamma d\phi &= C \int \Pi d\phi - D \int \Gamma d\phi. \end{aligned}$$

Supposing the solution in the form:

$$\int \Pi d\phi \sim \int \Gamma d\phi \sim e^{\nu},$$

we obtain the equations:

$$\begin{aligned} (A_s - \lambda_s) \int \Pi d\phi &= B_s \int \Gamma d\phi, \\ (D - \lambda_s) \int \Gamma d\phi &= C_s \int \Pi d\phi, \end{aligned}$$

which coincide, of course, with (17) in I. From the first one we have

$$\frac{\int \Gamma d\phi}{\int \Pi d\phi} = \frac{A_s - \lambda_s}{B_s}. \quad (15)$$

Inserting this into (13) we get:

$$\begin{aligned} (A_{s+2} - \lambda_s) \overline{\theta_H^2} - \frac{B_{s+2}}{B_s} (A_s - \lambda_s) \overline{\theta_T^2} &= 4\delta, \\ -C_{s+2} \overline{\theta_H^2} + (D - \lambda_s) \frac{A_s - \lambda_s}{B_s} \overline{\theta_T^2} &= 0. \end{aligned}$$

From these two equations we get finally the expressions for $\overline{\theta_H^2}$ and $\overline{\theta_T^2}$:

$$\overline{\theta_H^2} = \frac{4\delta(D - \lambda_s)}{(A_{s+2} - \lambda_s)(D - \lambda_s) - B_{s+2} C_{s+2}}, \quad (16)$$

$$\overline{\theta_T^2} = \frac{4\delta C_{s+2} B_s}{(A_s - \lambda_s)[(A_{s+2} - \lambda_s)(D - \lambda_s) - B_{s+2} C_{s+2}]}. \quad (17)$$

The expressions thus obtained depend (for a given energy) only slightly (by means of s) on the co-ordinate of the point in the shower. In other words the angle of the scattering of the shower is almost constant along its whole length, e.g. we find from (16) (using the numerical table in I) that $\overline{\theta_H^2}$ changes only by 1.7 times from the point $t = 0.1 y$ in the shower (being near its origin) up to its end ($t = 3.8 y$). The cause is that mainly the particles with small energies are scattered, and on the other hand, every such particle originates from particles of greater energy at a distance of the order of unity (in our conventional units (10) in I). Due to the same cause the scattering of the particles in a shower does not depend on the energy of the particle which generated the shower.

At the maximum point of the shower ($s = 1$) the formulae (16), (17) give:

$$\overline{\theta_H^2} = 2.3 \delta, \quad \overline{\theta_F^2} = 0.72 \delta. \quad (18)$$

Inserting the numerical value of δ we get

$$\overline{\theta_H^2} = \frac{900}{E^2} \quad (19)$$

(E in MeV). Already for $E = 30$ MeV the angle $\theta_H \sim 1$. We also note, that due to a slight dependence of δ on Z the mean angle of scattering of the shower (for the particles of a given energy) does practically not depend on the matter in which it propagates. But the mean scattering angle of the greater part of the particles, corresponding to $E \sim z$ increases evidently with increasing atomic number. However, (19) shows that already in air the mean angle is of the order of unity and in lead most of the particles are fully scattered over the directions. This fact restricts the accuracy of the formula (3) for heavy elements.

3. THE WIDTH OF THE SHOWER

A similar method can be used for a calculation of the mean horizontal width of a shower at a given level.

Let y and z be the co-ordinates of a shower particle in the horizontal plane. The plane vector with these components we denote as r . In the kinetic equations (8), (9) we must add now to their left-hand side the derivatives with respect to y and z (the distribution functions Π and Γ are now function also of r).

Hence the initial equations are now:

$$\frac{\partial \Pi}{\partial t} + \left(\theta \cdot \frac{\partial \Pi}{\partial r} \right) = -A\Pi + B\Gamma + \delta \Delta_\theta \Pi, \quad \frac{\partial \Pi}{\partial t} + \left(\theta \cdot \frac{\partial \Gamma}{\partial r} \right) = C\Pi - D\Gamma \quad (20)$$

(differentiation with respect to r means a gradient in the plane y, z).

We multiply (20) by $(\theta \cdot r)$ and integrate it with respect to the angles $d\theta$ and the plane $dy dz$. The third term from the right-hand side of (20) vanishes

and the second terms on the left-hand side we integrate by parts and obtain:

$$\left. \begin{aligned} \frac{\partial}{\partial t} \int \Pi(\theta \cdot r) d\theta dy dz &= -A \int \Pi(\theta \cdot r) d\theta dy dz \\ &+ B \int \Gamma(\theta \cdot r) d\theta dy dz + \int \theta^2 \Pi d\theta dy dz, \\ \frac{\partial}{\partial t} \int \Gamma(\theta \cdot r) d\theta dy dz &= C \int \Pi(\theta \cdot r) d\theta dy dz \\ &- D \int \Gamma(\theta \cdot r) d\theta dy dz + \int \Gamma \theta^2 d\theta dy dz. \end{aligned} \right\} \quad (21)$$

We shall not perform here the calculations in full detail and shall only indicate their course. The integrals of $\Pi\theta^2$ and $\Gamma\theta^2$ were already calculated. Therefore, using (21) we can calculate the integrals of $(\theta \cdot r) \Pi$ and $(\theta \cdot r) \Gamma$. Further on we multiply (20) by r^2 and integrate with respect to $d\theta dy dz$ thus obtaining:

$$\left. \begin{aligned} \frac{\partial}{\partial t} \int \Pi r^2 d\theta dy dz &= -A \int \Pi r^2 d\theta dy dz \\ &+ B \int \Gamma r^2 d\theta dy dz + 2 \int (\theta \cdot r) \Pi d\theta dy dz \\ \frac{\partial}{\partial t} \int \Gamma r^2 d\theta dy dz &= C \int \Pi r^2 d\theta dy dz \\ &- D \int \Gamma r^2 d\theta dy dz + 2 \int (\theta \cdot r) \Gamma d\theta dy dz. \end{aligned} \right\} \quad (22)$$

The integrals of $\Pi(\theta \cdot r)$ and $\Gamma(\theta \cdot r)$ being known we can find from (22) the integrals of Πr^2 , Γr^2 and then directly the transverse section of the shower.

The formulae for the transverse widths $\overline{r_T^2}$, $\overline{r_H^2}$ are quite cumbersome and we shall not write them up here. We shall only note that these widths too turn out to be almost independent of the length of the path passed by the shower and of the kind of the particle which generates it.

In the point of the maximum ($s = 1$) we get:

$$\overline{r^2} = \frac{2500}{E^2} \quad (23)$$

(E in MeV, the length in the conventional units (10) in I). Hence we see that the particles with the energy E are displayed over a region with a width increasing with the decrease of their energy. The formula (23) is, of course, valid only for $E > \varepsilon$. For smaller energies it can be shown that the width tends to a constant value. In the ordinary units the width of the shower in the air turns out to be of the order of 250 m.

It is a pleasure to me to print this paper in the issue of the *Journal of Physics* dedicated to A. Joffe, who played such an eminent role in the development of the Soviet physics.

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44. ON THE THEORY OF SECONDARY SHOWERS

The number of showers produced by mesotrons is calculated. The maximum of the Rossi curve is also discussed.

It is known that a mesotron gives rise to a shower by means of ionisation, that producing δ -electrons which in their turn produce showers. We shall calculate here the probability of observing in a certain place a shower with more than a given number n of charged particles, i.e. we shall calculate the number N of such showers referred to a single mesotron.

A mesotron produces δ -electrons along its whole track. Every one of these δ -electrons produces, in general, a shower. Therefore, in a given place one observes showers originated by δ -electrons, which were produced on the whole upper part of the mesotron track. In every shower the total number of charged particles has a maximum at some distance traversed by the shower from the place of its origin. The magnitude of this maximum is larger, the greater the energy of the primary particle—in our case of the δ -electron. On the other hand, the probability of the δ -electron production decreases, as is known, with increasing energy of the δ -electron. It is, therefore, evident that in a given place those showers will be of importance which possess a maximum exactly on the place considered. Every other shower, which contains in the place considered no less than n particles, must have a maximum greater than n and is, therefore, less probable.

It was shown elsewhere¹ that the total number of particles in a shower is of the form $a e^{\varphi(t, \eta)}$, a being a slowly varying function (t is the length of the path measured in conventional units, cf. (11) in ref. 1; $\eta = \ln(E_0/\varepsilon)$, E_0 being the energy of the primary particle, i.e. of the δ -electron, ε is the ionisation energy loss per unit path). We are interested in the showers which contain in the given place (i.e. at a given t) a number of charged particles larger than n :

$$a e^{\varphi(t, \eta)} > n. \quad (1)$$

Let η_0 be the value of η giving a shower with the maximum equal to n . According to what was said above we are interested in the η values near to η_0 . Hence we can expand $\varphi(t, \eta)$ in a power series of $\eta - \eta_0$. According to (30) in ref. 1 the derivative $d\varphi(t, \eta)/d\eta$ is equal to 1 in the maximum point i.e. for $\eta = \eta_0$. Therefore, we obtain (1) in the form:

$$a e^{(\eta - \eta_0) + \varphi(t, \eta_0)} > n. \quad (2)$$

Л. Д. Ландау, К теории вторичных ливней, *Журнал Экспериментальной и Теоретической Физики* 11, 32 (1941).

L. Landau, On the theory of secondary showers, *J. Phys. U.S.S.R.* 4, 375 (1941).

In the following we shall need the probability of the δ -electron production by a mesotron. One gets easily the probability of the production of a δ -electron with an energy in the interval dE if one uses the well-known formulae (cf. e.g. ref. 2). Thus we find for the probability concerned:

$$\frac{\varepsilon}{L} \cdot \frac{dE}{E^2}; \quad (3)$$

L is here the logarithm which stands in the formula for the ionisation energy loss. With Fermi's correction³ we obtain for L :

$$L = 27.5 + \ln \frac{V}{\sum_i Z_i(Z_i + 1)}, \quad (4)$$

V being the molecular volume in cm^3 and Z_i the atomic numbers of atoms in the molecule. This gives numerically for

	Air	Water	Al	Pb
L	33	26	25	22

(5)

The energy ε is determined by means of the formula:

$$\varepsilon = \frac{26 L \sum_i Z_i}{\sum_i Z_i(Z_i + 1)} \quad (6)$$

(ε is measured in MeV).

The probability of the production of a δ -electron with an energy exceeding E is, according to (3):

$$\frac{\varepsilon}{LE} = \frac{1}{L} e^{-\eta}. \quad (7)$$

If the δ -electron is at the distance t from the observation point, it must, in order to produce a shower of more than n particles, have at least an energy η which is determined from (2) as

$$e^{-\eta} = \frac{a}{n} e^{-\eta_0 + \varphi(t, \eta_0)}. \quad (8)$$

We insert this into (7) and find the probability of the production of a δ -electron with an energy exceeding this value:

$$\frac{a}{nL} e^{-\eta_0 + \varphi(t, \eta_0)}.$$

The total number of the showers N with the number of particles exceeding n is hence obtained by means of an integration over the distances t :

$$N = \frac{1}{nL} \int a e^{-\eta_0 + \varphi(t, \eta_0)} dt.$$

This integral was calculated in ref. 4, formula (2):

$$\int_0^{\infty} a e^{\varphi(t, \eta_0)} dt = e^{\eta_0}.$$

Hence we find finally

$$N = \frac{1}{nL}. \quad (9)$$

This expression shows that N is almost independent of the substance which mesotrons traverse, because in all condensed bodies (N can be measured only in such bodies) L has almost equal values. Practically it suffices to write

$$N = \frac{1}{24n}. \quad (10)$$

Consider now the maximum of the Rossi curve which gives the number of showers produced by a mesotron in a thin plate (for instance, a Pb plate) as a function of the thickness of the plate. The most part of the showers observed behind the plate are produced in the lead by the δ -electrons produced in the air above the plate. The probability of the δ -electron production in the air (per a conventional unit path), being proportional to the ionisation energy loss ε (6), is in fact larger than that in the lead. On the other hand, the showers produced by δ -electrons in the lead contain more particles than the showers produced in the air (the number of particles in a shower produced by an electron of the energy E is of the order of E/ε).¹

In calculating the number N_{\max} of the showers we must correspondingly take the δ -electron production probability in the air and the limiting energy η (8) in the lead. This gives

$$N_{\max} = \frac{\varepsilon_1}{\varepsilon_2 L_1} \cdot \frac{1}{n}, \quad (11)$$

the index 1 standing for air, 2—for lead. The equilibrium number N_2 of showers in lead (i.e. in a thick layer of lead) is according to (9) $N_2 = 1/n L_2$. The ratio of these both numbers is

$$\frac{N_{\max}}{N_2} = \frac{\varepsilon_1 L_2}{\varepsilon_2 L_1},$$

or, inserting the expressions for L and ε :

$$\frac{N_{\max}}{N_2} = \frac{\bar{Z}_2}{\bar{Z}_1}, \quad (12)$$

\bar{Z} is here (as in ref. 1) the mean atomic number defined as $\sum Z_i(Z_i + 1)/\sum Z_i$. The ratio (12) for lead is equal to 10, for aluminium to 1.7. Of course, we did not take here into account that apart from the δ -electrons there exist also the mesotron decay electrons; this must somewhat increase the ratio (12).

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45. ON THE SCATTERING OF LIGHT BY MESOTRONS

Landau's criterion is used for estimating the cross-sections at high energies for the following processes involving mesotrons: (1) Compton effect on a mesotron, (2) mesotron pair production by a collision of two photons, (3) Coulomb scattering of mesotrons. These cross-sections do not increase with energy, as they do when calculated by means of perturbation theory methods.

1. INTRODUCTION

When investigating processes, involving mesotrons, we have to deal with difficulties which do not occur in processes involving electrons. This difficulty is the well-known increase of the scattering cross-section of mesotrons with increasing energies.

It was shown recently¹ that this increase is connected with the fact that one must ascribe to a mesotron a finite radius which is equal (for a mesotron at rest) to its "classical electromagnetic" value $e^2/\mu c^2$ (μ —the mass of a mesotron), whereas an electron may be considered as a point in quantum mechanics. This is to be understood in the sense that in interacting, say, with an electromagnetic field an electron acts as a point charge up to arbitrarily small wave lengths. But a mesotron acts as a point charge and can be described by Proca's equation only for wave lengths larger than its electromagnetic radius.

The existing theory breaks down for smaller wavelengths. This is seen at once if we notice that the effective cross-sections are so large in this region that the number of scattered particles with a given angular momentum turns out to be larger than that in the incident stream, this being absurd.

At present it is impossible to calculate exactly the effective cross-sections for very large energies. Using Landau's criterion it is, however, possible to give an estimate of these cross-sections for any energy, however great it is. The cross-sections thus obtained do not increase with energy. In this paper Landau's criterion is applied to processes in which mesotrons interact with an electromagnetic field, but which are not connected with mesotron emission.

2. COMPTON EFFECT

The Compton effect cross-section was calculated by one of us² and also by Booth and Wilson³ and Kobayasi and Utiyama⁴. For the energies $\mu c \ll E < 137 \mu c^2$ of the incident photon and in the frame of reference where the meso-

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tron was (before the collision) at rest, this cross-section is equal to

$$\frac{5\pi}{18} \left(\frac{e^2}{\mu c^2} \right)^2 \frac{E}{\mu c^3} \quad (2.1)$$

To find the limits of applicability of this formula and to estimate the cross-section for energies larger than $137 \mu c^2$ it is necessary to calculate the scattering cross-sections for photons with a given angular momentum. The maximum value of such a cross-section (in the frame of reference where the centre of inertia of both particles is at rest) is

$$S_e = 4\pi(2l + 1) \lambda^2 \quad (2.2)$$

The transition current is, on the same frame of reference (cf. ref. 2, formula (3.14)):

$$\begin{aligned} j_{IF} = e^2 \{ & (ik/\kappa^2 \delta') [(H^+ \cdot \bar{G}) - (E^+ \cdot \bar{F})] [(1 - \omega/c k) F + [n + n' \wedge G]] \\ & + (1/\delta') [(\bar{v} \wedge H^+) \wedge G] + (1/\delta') \bar{v}_0 [E^+ \wedge G] + (1/\delta') [(\bar{v} \wedge E^+) \wedge F] \\ & + (2ik/\delta') (A^+ \cdot n') (\bar{v} \wedge G) + \bar{v}_0 F - (ik/\delta) (1 + (\omega/c k)) \bar{v}_0 [(v \wedge H^+) + v_0 E^+] \\ & + (ik/\kappa^2 \delta) (1 + (\omega/c k)) \bar{F} [(H^+ \cdot G) - (E^+ \cdot F)] + (1/\delta) [(v \wedge H^+) \wedge \bar{G}] \\ & + (1/\delta) v_0 [E^+ \wedge \bar{G}] + (1/\delta) (v \cdot E^+) \bar{F} - (ik/\delta') n [(v \cdot \bar{v} \wedge H^+) + v_0 (E^+ \cdot v)] \\ & + (ik/\delta') n \bar{v}_0 (v \cdot E^+) + (ik/\delta') [(v \cdot n + n') - v_0 (1 - (\omega/c k))] (\bar{v} \wedge H^+) + \bar{v}_0 E^+ \\ & + (2k^2/\delta') (A^+ \cdot n') [(\bar{v} \cdot v) - \bar{v}_0 v_0] n - (2k^2/\delta') (A^+ \cdot n') [(v \cdot n + n') \\ & - v_0 (1 - (\omega/c k))] \bar{v} - 2[(\bar{v} \cdot v) - \bar{v}_0 v_0] A^+ + (\bar{v} \cdot A^+) v + (v \cdot A^+) \bar{v} \\ & + \text{compl conj. term} \}, \quad (2.3) \end{aligned}$$

where $k n$ —the wave vector of the incident photon; $k n'$ —that of the scattered one; v, v_0, F, G —the wave functions of a free mesotron (one four-dimensional and two three-dimensional vectors). The conjugated complex functions refer to the final state; ω —energy of the mesotron;

$$\delta = -2k^2 \left(1 + \frac{\omega}{c k} \right), \quad \delta' = 2k^2 \left(\cos \vartheta + \frac{\omega}{c k} \right), \quad \vartheta = (n \cdot n'), \quad \kappa = \frac{\mu c}{\hbar};$$

A —vector potential of the electromagnetic field

$$A = A^+ e^{i[(k \cdot r) - c k t]} + A^- e^{-i[(k \cdot r) - c k t]}, \quad H^+ = i[k \wedge A^+], \quad E^+ = i k A^+.$$

We see that this expression contains terms of two kinds. In terms which do not contain δ' the dependence on scattering angle is due only to the numerator, which contains sine or cosine of this angle. Such terms correspond only to small values of the angular momentum l (namely to $l = 0$ or $l = 1$); therefore, they are of no importance for us and we can omit them. The rest of the terms contain $1 + \cos \vartheta$ in denominator and, being expanded in spherical harmonics give terms which correspond to all values of l . Neglecting the

terms with small l , we thus obtain the following expressions for the vector product of the transition current with a unit vector n' in the direction of the quantum emitted:

$$B = \frac{1}{A^+} [\mathbf{j}_{IF} \wedge n'].$$

1. The primary and the final polarisation of the mesotron are longitudinal

$$B^{L-L} = -\frac{e^2}{2\mu c^2} \left(\frac{k}{\kappa}\right) \frac{1}{1 + (n \cdot n')} (e \cdot n') [n \wedge n']. \quad (2.4)$$

2. Both are transversal

$$B^{T-T} = \frac{e^2}{4\mu c^2} \left(\frac{k}{\kappa}\right) \frac{1}{1 + (n \cdot n')} (\alpha \cdot n') (e \cdot n') (\alpha' \cdot n) [n \wedge n']; \quad (2.5)$$

α and α' are unit vectors of the primary and final polarisations. The contributions to the effective cross-sections due to transitions with a change of the mesotron polarisation are smaller and we can neglect them here.

This property of the mesotron transitions was repeatedly noted. It corresponds to matrix elements of the first order being larger (for large energies) for the processes with a change of polarisation than for ones with no such change. Hence in the Compton effect which is a second order process, the main role is played by the processes with the same polarisation in the primary and final states.

The differential cross-section is (cf. ref. 2, formula (2.7)):

$$\sigma = \frac{1}{2} B^2. \quad (2.6)$$

We must expand expressions (2.4) and (2.5) in spherical harmonics. Such an expansion is, however, quite cumbersome. Therefore, we shall use another method, which is sufficient for our purpose.

It is easily seen that the scattering due to the term $(1 + \cos \vartheta)$ in the denominator occurs mainly in the angles near $\vartheta = \pi$. But for such angles we can replace a spherical harmonic by a plane wave with a wave vector in the plane, normal to the wave vector of the quantum, scattered at an angle π .

The spherical harmonics Y_l^m satisfies, actually, the equation

$$\Delta^* Y_l^m = -l(l+1) Y_l^m, \quad (2.7)$$

Δ^* being Laplace operator on a sphere:

$$\Delta^* = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2}.$$

If $\vartheta = \pi - \theta$ with a small θ , this operator can be written as

$$\Delta^* = \frac{1}{\theta} \frac{\partial}{\partial \theta} \theta \frac{\partial}{\partial \theta} + \frac{1}{\theta^2} \frac{\partial^2}{\partial \varphi^2}.$$

This operator is identical with the Laplace operator on a plane θ standing for the radius vector. If we also replace $l(l+1)$ for large l by l^2 , the equation (2.7) thus becomes a wave equation in a plane.

The solution of this equation is a plane wave with a wave vector l . The absolute value of this vector is l (the index of the spherical harmonic) and the different values of the index m correspond to different orientations of the wave vector in a plane.

Thus, the expansion over the spherical harmonics can be replaced in our case by an expansion on a two-dimensional Fourier integral.

Let $f(\theta, \varphi)$ be a wave function with a maximum at $\vartheta = \pi$, describing some scattering process. The expansion of $f(\theta, \varphi)$ is

$$f(\theta, \varphi) = \frac{1}{(2\pi)^2} \iint f_1 e^{i(\theta \cdot \vartheta)} d l_x d l_y,$$

$\vartheta = (\vartheta_x, \vartheta_y)$ being cartesian co-ordinates corresponding to polar co-ordinates θ, φ . The integral of the square of the modulus of $f(\theta, \varphi)$ is equal to

$$\iint |f(\theta, \varphi)|^2 d\Omega = \frac{1}{(2\pi)^2} \iint |f_1|^2 d l_x d l_y.$$

If we introduce polar co-ordinates in the l_x, l_y plane and integrate over the angle, we get

$$\iint |f(\theta, \varphi)|^2 d\Omega = \frac{1}{2\pi} \int |f(l)|^2 dl, \quad (2.8)$$

where

$$|f(l)|^2 = l |f_l|^2, \quad (2.9)$$

which determines the scattering corresponding to an angular momentum l .

The case of a vector function (in the scattering of light) is somewhat different from that of a scalar function. For large angular momenta, however, we can pay no attention to this circumstance and expand every component as a scalar.†

To expand the vectors (2.4) and (2.5) we choose the z axis in a direction opposite to that of the vector n , and the x axis along the vector l .

The components of the vector B^{L-L} are then of the form:

$$B_x^{L-L} = \frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \cdot \frac{n'_x n'_y}{n'^2_x + n'^2_y}, \quad B_y^{L-L} = -\frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \cdot \frac{n'^2_x}{n'^2_x + n'^2_y}, \quad B_z^{L-L} = 0. \quad (2.10)$$

We used here the relation

$$1 - n'_z \approx \frac{1}{2} (n'^2_x + n'^2_y),$$

which is valid for n'_z close to unity.

† The electric multipole emission with a momentum l , for instance, is described by a linear combination of spherical harmonics with indexes $l+1, l-1$ but not l .

Their Fourier components are equal to

$$\left. \begin{aligned} B_x^{L-L}(l) &= \frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \iint \frac{n'_x n'_y}{n_x'^2 + n_y'^2} e^{i(n'_x l_x + n'_y l_y)} dn'_x dn'_y = \\ &= \frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \cdot \frac{4\pi}{e^4} l_x l_y, \\ B_y^{L-L}(l) &= -\frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \iint \frac{n_x'^2}{n_x'^2 + n_y'^2} e^{i(n'_x l_x + n'_y l_y)} dn'_x dn'_y = \\ &= \frac{e^2}{\mu c^2} \cdot \frac{k}{\kappa} \cdot \frac{2\pi}{e^4} (l_x^2 - l_y^2) \\ B_z^{L-L}(l) &= 0. \end{aligned} \right\} \quad (2.11)$$

The square of the vector $B^{L-L}(l)$ corresponding to an angular momentum l is equal to

$$l |B^{L-L}(l)|^2 = \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{\varepsilon}{\mu c^2} \right)^2 \cdot \frac{2\pi}{e^2},$$

$\varepsilon = \hbar c k$ being the photon energy.

Thus, the effective cross-section for the scattering of photons with an angular momentum l is, according to (2.6):

$$\sigma_l \sim \frac{1}{l^3} \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{\varepsilon}{\mu c^2} \right)^2. \quad (2.12)$$

It is easily seen that the expansion of B^{T-T} leads to $l |B^{T-T}(l)|^2$ which is proportional $1/l^7$. Hence we can omit these terms.

Expression (2.12) determines, therefore, the total scattering cross-section. We use now Landau's criterion. According to Landau the formula is correct up to energies defined by the quality

$$\frac{1}{l^3} \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{\varepsilon}{\mu c^2} \right)^2 \sim l \lambda^2 \sim \frac{\hbar^2 c^2}{\varepsilon^2} l$$

(we omit numerical coefficients), i.e. up to the energy

$$\varepsilon \sim \sqrt{\frac{\hbar c}{e}} \cdot l. \quad (2.13)$$

For a given energy (larger than $\sqrt{137} \mu c^2$) (2.12) is valid, *vice versa*, for all l satisfying the inequality

$$l \geq l_0 = \sqrt{\frac{e^2}{\hbar c}} \cdot \varepsilon. \quad (2.14)$$

It is impossible to find cross-sections for smaller l . One can, however, affirm that these cross-sections must satisfy in any case the inequality

$$\sigma_l \leq l \lambda^2. \quad (2.15)$$

The total cross-section is a sum of two parts. The first is the integral of (2.12) over all $l > l_0$. This gives

$$\sigma \sim \frac{\hbar c}{e^2} \left(\frac{e^2}{\mu c^2} \right)^2 = \frac{\hbar}{\mu c} \cdot \frac{e^2}{\mu c^2}. \quad (2.16)$$

The second part is at any rate not larger than the integral of (2.2) over all $l < l_0$, which is $(\hbar/\mu c) \cdot e^2/\mu c^2$. This expression coincides with (2.16). Hence one can think that (2.16) is the correct order of magnitude for the scattering cross-section.

Thus, the scattering is proportional to the square of the energy for photon energies less than $\sqrt{137} \mu c^2$, and is constant for larger energies.

In the frame of reference of the centre of inertia this scattering takes place only in a small angle around $\vartheta = \pi$. This is a result of the fact that in scattering mainly the waves with large angular momenta take part. One can hence conclude in the usual manner that in this case the amplitudes of the scattered wave (2.11) will be different from zero only for angles

$$l_0 \theta_0 \sim 1,$$

l_0 being the least of the angular momenta, taking part in the scattering.

For angles near π the scattering is determined by

$$d\sigma(\vartheta) = \frac{\pi}{6} \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{\varepsilon}{\mu c^2} \right)^2 \theta \quad (\theta = \pi - \vartheta).$$

We integrate up to the angle

$$\theta_0 \sim \frac{1}{l_0} \sim \sqrt{\frac{\hbar c}{e^2}} \cdot \frac{\mu c^2}{\varepsilon} \quad (2.17)$$

and obtain a cross-section which is independent of the energy, this being in agreement with our estimate. Hence, in the angle within which the scattering takes place it is described by the ordinary formula derived by perturbation theory methods.

In the frame of reference, where the mesotron is at rest, the energy transferred from a photon to the mesotron is equal to

$$\Delta E = \gamma p \cos^2 \vartheta,$$

ϑ being the scattering angle in the frame of reference of the centre of inertia, and p —the momentum in this frame; γ is

$$\gamma = \sqrt{1 - \frac{u^2}{c^2}},$$

with u being the relative velocity of the both systems.

When $\theta = \theta_0$ the transferred energy is

$$\Delta E \sim E \left(1 - \frac{2\hbar c}{e^2} \cdot \frac{\mu c^2}{E} \right).$$

Hence, in the frame considered the photon transfers nearly all its energy to the mesotron. After being scattered the photon is left with an energy of the order of magnitude not exceeding $137\mu c^2$.

Hence the scattering occurs always in such a manner that photons are left with energies less than the critical one. As the scattering is then described by an ordinary formula (in the frame of reference where the mesotron was initially at rest)

$$d\sigma = \frac{1}{96} \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{E}{\mu c^2} \right)^2 \left\{ 11 - 10 \frac{\cos\theta - \frac{E}{\mu c^2} (1 - \cos\theta)}{1 + \frac{E}{\mu c^2} (1 - \cos\theta)} + 7 \left(\frac{\cos\theta - \frac{E}{\mu c^2} (1 - \cos\theta)}{1 + \frac{E}{\mu c^2} (1 - \cos\theta)} \right)^2 \right\} \frac{d\Omega}{\left[1 + \frac{E}{\mu c^2} (1 - \cos\theta) \right]^2}, \quad (2.18)$$

our criterion can be formulated as follows: (2.18) describes the photon scattering correctly if at least one of the photons (the incident or the scattered one) has an energy less than $137\mu c^2$.

3. MESOTRON PAIR PRODUCTION BY TWO PHOTONS

The results of the preceding section can be applied to a mesotron pair production by a collision of two photons. In the co-ordinate system, where the total momentum of both is zero, this process is determined by formula⁴:

$$d\sigma = \frac{1}{32} \left(\frac{e^2}{\mu c^2} \right)^2 \left(\frac{\varepsilon}{\mu c^2} \right)^2 (13 + \cos^2\theta) d\Omega, \quad (3.1)$$

ε being the energy of the particles ($\varepsilon \gg \mu c^2$).

This formula is derived as a Compton effect on a mesotron with a negative energy accompanied by a transition of the mesotron into a positive energy state. This is seen from a comparison of matrix elements for photon absorption and pair creation, which turn out to be equal (up to the sign) [cf. ref. 3, p. 90].

Bearing this in mind, we can apply now to the pair production the criterion obtained for the Compton effect.

In the co-ordinate system, where the mesotron is at rest, pair production occurs only when either of the photons has a wave length which is less than the mesotron radius.

We can write the applicability limits in the system where the total momentum is zero if we use the invariance of the scalar product of the mesotron and the photon four-dimensional momenta. In the initial system (of the mesotron at rest)

this four-dimensional product is equal to

$$\mu c^2 \frac{\hbar c}{e^2};$$

the energy has its maximum value.

Equalizing this quantity to the same scalar product in the new co-ordinate system, we get a condition which connects the maximum photon energy E_0 with the maximum angle θ_0 between mesotron and photon directions

$$\mu c^2 \frac{\hbar c}{e^2} = E_0^2 (1 - \cos \theta_0). \quad (3.2)$$

Hence for large energies the scattering takes place only inside the angle

$$\theta_0 \sim \frac{\mu c^2}{E_0} \sqrt{\frac{\hbar c}{e^2}}.$$

We obtain the cross-section for large energies if we integrate (3.1) over all angles up to θ_0

$$\sigma \sim \frac{\hbar c}{e^2} \left(\frac{e^2}{\mu c^2} \right)^2 = \frac{\hbar}{\mu c} \cdot \frac{e^2}{\mu c^2}. \quad (3.3)$$

4. THE SCATTERING OF MESOTRONS IN A COULOMB FIELD

The mesotron scattering in a Coulomb field cannot be reduced to a Compton effect. It is, however, possible to make some conclusions as to the applicability limits of the formula (derived by Massey and Corben⁶) in this case also.

Consider the scattering in the co-ordinate system where the mesotron is at rest. The field of the incident proton or electron can be expanded in a Fourier integral. The component of such an expansion differs from that of the field in vacuum by having a zero frequency. It is sometimes called a pseudo-photon. The scattering consists of an absorption of either of Fourier components by a mesotron, which thus gets a momentum equal to that of the pseudo-photon absorbed.

Hence we have here to deal with a process in which only one pseudo-photon takes part. One can assume, in analogy with the Compton effect, that the Massey-Corben formulae are valid up to the pseudo-photon energies equal to $137 \mu c^2$. If the energy of the quantum is larger, the mesotron will interact with it much more weakly and there will be almost no scattering.

In the co-ordinate system where the scattering particle is at rest this corresponds to the mesotron being incapable of giving up an energy larger than $E_c = (\mu/M_0) 137 \mu c^2$, M_0 being the mass of the scatterer. The Massey-Corben formula is valid in this co-ordinate system if the mesotron energy is less than E_c . If the energy is larger the scattering will take place only inside an angle

$$\theta \sim \sqrt{\frac{\hbar c}{e^2}} \frac{\mu c^2}{E}.$$

In the case of scattering on electrons this critical energy is $E_c \sim 2.8 \times 10^{12}$ eV for the scattering on protons $E_c \sim 1.4 \times 10^9$ eV.

Due to the high critical energy value the variations of Massey-Corben formula are of no importance in the case of electrons. They are, however, important for the scattering on protons.

For mesotron energies $> Mc^2$ (M —proton mass) the cross-section for the scattering on protons is

$$S(\varepsilon) d\varepsilon = \frac{2\pi}{3} \left(\frac{e^2}{\mu c^2} \right)^2 \frac{d\varepsilon}{s} \left[1 - \frac{\varepsilon}{E} + \frac{\varepsilon^2}{2E^2} \right]; \varepsilon \gg \frac{\mu}{M} \mu c^2, \quad (4.1)$$

ε being the energy transferred in a collision, E —the mesotron energy. For small ε we have the usual ionization losses.

The specific mesotron energy loss is according to (4.1)

$$\int_0^E \varepsilon S(\varepsilon) d\varepsilon = \frac{4\pi}{9} \left(\frac{e^2}{\mu c^2} \right)^2 E, \quad (4.2)$$

for

$$E < \frac{\mu}{M} 137 \mu c^2 = E_c,$$

and

$$\int_0^{E_c} \varepsilon S(\varepsilon) d\varepsilon \sim \left(\frac{e^2}{Mc^2} \right)^2 137 \mu c^2, \quad (4.3)$$

for

$$E > E_c.$$

It is interesting to compare the energy loss in a collision with a proton with ionisation energy loss

$$\frac{2\pi e^4}{m c^2} L$$

(m —electron mass, L —the ionisation logarithm, which is in any case not less than unity).

It is seen from (4.1) and (4.3) that the energy loss in the scattering on protons is less than the ionization energy loss*.

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† If the Massey-Corben formula would be valid for arbitrary energies, these energy losses would become equal for mesotron energies $(\mu/M) \cdot \mu c^2$, having then the value (4.2).

46. THE THEORY OF SUPERFLUIDITY OF HELIUM II

The quantisation of an arbitrary system of interacting particles (a liquid) is performed by means of introducing the operators of the density and of the velocity of the liquid; the commutation rules between these operators are determined (§ 1). From the results of this quantisation the general character of the distribution of the energy levels in the spectrum of a quantum liquid is determined (§ 2). The temperature dependence of the heat capacity of helium II is investigated (§ 3). It is shown that at absolute zero a quantum liquid can possess the property of superfluidity (§ 4). At non-zero temperatures it is found that two motions—a superfluid and a normal—can simultaneously exist in helium II. This can be described by means of the conception of the superfluid and normal parts of the liquid; the λ -point in helium II is connected with the disappearance of the “superfluid” part of the liquid (§ 5). The experiments made to measure the heat conductivity and viscosity of helium II are interpreted; the thermomechanical effects in helium II are considered (§ 6). A system of hydrodynamic equations is advanced describing the macroscopic motion of helium II (§ 7). By means of these equations the propagation of sound is investigated and it is shown that two velocities of sound must exist in helium II (§ 8).

Liquid helium is known to possess a number of peculiar properties at temperatures lower than the λ -point. Of these properties the most important one is superfluidity discovered by P. L. Kapitza¹—the lack of viscosity during the flow of helium through a thin capillary or slit.

All these properties, including the fact that helium exists as a liquid right down to absolute zero, obviously cannot be explained by the classical theory and are connected with quantum phenomena.

L. Tisza² suggested that helium II should be considered as a degenerate ideal Bose gas. He suggested that the atoms found in the normal state (a state of zero energy) move through the liquid without friction. This point of view, however, cannot be considered as satisfactory. Apart from the fact that liquid helium has nothing to do with an ideal gas, atoms in the normal state would not behave as “superfluid”. On the contrary, nothing could prevent atoms in a normal state from colliding with excited atoms, i.e. when moving through the liquid they would experience a friction and there would be no superfluidity at all. In this way the explanation advanced by Tisza not only has no foundation in his suggestions but is in direct contradiction to them.

1. THE QUANTISATION OF THE MOTION OF LIQUIDS

An arbitrary system of interacting particles (a liquid) can be described in classical theory by means of the density ρ and the flow of mass \mathbf{j} , which are determined in the following manner. Let \mathbf{R} be the radius-vector of an arbitrary

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point in space and \mathbf{r}_α —the radius-vector of a particle with a mass m_α . Then ρ is determined as

$$\rho = \sum m_\alpha \delta(\mathbf{r}_\alpha - \mathbf{R}), \quad (1.1)$$

δ being the three-dimensional δ -function and the summation is extended over all particles in the system. The volume-integral $\int \rho dV$ gives the total mass of the system. Similarly the density \mathbf{j} of the flow of the mass is determined as

$$\mathbf{j} = \sum m_\alpha \mathbf{v}_\alpha \delta(\mathbf{r}_\alpha - \mathbf{R}) = \sum \mathbf{p}_\alpha \delta(\mathbf{r}_\alpha - \mathbf{R})$$

(\mathbf{v}_α , \mathbf{p}_α are the velocity and the momentum of the particle m_α).

It must be emphasised that in such a description of a liquid there is no averaging in that sense in which it is done in statistics. This description proceeds from the microscopic picture as all the particles possess (at a given moment) definite co-ordinates \mathbf{r}_α and velocities \mathbf{v}_α .

When passing over to the quantum theory ρ and \mathbf{j} must be regarded as certain operators the form of which must be determined. For the sake of simplicity suppose that the system consists of one particle only. Then the classical density is $\rho = m \delta(\mathbf{r} - \mathbf{R})$. The operator ρ must be determined in such a way that its mathematical expectation $\int \psi^*(\mathbf{r}) \rho \psi(\mathbf{r}) dV$ [$\psi(\mathbf{r})$ being the wave function of the particle] equals the density of the mass at the point \mathbf{R} , i.e. $m|\psi(\mathbf{R})|^2$. From this it follows that the operator ρ must have the same form $\rho = m \delta(\mathbf{r} - \mathbf{R})$ and in the case of an arbitrary system of particles—correspondingly the form (1.1).

The classical density of the flow for one particle is $\mathbf{j} = \mathbf{p} \delta(\mathbf{r} - \mathbf{R})$. It is easy to see that the corresponding operator is

$$\mathbf{j} = \frac{1}{2} [\mathbf{p} \delta(\mathbf{r} - \mathbf{R}) + \delta(\mathbf{r} - \mathbf{R}) \mathbf{p}],$$

where \mathbf{p} is the usual operator of momentum:

$$\mathbf{p} = \frac{\hbar}{i} \nabla$$

(∇ denotes the differentiation with respect to \mathbf{r}). Actually the mathematical expectation of \mathbf{j} is

$$\int \psi^*(\mathbf{r}) \mathbf{j} \psi(\mathbf{r}) dV = \frac{\hbar}{2i} \int \psi^* \nabla \delta(\mathbf{r} - \mathbf{R}) \psi dV + \frac{\hbar}{2i} \int \psi^* \delta(\mathbf{r} - \mathbf{R}) \nabla \psi dV$$

or, integrating the first term by parts:

$$\begin{aligned} \int \psi^* \mathbf{j} \psi dV &= -\frac{\hbar}{2i} \int \psi \delta(\mathbf{r} - \mathbf{R}) \nabla \psi^* dV + \frac{\hbar}{2i} \int \psi^* \delta(\mathbf{r} - \mathbf{R}) \nabla \psi dV \\ &= \frac{\hbar}{2i} \{ \psi^*(\mathbf{R}) \nabla \psi(\mathbf{R}) - \psi(\mathbf{R}) \nabla \psi^*(\mathbf{R}) \}, \end{aligned}$$

i.e. exactly what it ought to be. For an arbitrary system we have, similarly

$$j = \frac{1}{2} \sum [p_\alpha \delta(r_\alpha - R) + \delta(r_\alpha - R) p_\alpha], \quad (1.2)$$

$$p_\alpha = \frac{\hbar}{i} \nabla_\alpha$$

We now determine the commutation rules. For the density ϱ we obviously have

$$\varrho_1 \varrho_2 - \varrho_2 \varrho_1 = 0 \quad (1.3)$$

(ϱ_1, ϱ_2 denote $\varrho(R_1), \varrho(R_2)$, respectively).

For the sake of brevity let us consider only one term from each of the sums (1.1) and (1.2) when determining the commutation rules, as the operators corresponding to different particles commute with each other. To determine the commutation of ϱ with j we write

$$\begin{aligned} j_1 \varrho_2 - \varrho_2 j_1 = \frac{m \hbar}{2i} \{ [\nabla \delta(r - R_1) + \delta(r - R_1) \nabla] \delta(r - R_2) \\ - \delta(r - R_2) [\nabla \delta(r - R_1) + \delta(r - R_1) \nabla] \} \end{aligned}$$

To simplify the expression on the right-hand side we note that the operators of the form

$$\delta(r - R_1) \nabla \delta(r - R_2)$$

can be transformed in the following way:

$$\delta(r - R_1) \nabla \delta(r - R_2) = \delta(r - R_1) (\nabla \delta(r - R_2)) + \delta(r - R_1) \delta(r - R_2) \nabla,$$

where in the first term $(\nabla \delta(r - R_2))$ denotes simply a gradient of the δ -function, i.e. ∇ is no longer an operator. Owing to the presence of the factor $\delta(r - R_1)$ in this term one can write $(\nabla \delta(R_1 - R_2))$ instead of $(\nabla \delta(r - R_2))$. In this way

$$\delta(r - R_1) \nabla \delta(r - R_2) = \delta(r - R_1) (\nabla \delta(R_1 - R_2)) + \delta(r - R_1) \delta(r - R_2) \nabla.$$

Similarly

$$\nabla \delta(r - R_1) \delta(r - R_2) = \delta(r - R_2) \nabla \delta(r - R_1) + \delta(r - R_1) (\nabla \delta(R_1 - R_2)).$$

The result is

$$j_1 \varrho_2 - \varrho_2 j_1 = \frac{\hbar}{i} m \delta(r - R_1) \nabla \delta(R_1 - R_2)$$

or for an arbitrary system

$$j_1 \varrho_2 - \varrho_2 j_1 = \frac{\hbar}{i} \varrho_1 \nabla \delta(R_1 - R_2) \quad (1.4)$$

(It makes no difference whether we write ϱ_1 or ϱ_2 on the right-hand side in view of the presence of the δ -function of $R_1 - R_2$.)

In a similar way the commutation rules between the components of the vector j with each other can be obtained. The calculation in this case is longer and we will not enter into it here.

We introduce the operator v of the velocity of the liquid according to

$$j = \frac{1}{2} (\varrho v + v \varrho), \quad (1.5)$$

$$v = \frac{1}{2} \left(\frac{1}{\varrho} j + j \frac{1}{\varrho} \right). \quad (1.6)$$

It will be more convenient to use the operator v instead of the operator of the flow j .

For the commutation rule of ϱ with v we have

$$\begin{aligned} v_1 \varrho_2 - \varrho_2 v_1 &= \frac{1}{2} \left(\frac{1}{\varrho_1} j_1 + j_1 \frac{1}{\varrho_1} \right) \varrho_2 - \frac{1}{2} \varrho_2 \left(\frac{1}{\varrho_1} v_1 + v_1 \frac{1}{\varrho_1} \right) \\ &= \frac{1}{2\varrho_1} (j_1 \varrho_2 - \varrho_2 j_1) + \frac{1}{2} (j_1 \varrho_2 - \varrho_2 j_1) \frac{1}{\varrho_1} \end{aligned}$$

or, on inserting (1.4):

$$v_1 \varrho_2 - \varrho_2 v_1 = \frac{\hbar}{i} \nabla \delta(\mathbf{R}_1 - \mathbf{R}_2). \quad (1.7)$$

The commutation rules for the components of v are found to be

$$v_{1i} v_{2k} - v_{2k} v_{1i} = \frac{\hbar}{i} \delta(\mathbf{R}_1 - \mathbf{R}_2) \frac{1}{\varrho_1} (\text{curl } v)_{ik}, \quad (1.8)$$

where $(\text{curl } v)_{ik}$ denotes the difference

$$\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}.$$

Further on we shall also need the commutation rule between ϱ and $\text{curl } v$. By applying the operation curl (with a differentiation with respect to coordinates \mathbf{R}_1) to both sides of the equation (1.7) we get

$$\text{curl } v_1 \cdot \varrho_2 - \varrho_2 \cdot \text{curl } v_1 = 0. \quad (1.9)$$

It is easy to see that by applying the formulae obtained to the macroscopic movement of the liquid we get, as required, the usual hydrodynamic equations written in an operational form. The energy of a unit volume of a classical liquid considered macroscopically is

$$\frac{\varrho v^2}{2} + \varrho \varepsilon(\varrho),$$

where $\varepsilon(\varrho)$ is the internal energy of a unit mass of the liquid. It is supposed that the energy ε depends only on the density ϱ of the liquid; this corresponds

to the macroscopic character of the consideration and is connected with a statistical averaging. For a microscopic investigation this supposition is, of course, invalid.

The corresponding quantum operator is†

$$\frac{\mathbf{v} \varrho \mathbf{v}}{2} + \varrho \varepsilon(\varrho).$$

The Hamiltonian H of the liquid is an integral over the volume

$$H = \int \left\{ \frac{\mathbf{v} \varrho \mathbf{v}}{2} + \varrho \varepsilon(\varrho) \right\} dV. \quad (1.10)$$

For the derivative of the density ϱ with respect to time one has

$$\dot{\varrho} = \frac{i}{\hbar} (H \varrho - \varrho H).$$

We shall denote temporarily the co-ordinates of the point at which ϱ is taken by the index 1, and the co-ordinates of the variable point in the region of integration in (1.10) by the index 2. Then

$$\dot{\varrho}_1 = \frac{i}{\hbar} \int \left\{ \frac{1}{2} [\mathbf{v}_2 \varrho_2 \mathbf{v}_2 \varrho_1 - \varrho_1 \mathbf{v}_2 \varrho_2 \mathbf{v}_2] + [\varrho_2 \varepsilon(\varrho_2) \varrho_1 - \varrho_1 \varrho_2 \varepsilon(\varrho_2)] \right\} dV_2.$$

In view of (1.3) the second term under the sign of the integration vanishes, and the first can be written as

$$\frac{1}{2} [\mathbf{v}_2 \varrho_2 (\mathbf{v}_2 \varrho_1 - \varrho_1 \mathbf{v}_2) + (\mathbf{v}_2 \varrho_1 - \varrho_1 \mathbf{v}_2) \varrho_2 \mathbf{v}_2]$$

or, by introducing (1.7):

$$\frac{\hbar}{2i} \nabla \delta(\mathbf{R}_2 - \mathbf{R}_1) (\mathbf{v}_2 \varrho_2 + \varrho_2 \mathbf{v}_2) = \frac{\hbar}{i} \nabla \delta(\mathbf{R}_2 - \mathbf{R}_1) \mathbf{j}_2.$$

In this way

$$\dot{\varrho}_1 = \int \nabla \delta(\mathbf{R}_2 - \mathbf{R}_1) \mathbf{j}_2 dV_2 = - \int \delta(\mathbf{R}_2 - \mathbf{R}_1) \operatorname{div} \mathbf{j}_2 dV_2 = - \operatorname{div} \mathbf{j}_1,$$

i.e. we come to the continuity equation in operational form:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \frac{\varrho \mathbf{v} + \mathbf{v} \varrho}{2} = 0. \quad (1.11)$$

† The operator $\frac{\mathbf{v} \varrho \mathbf{v}}{2}$ can also be written in the form

$$\frac{\varrho \mathbf{v}^2 + \mathbf{v}^2 \varrho}{4}$$

In a similar way the derivative:

$$\dot{v} = \frac{i}{\hbar} (Hv - vH),$$

can be calculated, which brings us to the equation

$$\frac{\partial v_i}{\partial t} + \frac{1}{2} \left(v_k \frac{\partial v_i}{\partial x_k} + \frac{\partial v_i}{\partial x_k} v_k \right) = - \frac{1}{\rho} \frac{\partial}{\partial x_i} \frac{\partial \varepsilon}{\partial \rho}, \quad (1.12)$$

i.e. Euler's equation in an operational form ($d\varepsilon/d\rho$ is the pressure p of the liquid).

It must be again emphasised that the equations (1.11), (1.12) are less general than the commutation rules (1.3–1.9), which are also valid for an exact, microscopical investigation of the liquid.

2. THE ENERGY SPECTRUM OF A QUANTUM LIQUID

In the classical hydrodynamics of ideal liquids it is shown that if, at a certain moment of time, the motion is potential ($\text{curl } v = 0$) in the whole volume of the liquid, it will be potential for all other moments of time (Lagrange's theorem). It appears that this classical theorem finds its analogy in quantum hydrodynamics.

According to the commutation rules (1.9), $\text{curl } v$ always commutes with the density ρ . The components of $\text{curl } v$, however, do not commute, generally speaking, either with each other or with the components of velocity v (when the operation curl is applied to the equation (1.8) the right-hand side does not vanish). Therefore, $\text{curl } v$ does not, generally speaking, commute with the Hamiltonian, i.e. is not conserved.

An exception is the case when over the whole volume of the liquid $\text{curl } v = 0$. In this case we have zero in the right-hand side of (1.8) and $\text{curl } v$ commutes with ρ and v and, therefore, also with the Hamiltonian.†

In this way $\text{curl } v$ is conserved if it is zero. In other words, a quantum liquid always possesses stationary states in which $\text{curl } v$ equals zero over the whole volume of the liquid. Such a state might be called, by analogy to classical hydrodynamics, a state of potential motion of the liquid.

Concerning these results an analogy can be made with the angular momentum M in quantum mechanics. The commutation of two components of M with each other leads to the third component of M , with the result that all the components of M commute with each other if they are all equal to zero. It is also known that there exist no states with an infinitely small angular momentum, its first non-zero eigenvalues are of the order of \hbar . This is a consequence of the fact that the commutation rules are inhomogeneous—their left-hand sides are quadratic in M and the right-hand sides are linear.

† Not only with the Hamiltonian (1.10), but also with all other functions containing v , ρ and their derivatives of any order with respect to the co-ordinates.

A similar statement can be advanced concerning curl v in quantum hydrodynamics. Namely, no states can exist in which curl v would be non-zero, but arbitrarily small over the whole volume of the liquid. In other words, between the states of the potential (curl $v = 0$) and vortex (curl $v \neq 0$) motions of a quantum liquid there is no continuous transition.

From this the principal features of the energy spectrum of a liquid directly follow. The presence of a gap between the states of the potential and vortex motions means that between the lowest energy levels of vortex and potential motions a certain finite energy interval must exist. As to the question which of these two levels lies lower, apparently both cases are logically possible. It will be shown below that we get the phenomenon of superfluidity if we suppose that the normal level of the potential motions lies lower than the normal level of vortex motions. Hence we must suppose that this very case exists in liquid helium. It must be remarked, however, that, as only one quantum liquid exists, liquid helium, the question as to whether such a distribution of the levels and hence the property of superfluidity is a general property of a quantum liquid cannot be solved experimentally.

This brings us to the following picture of the distribution of the levels in the energy spectrum of liquid helium (it must be emphasised that we do not here refer to the levels for single helium atoms but to the levels corresponding to the states of the whole liquid). This spectrum is made up of two superimposed continuous spectra. One of them corresponds to the potential motions and the other—to vortex motions. The lowest level of the vortex spectrum is situated above the lowest level of the potential spectrum, this latter level being the normal unexcited state of a liquid; the energy interval between these two levels we denote by Δ .

The value of the energy gap Δ cannot be calculated exactly. Its order of magnitude is

$$\Delta \sim \frac{\hbar^2 \rho^{2/3}}{m^{5/3}} \quad (2.1)$$

(m being the mass of the helium atom and ρ —the density of the liquid). This is the only quantity of the dimension of energy which can be built up from m , ρ and \hbar . This gives numerically $\Delta/k \sim 1^\circ$, i.e. Δ , as was expected, is of the order of kT_λ , T_λ being the temperature of the λ -point of helium (cf. (3.8)).

Consider an excited level which is situated not too high above the beginning of the spectrum (vortex or potential one).

Every weakly excited state can be considered as an aggregate of a number of single "elementary excitations". As far as the excited levels of the potential spectrum are concerned, the potential internal motions of the liquid are longitudinal waves, i.e. these motions are sound waves. Therefore, the corresponding elementary excitations are simply sound quanta, i.e. phonons. The energy of the phonons is known to be a linear function of their momentum p :

$$\varepsilon = c p, \quad (2.2)$$

c being the velocity of sound. Thus, at the beginning of the potential spectrum the energy is proportional to the first power of the momentum.

An "elementary excitation" of the vortex spectrum might be called a "roton"[†]. Those special reasons which stipulate a linear dependence of ε on p for phonons do not exist for rotons. For small momenta p the energy of the roton can be simply expanded in powers of p ; in view of the isotropy of the liquid the expansion of the scalar ε in powers of the vector p only contains terms with even powers, so one may write:

$$\varepsilon = \Delta + \frac{p^2}{2\mu}, \quad (2.3)$$

where μ is an "effective mass" of the roton (in (2.2) and (2.3) the energy is measured from the normal state)[‡].

If the number of phonons and rotons (per unit volume of the liquid) is not large, their aggregate can be regarded as a mixture of two ideal gases—a phonon gas and a roton gas. It is known that the phonon gas obeys Bose statistics. As to the rotons, they too probably obey Bose statistics. It must, however, be remarked that inasmuch as the energy of a roton always contains a quantity Δ large compared with kT (at low temperatures only when the aggregate of rotons can be treated as a gas) the difference between the Bose and Fermi statistics is not essential and one can use Boltzmann's distribution for the rotons.

3. THE HEAT CAPACITY OF HELIUM II

From the properties of the energy spectrum one can come to certain conclusions concerning the temperature dependence of the heat capacity of helium II.

At sufficiently low temperatures, when the abnormal behaviour of the heat capacity near the λ -point is no longer essential, the heat capacity may be considered as consisting of two parts—a phonon and a roton part. The first (C_{ph}) is no other than the Debye heat capacity proportional to the cube of the temperature. This part of the heat capacity can be calculated exactly^{††} as the Debye temperature can be calculated from the data obtained⁴ for the compressibility of helium from which, in its turn, the velocity of sound in helium can be calculated (it turns out to be 250 m/sec). According to the known formula we get, for the heat capacity per 1 g of helium

$$C_{ph} = 4.4 \times 10^{-3} T^3 \frac{\text{cal}}{\text{deg} \cdot \text{g}}. \quad (3.1)$$

[†] This name was suggested by I. E. Tamm.

[‡] In a recent paper A. Bijl³ investigated the properties of the energy spectrum of a liquid and came to the conclusion that there must be an energy gap between the normal and all excited states. This result does not seem to be plausible as it would mean, in particular, the impossibility of the propagation of sound waves with small frequencies in liquids.

^{††} This calculation had already been made in 1940 by A. Migdal whom I wish to thank for informing me of his results.

The roton part of the heat capacity (C_r) has, owing to the presence of the energy gap Δ between the normal and vortex levels, an exponential temperature dependence. The coefficient before the exponential factor can be expressed through Δ and the effective mass μ of the roton. For the temperatures considered one can, as has already been pointed out, apply Boltzmann's distribution to the roton gas. The free energy of the gas with an undetermined number of particles (the number of particles in the roton gas is itself a temperature function determined from the condition of the minimum of the free energy) is

$$F_r = -kTV \int e^{-\epsilon/kT} \frac{d\tau_p}{(2\pi\hbar)^3} \\ (d\tau_p = dp_x dp_y dp_z)$$

(V is the volume). Below we shall refer the free energy to 1 g of helium corresponding to which we shall put $V = 1/\rho$ (ρ being the helium density), so that

$$F_r = -\frac{kT}{\rho(2\pi\hbar)^3} \int e^{-\epsilon/kT} d\tau_p. \quad (3.2)$$

The number of rotons is

$$N_r = \frac{1}{\rho(2\pi\hbar)^3} \int e^{-\epsilon/kT} d\tau_p. \quad (3.3)$$

Putting $\epsilon = \Delta + p^2/2\mu$ we get:

$$F_r = -\frac{kT}{\rho} \left(\frac{\mu kT}{2\pi\hbar^2} \right)^{3/2} e^{-\Delta/kT}; \quad (3.4)$$

$$N_r = \frac{1}{\rho} \left(\frac{\mu kT}{2\pi\hbar^2} \right)^{3/2} e^{-\Delta/kT}. \quad (3.5)$$

Thus the entropy is

$$S_r = -\frac{\partial F_r}{\partial T} = \frac{1}{\rho} \left(\frac{\mu k}{2\pi\hbar^2} \right)^{3/2} T^{1/2} \Delta e^{-\Delta/kT} \left(1 + \frac{5kT}{2\Delta} \right), \quad (3.6)$$

and the heat capacity is

$$C_r = T \frac{\partial S_r}{\partial T} = \frac{1}{\rho} \left(\frac{\mu}{2\pi\hbar^2} \right)^{3/2} \frac{k^{1/2} \Delta^2}{T^{1/2}} e^{-\Delta/kT} \left[1 + 3 \frac{kT}{\Delta} + \frac{15}{4} \left(\frac{kT}{\Delta} \right)^2 \right]. \quad (3.7)$$

If we use the data obtained by Keesom and Keesom⁵ for the heat capacity in the interval 1.2–1.5°K from the formulae derived we get for μ and Δ the values:

$$\frac{\Delta}{k} = 8 - 9^\circ, \quad \mu = 7 - 8 m_{\text{He}} \quad (3.8)$$

(m_{He} is the mass of a helium atom).

At sufficiently low temperatures C_r must become smaller than C_{ph} . It must, however, be remarked that the calculated phonon heat capacity according to (3.1) is found to be at 1.2° K about 22 times less than its experimental value. In this way at temperatures even twice as low as the λ -point the roton gas still plays a predominant role in the heat capacity.

Up to the present time the heat capacity at lower temperatures has only been measured by B. Bleaney and F. Simon⁶ (in a temperature interval from 0.25 to 0.8° K). Their results still give a value *ca.* 10 times larger than that given by the formula (3.1). On the other hand, Bleaney and Simon found it possible to describe the heat capacity in the temperature interval 0.25–0.8° K by a T^3 -law. From this it would follow that the surplus part of the heat capacity over C_{ph} , i.e. the heat capacity C_r , also obeys the T^3 -law, whereas according to the theory one would expect it to have an exponential temperature dependence. It must be mentioned, however, that Bleaney and Simon did not publish details of their measurements and so the possibility is not excluded that their results ought to be corrected. On the other hand, P. L. Kapitza's⁹ measurements show that the value of the entropy at 1.33° K calculated from the formula (3.6) with μ and Δ from (3.8) agree well with experiment, whereas the entropy value calculated from the data of Bleaney and Simon does not agree with Kapitza's results.

4. SUPERFLUIDITY OF HELIUM II AT ABSOLUTE ZERO

It will now be shown that the superfluidity of helium II is a consequence of the properties of the energy spectrum described above. Consider first liquid helium at absolute zero. At this temperature the liquid is in its normal, unexcited state. Let us consider the helium when flowing through a capillary at a constant velocity V . The presence of viscosity would be manifested by the fact that owing to the friction between the liquid and the walls of the capillary and the friction inside the liquid itself the kinetic energy of the movement of the liquid would be dissipated and the flow gradually slowed down.

It is convenient to consider this flow in the co-ordinate system moving together with the helium. In this system the helium is at rest and the walls of the capillary move with a velocity $-V$. In the presence of viscosity the helium at rest should also begin to move. Physically it is obvious that the interaction of the liquid with the moving walls of the tube cannot cause a motion of the whole liquid at once. The motion must begin with the excitation, of the inner movements in the layers of liquid close to the walls of the tube i.e. with the excitation of rotons and phonons in the liquid.

Let us suppose that a phonon can be excited in the liquid. Then the energy E_0 of the liquid (in the co-ordinate system in which it was initially at rest) is equal to the energy ε of the phonon (we will agree to take the energy of the normal state as zero) and the momentum P_0 of the liquid is the momentum p of the phonon. As $\varepsilon = cp$ we have:

$$E_0 = cp, \quad P_0 = p.$$

We will now return to the co-ordinate system in which the capillary is at rest. According to the known formulae for the transformation of energy and momentum in classical mechanics we have, for the energy E and the momentum P in this system

$$E = E_0 + (P_0 \cdot V) + \frac{M V^2}{2}, \quad P = P_0 + M V,$$

M being the mass of the liquid. Inserting E_0 and P_0 we find for the energy

$$E = c p + (p \cdot V) + \frac{M V^2}{2}. \quad (4.1)$$

The term $M V^2/2$ is the initial kinetic energy of the flowing helium, the expression $c p + (p \cdot V)$ being the change of energy owing to the excitation of the phonon. This change must be negative as the energy of the flowing liquid must decrease:

$$c p + (p \cdot V) < 0.$$

To fulfil this condition the absolute value of the velocity V must be, in any case, greater than the velocity of sound

$$V > c. \quad (4.2)$$

At smaller velocities the interaction with the walls of the tube cannot give rise to an excitation of phonons.

An analogous consideration can be applied to the excitation of a roton. In this case

$$E_0 = \varepsilon = \Delta + \frac{p^2}{2\mu},$$

and instead of (4.1) we get:

$$E = \Delta + \frac{p^2}{2\mu} + (V \cdot p) + \frac{M V^2}{2}. \quad (4.3)$$

For $E < M V^2/2$ it is at any rate necessary that

$$\Delta + \frac{p^2}{2\mu} - V p < 0,$$

$$V > \frac{p}{2\mu} + \frac{\Delta}{p}.$$

The expression on the right-hand side has a minimum at $p^2 = 2\mu\Delta$ so that for the excitation of rotons the velocity must be greater than

$$V > \sqrt{\frac{2\Delta}{\mu}}. \quad (4.4)$$

In this way we see that neither phonons nor rotons can be excited if the velocity of flow in helium II is not too large. This means that the flow of the liquid does not slow down, i.e. helium II discloses the phenomenon of superfluidity †.

It must be remarked that already the reasons given above are enough to make the superfluidity vanish at sufficiently large velocities. We leave aside the question as to whether superfluidity disappears at smaller velocities for some other reason (the velocity limit obtained from (4.2) is large—the velocity of sound in helium equals 250 m/sec; (4.4) gives a value only several times lower).

5. HELIUM II AT TEMPERATURES ABOVE ABSOLUTE ZERO

We now consider helium at non-zero temperatures. Helium is not then in its normal state and at sufficiently low temperatures its excitation can be considered as a gas consisting of phonons and rotons. The results of section 4 still remain valid as the circumstance that liquid helium is initially in the normal state was not used directly in our derivation. Therefore, new phonons and rotons cannot be excited owing to the movement of the helium relative to the walls of the vessel also at temperatures greater than 0° K. It is necessary, however, to find out how the phonons and rotons that are already in the liquid will manifest themselves.

To investigate this problem we shall consider helium in an axial-symmetric vessel rotating round its axis at a constant angular velocity Ω . We go over to the co-ordinate system rotating together with the vessel. In this system the vessel is at rest, i.e. the external conditions in which the liquid is found are stationary. Therefore, the Gibbs distribution here holds, i.e. the probability of finding helium in an excited state is determined by the formula

$$\text{const } e^{-E'/kT},$$

E' being the energy of the excited state of the liquid in the rotating co-ordinate system. It is known that the energy E' of a body in the rotating co-ordinate system is connected with the energy in the stationary system by means of

$$E' = E - (M \cdot \Omega),$$

where M is the angular momentum of the body in a given excited state (in the normal state $M = 0$). Thus, the Gibbs distribution in the stationary co-ordinate system is

$$\text{const } e^{-[E + (M \cdot \Omega)]/kT}. \quad (5.1)$$

For the sake of clarity let us suppose that the temperature is low enough so that one can speak of an ideal gas of phonons and rotons in helium. Then the

† It must be mentioned that for an ideal gas $\lambda = 0$ and, therefore, even at absolute zero it would not disclose the phenomenon of superfluidity at any velocities of the flow contrary to Tisza's suggestion.

energy E and angular momentum M of the excited state can be written as

$$E = \sum \varepsilon, \quad M = \sum m, \quad (5.2)$$

ε and m being the energy and angular momentum respectively of single phonons and rotons.

As is well known, if we substitute $E = \sum \varepsilon$ in the Gibbs distribution $e^{-E/kT}$ we can pass over to the distribution for single "particles" of the gas, in the given case to the Bose distribution:

$$\frac{1}{e^{\varepsilon/kT} - 1},$$

for phonons and probably for rotons (which particular distribution is not, however, essential to what will come).

Similarly, by inserting (5.2) into (5.1) we get in the same way the same distribution for phonons and rotons in the rotating vessel with the only difference that instead of ε enters $\varepsilon - (m \cdot \Omega)$, so that the Bose distribution will take the form:

$$\frac{1}{e^{[\varepsilon - (\Omega \cdot m)]/kT} - 1}.$$

But such a distribution is no other than the distribution for the rotating as a whole (at an angular velocity Ω). And so we come to the result that during the rotation of the vessel containing helium II a statistical equilibrium is established which differs from the equilibrium in a vessel at rest only in that the photon and roton gas rotates together with the vessel as if it was carried along by the walls of the vessel.

If, from the above distribution one calculates the angular momentum of the helium in the rotating vessel at a given temperature, i.e. the quantity:

$$M = \frac{1}{(2\pi\hbar)^3} \int \int \frac{m \, d\tau_p \, dV}{e^{[\varepsilon - (\Omega \cdot m)]/kT} - 1}, \quad (5.3)$$

then at absolute zero, i.e. when phonons and rotons are completely absent we would obviously get zero. At higher temperatures the angular momentum will be non-zero but the momentum of inertia (i.e. the proportionality factor between M and Ω) will be at sufficiently low temperatures much lower than the usual one (which corresponds to the rotation of the total mass of the liquid with the vessel).

This brings us to the fundamental result that when the walls of the vessel are in motion only a part of the liquid helium mass is carried along by them, the other part "remaining stationary". And so we might regard liquid helium as if it was a mixture of two liquids, one of which is superfluid having no viscosity and not carried along by the walls of the vessel and the other—a "normal" one which is carried along by the moving walls and behaves itself like a normal liquid. It is most essential that "there is no friction" between these two liquids moving through each other, that is, that there is no transfer of momentum from one to the other. Actually we get this relative motion when considering the statistical equilibrium in an uniformly rotating vessel.

But if there can be some sort of relative motion in the state of statistical equilibrium it means that it cannot be accompanied by friction.

It must be stressed that when we look upon helium as a mixture of two liquids it is no more than a method of expression, convenient for describing phenomena which take place in helium II. Like every description of quantum phenomena in classical terms it is not quite adequate. Actually one must say that in a quantum liquid two movements can exist simultaneously each of which is connected with its own "effective mass" (so that the sum of both these masses equals the total real mass of the liquid). One of these movements is normal, i.e. possesses the same properties as the movements of usual liquids; the other is superfluid. Both these motions take place without a transfer of the momentum from one to the other. We particularly emphasize that there is no division of the real particles of the liquid into "superfluid" and "normal" ones here. In a certain sense one can speak of "superfluid" and "normal" masses of liquid as of masses connected with the two simultaneously possible movements, but this by no means signifies the possibility of a real division of the liquid into two parts.

Keeping in mind these reservations as to the real nature of the phenomena taking place in helium II it is still convenient to use the terms "superfluid" and "normal" liquids as a short and convenient method of describing these phenomena and in the further discussion we shall do so.

During the rotation of the vessel containing helium the superfluid part remains stationary as has already been pointed out. It can be said that the superfluid liquid is not capable of rotation. Mathematically this means that the curl of the velocity of the superfluid motion equals zero. Hence the motion of the superfluid liquid is always potential†. As to the normal part of the liquid it can accomplish a potential or a vortex motion.

This brings us in particular to the following interesting property of the motion of helium II. From hydrodynamics we know that during the potential flow of a liquid no pressure is exerted on a body immersed in the liquid (the so-called Euler paradox). Therefore, the superfluid part will not, during its motion, exert any pressure on a body immersed in helium II; the body will only be influenced by the normal part of the liquid.

At low temperatures, when the excitation is not too great the normal part of the liquid is an aggregate of phonons and rotons. Therefore, the viscosity of the normal liquid can be calculated in the same way as that of a mixture of phonon and roton gases. Even at not very low temperatures, i.e. at such temperatures when the number of rotons is large compared to the number of phonons, we cannot completely disregard the phonons and calculate the viscosity as the viscosity of the roton gas alone because the "length of the free path" of phonons is, as can be shown, large as compared to that of the rotons.

The most important parameter which determines the properties of helium at every given temperature is the ratio of the masses of the superfluid and

† A more strict proof of this statement will be given in section 9 where the analogous questions, in superconductors are being investigated.

normal parts of the liquid. We introduce the density ϱ_n of the normal liquid and the density ϱ_s of the superfluid; the sum $\varrho = \varrho_n + \varrho_s$ is the total true density of the liquid.

At absolute zero the ratio ϱ_n/ϱ equals zero. As the temperature rises it increases until it becomes equal to unity after which, of course, it will remain constant. The temperature at which ϱ_n/ϱ becomes unity is the point of transition from helium II into helium I. In this way the phase transition in helium is connected with the disappearance of the superfluid part of the liquid. This disappearance takes place continuously, i.e. ϱ_n/ϱ becomes unity continuously, without a jump. Therefore, the transition is a phase transition of the second order, i.e. a λ -point (it is not accompanied by the absorption or giving out of latent heat). The presence of a jump in the heat capacity is a direct thermodynamical consequence of a phase transition of the second order (cf. e.g. ref. 7).

We will here point out the following possible method for a direct experimental determination of the curve of ϱ_n/ϱ as a function of the temperature. If the moment of inertia J of the vessel filled with helium II and rotating about its axis is measured the ratio of J to the moment of inertia J_0 , calculated on the supposition that the total mass of helium rotates with the vessel, determines the ratio ϱ_n/ϱ at a given temperature.

At sufficiently low temperatures the ratio ϱ_n/ϱ can be calculated theoretically. At these temperatures ϱ is built up of two independent parts—the effective mass of an ideal gas of phonons and the mass of a roton gas. We will first calculate the phonon part of ϱ_n . To do this we shall imagine that the phonon gas moves as a whole with a constant velocity V relative to the liquid. As is known, the distribution function for a gas, moving as a whole, is obtained from the distribution function of the stationary gas simply by inserting the quantity $\varepsilon - (\mathbf{p} \cdot \mathbf{V})$ in place of the energy ε of the particle (where \mathbf{p} is the momentum of the particle). Hence the distribution function for the phonons obeying Bose statistics is

$$\frac{1}{(2\pi\hbar)^3} \frac{1}{e^{\varepsilon - (\mathbf{p} \cdot \mathbf{V})/\hbar T} - 1}.$$

We now calculate the total momentum \mathbf{P} of the phonon gas (per unit volume). We have:

$$\mathbf{P} = \frac{1}{(2\pi\hbar)^3} \int \frac{\mathbf{p} d\tau_p}{e^{\varepsilon - (\mathbf{p} \cdot \mathbf{V})/\hbar T} - 1}.$$

Suppose that the velocity V is small and expand the integrated expression in powers of $(\mathbf{p} \cdot \mathbf{V})$. The zero order term vanishes after averaging over the directions of \mathbf{p} and we obtain:

$$\mathbf{P} = - \int \mathbf{p} (\mathbf{p} \cdot \mathbf{V}) \frac{\partial n_0}{\partial \varepsilon} d\tau_p, \quad ?$$

$$n_0 = \frac{1}{(2\pi\hbar)^3} \frac{1}{e^{\varepsilon/\hbar T} - 1}$$

As for the phonons we have $\varepsilon = c p$ and, therefore,

$$c \frac{p}{p} \frac{\partial n_0}{\partial \varepsilon} = \frac{\partial n_0}{\partial p},$$

so we can write

$$P = -\frac{1}{c} \int p (\mathbf{p} \cdot \mathbf{V}) \frac{\partial n_0}{\partial p} d\tau_p.$$

Integrating by parts and averaging over the directions of the momentum p we get:

$$P = \frac{4}{3c} V \int p n_0 d\tau_p = \frac{4}{3c^2} V \int \varepsilon n_0 d\tau_p.$$

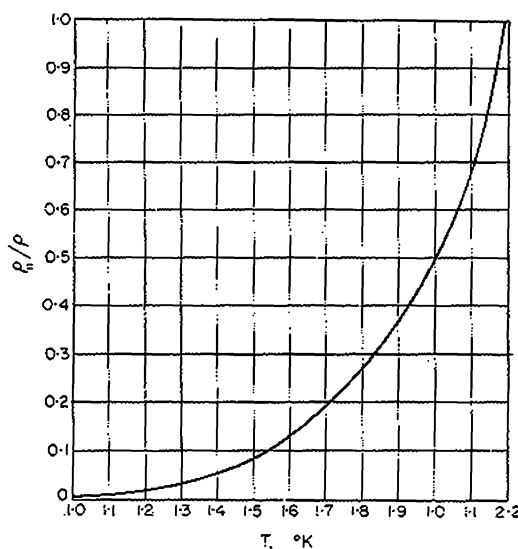


FIG. 1.

But the integral entering this formula is the energy ϱE_{ph} per unit volume of the phonon gas, so that

$$P = \frac{4}{3} \varrho V \frac{E_{ph}}{c^2}.$$

The coefficient of V is the effective mass of a unit volume of the phonon gas, i.e. the phonon part $\varrho_n^{(ph)}$ of the density ϱ_n :

$$\varrho_n^{(ph)} = \frac{4}{3} \frac{E_{ph}}{c^2} \varrho. \quad (5.4)$$

The energy E_{ph} of the phonon gas, and hence $\varrho_n^{(ph)}$ is proportional to T^4 .

A similar calculation gives the following expression for the roton part of the mass of the normal liquid at low temperatures (the Boltzmann distribution

with $\varepsilon = \Delta + p^2/2\mu$ is used for the rotons):

$$\varrho_n^{(r)} = N_r \mu \varrho \quad (5.5)$$

(N_r is taken from (3.5)). The number N_r of rotons, and hence also $\varrho_n^{(r)}$ depend on the temperature exponentially ($\sim e^{-\Delta/kT}$, cf. section 3).

ϱ_n/ϱ is plotted against the temperature in Fig. 1. Even at 1° K the phonon part of ϱ_n is thirty times less than the roton part. The part of the curve in Fig. 1 which is near to the λ -point cannot, of course, be calculated theoretically and is obtained by means of an interpolation.

One may, however, expect that owing to a very rapid increase of ϱ_n according to formula (5.5), the value of the temperature of the λ -point itself can be approximately obtained from $\varrho_n/\varrho = 1$ using formula (5.5) for ϱ_n . Such a calculation gives 2.3°K for the temperature of the λ -point which is in a good agreement with the known value 2.19°K.

6. THE FLOW THROUGH CAPILLARIES AND THE HEAT CONDUCTIVITY OF HELIUM II

The ideas developed above allow one to give a satisfactory explanation of a number of results experimentally observed and also to predict a number of new properties which one might expect to find in helium II.

P. L. Kapitza¹ showed that helium II discloses no viscosity when flowing through a capillary or a narrow slit. From the point of view of the theory advanced here, this is explained by the fact that when helium II flows out of a vessel through a narrow slit the superfluid part of the liquid flows through the slit disclosing no friction; the normal part remains in the vessel flowing through the slit incomparably slower with a velocity corresponding to its viscosity and the thickness of the capillary.

It is known that when the viscosity is measured by means of observing the damping of the torsional vibrations of a disk immersed in helium II (ref. 8) one gets a non-zero value. This result also appears to be perfectly natural; actually, if the disk rotates in the liquid consisting of a superfluid and a normal part it will stop owing to the viscosity of the normal liquid. In this way, the experiment in which the liquid flows through a capillary discloses the presence of the superfluid part of the liquid and the experiment with the disk rotating in the helium II discloses the normal part.

The entropy of helium II is determined by the statistical distribution of the rotons and phonons. Hence for all motions of the liquid in which the phonon and roton gas remains stationary there is no macroscopical entropy transfer. This brings us to the most important result that during the flow of the superfluid liquid no entropy transfer takes place. In other words, the motion of the superfluid liquid does not carry heat. From this it follows in its turn that the motion of helium II in which only the superfluid part of the liquid takes part is thermodynamically reversible.

It was shown above that during the flow of helium II through a thin capillary or through a slit we have to deal with just such a flow of the superfluid liquid.

Therefore, this flow must be reversible (to be more exact, the thinner the capillary and the less the normal liquid penetrates it the closer we are to a complete reversibility). This was disclosed during the recent ingenious experiments made by P. L. Kapitza⁹.

As the helium flowing through a thin capillary does not carry heat, we may come to an important conclusion. Namely, liquid helium flowing out of a vessel through a thin capillary must be at a temperature of absolute zero (it would be more true to say at a temperature lower than the temperature of the helium in the vessel and equal to zero only in the ideal case of an infinitely thin capillary).

The known property of helium of forming a moving film on a hard surface is an effect akin to the properties of its flow through a thin capillary. The fact itself that a film is formed is not a peculiar property only to be found in helium II. Films are formed by all liquids wetting a given hard surface. But in ordinary liquids the formation of the film takes place extremely slowly owing to the presence of viscosity. The formation and movement of the film of helium II takes place quickly owing to the superfluidity.

The so-called thermomechanical effect in helium II is known to consist in the fact that when helium flows out of a vessel through a thin capillary a heating is observed in the vessel; on the other hand, at the place where the helium flows into a vessel from a capillary a cooling takes place. The presence of the thermomechanical effect is not in itself peculiar only to helium[†], anomalous is only the large value of the effect. The thermomechanical effect in ordinary liquids is an irreversible phenomenon of the same type as the thermoelectric Peltier effect.

An effect of this type must also exist in helium II; however, in this case it is overlapped by a much stronger effect which is specific to helium II having nothing in common with the irreversible phenomena of the type of the Peltier effect. Namely, the heating when the helium flows out through a capillary is simply due to the fact, that this helium does not carry heat, therefore, the heat in the vessel is distributed over a smaller amount of helium. When the helium flows in we find the reverse phenomenon.

It is easy to write what heat Q is absorbed when 1 g of helium flows out of the vessel through a capillary. As the liquid flowing out does not contain phonons and rotons its entropy equals zero. If the helium in the vessel is to remain at its initial temperature T the amount of heat TS (S —entropy of 1 g of helium at a temperature T) must be conveyed to it to compensate for the decrease which takes place per unit mass of the entropy owing to the introduction of 1 g of helium with zero entropy. This means that if 1 g of helium flows into a vessel containing helium at a temperature T , the amount of heat absorbed is

$$Q = TS. \quad (6.1)$$

In the opposite case when 1 g of helium flows out of the vessel containing helium at a temperature T this amount of heat is given off. The heat Q was

[†] The thermomechanical effect was observed recently in water by B. V. Deriaguin.

recently measured by P. L. Kapitza⁹. The results obtained are found to be in excellent agreement with formula (6.1).

We shall now consider two vessels containing helium II at temperatures T_1 and T_2 and connected to each other by a thin capillary. As the superfluid liquid is able to flow freely along the capillary a mechanical equilibrium of the helium in the two vessel is quickly established. But as, however, the superfluid liquid does not carry heat, the heat equilibrium at which the temperature of the helium in the two vessels equalizes is only established much more slowly.

The conditions of the mechanical equilibrium of helium can be easily written down by using the fact that the establishment of this equilibrium takes place, according to what has gone before, at constant entropies S_1 and S_2 of helium in both vessels. If E_1 and E_2 are the internal energies per unit mass of helium at temperatures T_1 and T_2 the conditions of the mechanical equilibrium (i.e. the minimum energy) due to the transfer of the superfluid liquid is

$$\left(\frac{\partial E_1}{\partial N_1}\right)_{S_1} = \left(\frac{\partial E_2}{\partial N_2}\right)_{S_2},$$

N being the number of atoms in 1 g of helium. But the derivative $(\partial E/\partial N)_S$ is the chemical potential ζ . Therefore, the condition of equilibrium is obtained in the form:

$$\zeta(p_1, T_1) = \zeta(p_2, T_2)$$

(p_1, T_1 and p_2, T_2 being the pressure and temperature in the first and second vessel) or

$$\Phi(p_1, T_1) = \Phi(p_2, T_2), \quad (6.2)$$

where Φ is the thermodynamic potential of 1 g of helium II.

If the pressure p_1 and p_2 are small, we can expand Φ in their powers and remembering that $\partial\Phi/\partial p$ is the specific volume V we get:

$$V\Delta p = \Phi(T_1) - \Phi(T_2) = \int_{T_1}^{T_2} S dT \quad (6.3)$$

($\Delta p = p_2 - p_1$). If the temperature difference $\Delta T = T_2 - T_1$ is also small we can also expand in powers of ΔT and remarking that $\partial\Phi/\partial T = -S$ we get:

$$\frac{\Delta p}{\Delta T} = \frac{S}{V}. \quad (6.4)$$

As $S > 0$, and $V > 0$ also $dp/dT > 0$ in agreement with the experiment.

The formulae (6.1) and (6.4) were deduced already by H. London¹⁰ starting from Tisza's ideas, the verbal formulation of which coincides at this point with the theory here advanced. These formulae are fully confirmed by P. L. Kapitza's⁹ experiments.

We can note that the phenomena described can be regarded in the framework of the theory advanced here as the osmotic phenomena in the "solutions" of phonons and rotons in liquid helium, the narrow capillary or slit playing the role of a semi-permeable membrane.

Finally, we come to the heat conductivity in helium II. The process of heat transfer in helium II must be represented in the following form. Consider the heat conductivity of helium II in a capillary the large value of which was discovered by Keesom and Keesom¹¹ and studied in detail by P. L. Kapitza¹². If the helium is heated at one end of the capillary (or in a bulb soldered to it) two oppositely directed currents arise in helium. The normal liquid carrying the heat flows from the heated to the cold end; the heat which is transferred in this way is quite sufficient to explain the large value of the transfer experimentally observed. The current of the superfluid liquid flows in the opposite direction. The masses of liquid transferred by each current exactly compensate each other so that no real macroscopical flow in the helium takes place.

P. L. Kapitza¹² observed the deflection of a leaf hung in front of the open end of the capillary when the helium in the bulb at the other end of the capillary was heated. P. L. Kapitza suggested that the explanation of this phenomena might be that it was due to the axial flow of the helium from the heated to the cold end occupying practically the whole of the cross-section of the tube; the amount of helium in the tube was supposed to be sustained by means of the reverse flow of helium moving into the capillary along its surface.

According to the quantum representation developed above this phenomenon has a different appearance. The current of superfluid liquid flowing into the capillary exerts no pressure on the leaf (owing to the potentiality of the movement, cf. section 5). On the other hand, the current of normal liquid exerts pressure on the leaf deflecting it in the direction observed. Both these currents go along the whole cross-section of the capillary and there is no more need to introduce the concept of a current of liquid flowing along the walls with anomalous properties.

It must again be emphasised that the concept of a superfluid and normal liquid is only a convenient way of describing the phenomena, but one must in reality speak of two movements which take place simultaneously in one and the same liquid; one of these movements carries heat and the other does not.

We thus come to a peculiar picture; on a body immersed in helium II a force can act in the absence of any movement of the liquid as a whole.

Consider also heat conductivity in large volumes of helium. If a body with a heated surface is immersed in helium II again two currents arise in the volume of liquid—the current of normal liquid flowing from the heat surface and an opposite current of the superfluid liquid. The transition of the currents one into the other, i.e. the transition of the superfluid liquid into the normal one takes place in the thin layer near the surface. The thickness of this layer must be of the order of the length of the free path of the phonons and rotons forming the normal part of the liquid. Almost the whole temperature fall will

obviously take place in this layer and it follows that in the main volume of the liquid we can expect an almost total lack of temperature gradient. Such a distribution of temperature near the heated surface appears to be supported by P. L. Kapitza's¹² experiments.

7. EQUATIONS OF THE MACROSCOPIC HYDRODYNAMICS OF HELIUM II

Starting from the above considerations on the microscopic mechanism of the phenomenon of superfluidity a complete system of hydrodynamic equations can be built which would describe helium II on a macroscopic (phenomenological) way.

The starting point is the fundamental circumstance that the motion of helium II must be described not by one velocity as in ordinary hydrodynamics but by two. One of these is the "superfluid" velocity (denoted by v_s) satisfying the condition

$$\text{curl } v_s = 0. \quad (7.1)$$

On the boundary of a hard surface only the normal component of v_s becomes zero and not its tangential one corresponding to the fact that the superfluid liquid is not held back by friction against the walls of the vessel. For the "normal" velocity v_n of the liquid on the boundary with a hard surface the condition $v_n = 0$ (as in ordinary viscous liquids) must be fulfilled which expresses the fact that the normal liquid is brought to a standstill owing to the friction against the walls.

It turns out that the hydrodynamic equations with the two velocities v_s and v_n can be obtained absolutely unambiguously starting from the one condition only that they should satisfy all the conservation laws. These equations for the general case of arbitrary velocities are somewhat complicated and we shall not give them here and confine ourselves to a simplified deduction of the equations applicable to the motion with not too large velocities v_s and v_n .

Let j be the macroscopic current of the mass of liquid; it is the function of both the velocities v_s and v_n . For small velocities j can be expanded in powers of v_s and v_n . In the first approximation

$$j = \varrho_s v_s + \varrho_n v_n. \quad (7.2)$$

The coefficients ϱ_s and ϱ_n are obviously those which we called the densities of the superfluid and normal "parts" of the liquid. Their sum equals the real density ϱ of helium II:

$$\varrho = \varrho_s + \varrho_n. \quad (7.3)$$

ϱ_s and ϱ_n are, of course, functions of temperature. Note that the current j (7.2) is at the same time the density of the momentum, i.e. the momentum of a unit volume of liquid. ϱ and j must satisfy the continuity equation:

$$\frac{\partial \varrho}{\partial t} + \text{div } j = 0. \quad (7.4)$$

We shall here write the equations applicable to a motion in which the viscosity of the "normal liquid" plays no part. Then the equation for the momentum conservation is written in the form:

$$\frac{\partial j_k}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 \quad (7.5)$$

(the summation is extended over the indexes which are twice repeated), where the tensor Π_{ik} of the momentum current equals

$$\Pi_{ik} = p \delta_{ik} + \varrho_n v_i^{(n)} v_k^{(n)} + \varrho_s v_i^{(s)} v_k^{(s)}, \quad (7.6)$$

(p being the pressure). To take into account the viscosity of the normal liquid we must add to Π_{ik} the terms expressed in the ordinary way through the coefficients of viscosity and the derivatives of the velocity v_n with respect to the co-ordinates.

Further, the equation for the conservation of entropy takes the form:

$$\frac{\partial S_e}{\partial t} + \operatorname{div}(\varrho S v_n) = 0 \quad (7.7)$$

(S is the entropy per unit mass of helium II). The "entropy current" equals $\varrho S v_n$, as the entropy is only transferred by the normal part of the liquid. If the viscosity of the normal part is taken into account supplementary terms must be added to the right-hand side of (7.7) expressing the increase of the entropy owing to the irreversibility of the processes.

Finally, the last equation of the complete set of hydrodynamic equations we get equalizing the acceleration dv_n/dt to the force acting on a unit of the "superfluid" mass. To determine this force imagine that the unit mass of liquid is displaced from the point 1 to the point 2 in such a way that the distribution of phonons and rotons is not changed. In other words, one might say that during the transfer only the "superfluid liquid" is displaced and the distribution of the normal liquid remains unchanged. The energy E of the liquid changes during such a transfer by

$$\left(\frac{\partial E}{\partial M} \right)_1 - \left(\frac{\partial E}{\partial M} \right)_2$$

(M being the mass of the liquid). Derivatives must be taken here at constant entropy (because the entropy is connected only with the normal liquid) and at a constant momentum of the motion of the normal mass of the liquid relative to the superfluid†; besides this the volume of the liquid is considered as a constant.

† The motion of the superfluid liquid may be considered as external conditions in which the phonons and rotons move. Therefore, the "Lagrange function" for the motion of the normal liquid does not simply depend on its velocity v_n , but on the difference of the velocities $v_n - v_s$. The conserved momentum is, therefore, a derivative of the Lagrange function with respect to $v_n - v_s$, i.e. the momentum of the relative motion.

From the expression obtained for the change of energy it is seen that the quantity $\partial E/\partial M$ can be regarded as a "potential energy" of the superfluid liquid, so that the force acting upon it is

$$-\text{grad} \frac{\partial E}{\partial M}.$$

To calculate the derivative $\partial E/\partial M$ we notice that the derivative of the energy at constant entropy and volume is equal to the derivative of the thermodynamic potential at constant pressure and temperature. The thermodynamic potential $M\Phi$ of the liquid (Φ is the potential per unit mass) can be written in the form of the sum of the thermodynamic potential $M\Phi_0(p, T)$ of the stationary liquid and the kinetic energy $P^2/2M_n$ of the relative motion of the superfluid and normal "parts":

$$M\Phi = M\Phi_0(p, T) + \frac{P^2}{2M_n},$$

P is here the momentum of the motion of the normal mass M_n relative to the superfluid. By differentiating $M\Phi$ with respect to M at constant p, T and P and remembering that the normal mass M_n is proportional (at a given p and T) to the total mass M , we get

$$\Phi_0 - \frac{P^2}{2M_n M}.$$

If we insert $P = M_n(v_n - v_s)$ and put the ratio of the densities in place of the ratio of the masses we finally find, for the derivative $(\partial E/\partial M)_{S, V, P}$ the expression

$$\Phi_0 - \frac{\varrho_n}{2(\varrho_n + \varrho_s)}(v_n - v_s)^2.$$

It follows that the hydrodynamic equation for which we were looking is of the form:

$$\frac{dv_s}{dt} = \frac{\partial v_s}{\partial t} + (v_s \cdot \nabla) v_s = -\text{grad} \left\{ \Phi - \frac{\varrho_n (v_n - v_s)^2}{2(\varrho_n + \varrho_s)} \right\}$$

(the index of Φ_0 is left out). It can be written differently if we notice (7.1), then

$$(v_s \cdot \nabla) v_s = \text{grad} \frac{v_s^2}{2}.$$

In this way

$$\frac{\partial v_s}{\partial t} = -\text{grad} \left\{ \Phi + \frac{v_s^2}{2} - \frac{\varrho_n (v_n - v_s)^2}{2(\varrho_n + \varrho_s)} \right\} \quad (7.8)$$

The equations (7.1–7.8) are a complete set of hydrodynamic equations for helium II.

For a stationary flow the left-hand side of (7.8) is zero; hence:

$$\frac{v_s^2}{2} - \frac{\varrho_n(v_n - v_s)^2}{2(\varrho_n + \varrho_s)} + \Phi = \text{const.} \quad (7.9)$$

This equation together with the next one (7.10) plays here the role of the Bernoulli equation.

Consider now the motions at which liquid may be considered incompressible. If we take the densities ϱ_n and ϱ_s and entropy S as constants we find from (7.4) and (7.7) that

$$\text{div } v_s = 0, \quad \text{div } v_n = 0.$$

Now, for a stationary motion we have in (7.5) $\partial j / \partial t = 0$; by using

$$\frac{\partial v_k^{(s)}}{\partial x_k} \equiv \text{div } v_s = 0, \quad \frac{\partial v_k^{(n)}}{\partial x_k} = 0$$

we can re-write (7.5) in the form

$$\nabla p + \varrho_n(v_n \cdot \nabla) v_n + \varrho_s(v_s \cdot \nabla) v_s = 0.$$

Remembering that $\text{curl } v_s = 0$ we can write this equation as

$$\nabla \left(p + \varrho_n \frac{v_n^2}{2} + \varrho_s \frac{v_s^2}{2} \right) = \varrho_n [v_n \wedge \text{curl } v_n].$$

We project this equation on the line of the current of the normal motion, i.e. on the direction of v_n . Then on the right-hand side we get zero, so that

$$p + \varrho_n \frac{v_n^2}{2} + \varrho_s \frac{v_s^2}{2} = \text{const.} \quad (7.10)$$

It must be emphasised that the expression (7.10) is constant for a stationary flow only along each of the lines of current of the normal motion; and the expression (7.9) is constant over the whole volume of the liquid.

If the temperature and pressure change little over the volume of the liquid Φ can be expanded in powers of $\Delta T = T - T_0$, $\Delta p = p - p_0$; T_0 , p_0 being the temperature and pressure at a certain point in the liquid:

$$\Phi = \Phi(p_0, T_0) - S \Delta T + \frac{\Delta p}{\varrho}.$$

By inserting this into (7.9) we get

$$-\frac{v_s^2}{2} + \frac{\varrho_n(v_s - v_n)^2}{2\varrho} + S \Delta T - \frac{\nabla p}{\varrho} = \text{const.}$$

By combining this equation with equation (7.10)

$$\Delta p + \frac{\varrho_n v_n^2}{2} + \frac{\varrho_s v_s^2}{2} = \text{const}$$

we get

$$\Delta T + \frac{\varrho_n}{\varrho S} (\mathbf{v}_n \cdot \mathbf{v}_n - v_s) = \text{const.} \quad (7.11)$$

This relation, like (7.10), is valid along the current lines of the normal motion.

8. PROPAGATION OF SOUND IN HELIUM II

The equations obtained can be applied to the propagation of sound in helium II. The velocity of the motion in sound waves is as usual supposed to be small and the density, pressure and entropy are almost equal to their constant equilibrium values. The terms in (7.6) and (7.8) which are quadratic with respect to the velocities can be neglected, and in (7.7) we can take the entropy ϱS in the term $\text{div}(\mathbf{v}_n S \varrho)$ out of the sign of div as this term already contains the small quantity \mathbf{v}_n . In this way the system of hydrodynamic equations for sound waves acquires the form

$$\frac{\partial \varrho}{\partial t} + \text{div} \mathbf{j} = 0; \quad (8.1)$$

$$\frac{\partial \varrho S}{\partial t} + \varrho S \text{div} \mathbf{v}_n = 0; \quad (8.2)$$

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla p = 0; \quad (8.3)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \Phi = 0. \quad (8.4)$$

By differentiating (8.1) with respect to time and inserting (8.3) we get:

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p. \quad (8.5)$$

Further, we have

$$\frac{\partial S}{\partial t} = \frac{1}{\varrho} \frac{\partial \varrho S}{\partial t} - \frac{S}{\varrho} \frac{\partial \varrho}{\partial t} = -S \text{div} \mathbf{v}_n + \frac{S}{\varrho} \text{div} \mathbf{j},$$

or

$$\frac{\partial S}{\partial t} = \frac{S \varrho_s}{\varrho} \text{div}(\mathbf{v}_s - \mathbf{v}_n). \quad (8.6)$$

For the thermodynamic potential the relation

$$d\Phi = -S dT + V dp = -S dT + \frac{1}{\varrho} dp$$

holds (V being the specific volume). Hence we have

$$\nabla p = S \varrho \nabla T + \varrho \nabla \Phi,$$

or by introducing ∇p from (8.3) and $\nabla \Phi$ from (8.4)

$$\varrho_n \frac{\partial}{\partial t} (\mathbf{v}_n - \mathbf{v}_s) + \varrho S \nabla T = 0. \quad (8.7)$$

Differentiating (8.6) with respect to time and introducing (8.7) we find:

$$\frac{\partial^2 S}{\partial t^2} = \frac{S^2 \varrho_s}{\varrho_n} \nabla^2 T. \quad (8.8)$$

Two equations (8.5) and (8.8) determine the propagation of sound in helium II. It is already seen from the fact that there are two equations that there must be two velocities of sound in helium II.

Write S, ϱ, p, T in the form $S = S_0 + S', \varrho = \varrho_0 + \varrho',$ etc. where the quantities with a dash represent the small changes of the corresponding quantities stipulated by the sound wave and the quantities with index zero are their constant equilibrium values. Then we can write:

$$\varrho' = \frac{\partial \varrho}{\partial T} T' + \frac{\partial \varrho}{\partial p} p', \quad S' = \frac{\partial S}{\partial T} T' + \frac{\partial S}{\partial p} p',$$

and equations (8.5) and (8.8) take the form

$$\begin{aligned} \frac{\partial \varrho}{\partial p} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' + \frac{\partial \varrho}{\partial T} \frac{\partial^2 T'}{\partial t^2} &= 0, \\ \frac{\partial S}{\partial p} \frac{\partial^2 p'}{\partial t^2} + \frac{\partial S}{\partial T} \frac{\partial^2 T'}{\partial t^2} - \frac{S^2 \varrho_s}{\varrho_n} \nabla^2 T' &= 0. \end{aligned}$$

We look for a solution of these equations in the form of a plane wave in which p' and T' are proportional to a factor $e^{i\omega(t - \mathbf{z}/u)}$ (u being the velocity of sound) and then for the conditions of solubility we get the equation:

$$\begin{vmatrix} u^2 \frac{\partial p}{\partial \varrho} - 1 & u^2 \frac{\partial \varrho}{\partial T} \\ u^2 \frac{\partial S}{\partial p} & u^2 \frac{\partial S}{\partial T} - S^2 \frac{\varrho_s}{\varrho_n} \end{vmatrix} = 0$$

or

$$u^4 \frac{\partial(\varrho, S)}{\partial(p, T)} - u^2 \left(\frac{\partial S}{\partial T} + S^2 \frac{\varrho_s}{\varrho_n} \frac{\partial \varrho}{\partial p} \right) + S^2 \frac{\varrho_s}{\varrho_n} = 0$$

(where $\partial(\varrho, S)/\partial(p, T)$ denotes the Jacobian of the transformation from ϱ, S to p, T). By means of a simple transformation with the use of the thermodynamic relations this equation can be put in the form

$$u^4 - u^2 \left[\left(\frac{\partial p}{\partial \varrho} \right)_S + \frac{T S^2}{C_v} \frac{\varrho_s}{\varrho_n} \right] + \frac{S^2 \varrho_s T \left(\frac{\partial p}{\partial \varrho} \right)_T}{\varrho_s C_v} = 0 \quad (8.9)$$

(C_v being the heat capacity of a unit mass of helium II). This quadratic equation determines two velocities of sound in helium II.

If $\varrho_s = 0$, i.e. at the λ -point, one of the roots of the equation (8.9) becomes zero and we get, as we ought, only one ordinary velocity of sound

$$u = \sqrt{\left(\frac{\partial p}{\partial \varrho}\right)_s}.$$

Practically for all temperatures the heat capacities C_p and C_v are close to each other. According to the known thermodynamic formula in these conditions the isothermic and adiabatic compressibilities are also close to each other, i.e.

$$\left(\frac{\partial p}{\partial \varrho}\right)_s \approx \left(\frac{\partial p}{\partial \varrho}\right)_T.$$

If we denote the common value of $(\partial p/\partial \varrho)_T$ and $(\partial p/\partial \varrho)_s$ as $\partial p/\partial \varrho$ and the common value of C_p and C_v simply as C we get from the equation (8.9) two velocities of sound u_1 and u_2 in the form

$$u_1^2 = \frac{\partial p}{\partial \varrho}, \quad u_2 = \frac{TS^2}{C} \frac{\varrho_s}{\varrho_n}. \quad (8.10)$$

In this way one of the velocities (u_1) is almost constant and the other (u_2) strongly depends on the temperature, becoming zero at the λ -point. At a temperature 1.33°K we get a value of about 25 m/sec for u_2 . At extremely low temperatures, when $\varrho_n^{(ph)} \gg \varrho_n^{(r)}$ one gets

$$u_2 = \frac{c}{\sqrt{3}}. \quad (8.11)$$

In this way as the temperature tends to zero the velocity of sound tends to constant limits $u_1 = c$, $u_2 = c/\sqrt{3}$.

9. THE PROBLEM OF SUPERCONDUCTIVITY

The phenomenon of superconductivity is in many ways akin to the phenomenon of superfluidity. Superconductivity can also be explained by the supposition of the presence of an energy gap in the spectrum of the "electron liquid" in a metal. However, the character of this gap differs from that which is found in liquid helium. The raising of the question itself concerning the division of the motions of a liquid into potential and vortex is senseless when applied to the electron liquid in a metal.

The idea of a connection between superconductivity and the presence of a gap in the energy spectrum has been advanced several times (cf. for example, ref. 13). However, there has been no clear proof given that the presence of this gap really leads to the phenomenon of superconductivity.

In the electron liquid in a metal one can distinguish between the motions of a pure inner character and the motions in which the liquid moves as a whole, in a macroscopic manner. The latter correspond to the presence of a total current in the metal. The states with such a "macroscopic" motion begin directly from the normal unexcited state†. To explain superconductivity one must suppose that there is an energy interval between the normal state of the electron liquid and the beginning of the continuous spectrum of the states with an inner motion.

The value of this interval must be very small—in a temperature scale of the order of temperatures of the superconductive transition. What causes this gap is not, up to the present time, clear.

Let Ψ_0 be the wave function of the normal state of the electron liquid in which it is found at absolute zero. We shall consider a certain excited state which is close to the normal one. According to what was said this state is the state of the motion of the electron liquid as a whole. Corresponding to this the wave function Ψ of this state should have the form

$$\Psi = \Psi_0 \exp \left[\frac{i}{\hbar} \left(\mathbf{p} \cdot \sum_{\alpha} \mathbf{r}_{\alpha} \right) \right],$$

where \mathbf{p} is the momentum of the macroscopic motion per electron and the summation is extended over the radius-vectors of all the electrons.

Actually, however, one must keep in mind that a motion of the electron liquid as a whole must not necessarily have the same velocity in its whole volume. Therefore, in the wave function Ψ the exponential factor actually has the form

$$\exp \left(\frac{i}{\hbar} \sum \chi_{\alpha} \right) \text{ and not } \exp \left[\frac{i}{\hbar} \left(\mathbf{p} \cdot \sum \mathbf{r}_{\alpha} \right) \right]; \chi \text{ is}$$

a certain function of the co-ordinates of one of the electrons about which one can only say that it is nearly linear; different terms in the sum $\sum \chi_{\alpha}$ differ in that χ is taken as a function of the co-ordinates \mathbf{r}_{α} of different electrons

$$\Psi = \Psi_0 e^{\frac{i}{\hbar} \sum \chi_{\alpha}}. \quad (9.1)$$

We determine the density \mathbf{j} of the current in the electron liquid in the state Ψ . To do this we have to integrate the expression

$$\frac{\hbar e}{2im} (\Psi^* \nabla_{\alpha} \Psi - \Psi \nabla_{\alpha} \Psi^*) - \frac{e^2}{mc} \mathbf{A} \Psi \Psi^*$$

(m being an "effective mass" of the electron, \mathbf{A} —the vector potential of the magnetic field, e —the electron charge, c —the velocity of light) over the co-ordinates of all the electrons except the α th and then a summation must be made over all α (i.e. over all the electrons), taking the value of each term at the same point of the liquid. For $\Psi = \Psi_0$ and $\mathbf{A} = 0$, i.e. for an electron

† In this we have the essential distinction between superconductors and insulators in which there is also an energy gap, however, no states corresponding to a presence of a total current exist close to the normal state.

liquid at rest, the current j obviously equals zero. The result is

$$\mathbf{j} = N \frac{e}{m} \text{grad } \chi \cdot \int |\Psi_0|^2 d\tau' - N \frac{e}{m c} \mathbf{A} \int |\Psi_0|^2 d\tau'$$

($d\tau'$ is the product of the differentials of co-ordinates of all the electrons except of one of them; N is the total number of electrons). But the integral

$$\int |\Psi_0|^2 d\tau' = \int |\Psi|^2 d\tau'$$

is the probability density of finding electron at a given point of space. Therefore, the product $N \int |\Psi|^2 d\tau'$ is the number n of electrons in a unit volume of the electron liquid. In this way

$$\mathbf{j} = \frac{n e}{m} \text{grad } \chi - \frac{e^2}{m c} n \mathbf{A}. \quad (9.2)$$

Eliminating χ we find

$$\text{curl} \frac{m c \mathbf{j}}{e^2 n} = - \text{curl } \mathbf{A} = - \mathbf{B} \quad (9.3)$$

($\mathbf{B} = \text{curl } \mathbf{A}$ is the magnetic induction). This equation may be regarded as a supplementary condition which must be satisfied in the possible states of the macroscopic motion of the electron liquid. It is to be noted that this relation is a generalisation of the equation $\text{curl } \mathbf{v}_s = 0$ which is found for superfluidity and goes over into it if the magnetic field can be neglected.

If the metal is homogeneous the electron density can be considered constant and the equation (9.3) can also be written as

$$\text{curl } \mathbf{j} = - \frac{n e^2}{m c} \mathbf{B}. \quad (9.4)$$

From this equation (first put forward by R. Becker, G. Heller, and F. Sauter¹⁴ and first written in the form (9.4) by London¹⁵) the magnetic (Meissner effect) and the electric (lack of resistance) properties of superconductors follow. It is known also that (9.4) is also supported quantitatively by the experiments on the superconductivity of helium films¹⁶.

Similarly to the conclusions in section 5 one can come to the conclusion that at non-zero temperatures the equation (9.4) still holds with the only difference that $e n$ now equals not the total charge density of the electron liquid but only a certain portion of it. In this way one can in a certain sense speak of the division of the whole charge density into "superconducting" and "normal" parts, but not forgetting all the reservations mentioned for the similar division of the density of the mass of liquid helium. In particular, this does not mean that there are "superconducting" and "normal" electrons in metals; simply the motion of the electron liquid in the metal can be described as a combination of two simultaneous motions each of which is connected with a certain effective charge. The superconductivity vanishes at that temperature at which the "superconducting" charge becomes zero.

It must be noted, however, that in the equation (9.4) the charge $e n$ enters in a combination $n e/m$ with the mass m . Therefore, only that ratio has a physical sense and not each one of the values of $e n$ and m separately.

As in helium II we come to the conclusion that the superconducting current must not transfer heat. This is supported by the well-known fact that the thermoelectric phenomena are absent in superconductors¹⁷.

In conclusion, I wish to express my thanks to P. L. Kapitza for kindly informing me of his experimental results before publication.

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47. A THEORY OF THE STABILITY OF STRONGLY CHARGED LYOPHOBIC SOLS AND THE COALESCENCE OF STRONGLY CHARGED PARTICLES IN ELECTROLYTIC SOLUTION

1. INTRODUCTION

Any quantitative theory of the stability of lyophobic sols must consider as its main task a derivation of a quantitative criterion for the stability of the sols. This criterion gives a relation between quantities characterising the properties and state of the sol at the boundary between its unstable and stable states.

By the stable state we shall imply not states which are thermodynamically stable, but states which are stable in the sense of having a practically zero rate of coagulation of the system. In accordance with this, calculations of the stability criterion should be based on a clarification of the kinetics of the irreversible process of the transition from an unstable state of the sol to the more stable coagulate state, when there is no equilibrium between these two states. In this there is a difference of principle between our treatment of coagulation and the treatment of A. March¹ and also the treatment of coagulation given by I. Langmuir², who considered coagulation as a transition between two states which can be in equilibrium between themselves, like two phases.

In order that a quantitative criterion shall exist, it is obviously necessary that the transition from kinetically stable states of the sol to unstable states should be sufficiently sharp. The calculations which are the subject of both this and the previous work³ show that this transition is indeed quite sharp, its sharpness being greater when the radius of the particles of the sol is larger (for more detail see section 7 below). In connection with this last fact, the boundary of the completely unstable states is especially sharp for emulsions, the particles of which have a radius which grows during the coagulation process. Because of this the conditions for complete coagulation coincide in this case with the conditions for the coagulation of crudely dispersed emulsions.

Б. Дерягин и Л. Ландау, Теория устойчивости сильно заряженных лиофобных золей и слипания сильно заряженных частиц в растворах электролитов, *Журнал Экспериментальной и Теоретической Физики* 15, 663 (1945).

B. Deryagin and L. Landau, Theory of the stability of strongly charged lyophobic sols and of the adhesion of strongly charged particles in solutions of electrolytes, *Acta Phys.-chim. USSR*. 14, 633 (1941).

Without doubt, the coagulation of the sol takes place as a result of the interaction between mycellae during their approach. Also, for a calculation of the rate of coagulation and of the degree of stability when finding the criterion for stability, it is necessary to have as a basis a quantitative expression for the interaction of two spherical (in the simplest case) particles as a function of their separation. After this, as one can show, a calculation of the stability criterion is devoid of any special difficulty. Thus the principal, and the most difficult, part of the theory of stability of sols (and not only lyophobic sols) is the calculation of the force or the interaction energy between the particles of the sol as a function of the interparticle separation. Such a problem was first proposed by H. Kallmann and M. Willstätter in a very short note.⁴ These authors consider the interparticle interaction as an algebraic sum of two terms; an attraction which depends on the van der Waals forces acting between the molecules of the mycellae and a repulsion which depends on the electrostatic forces between the ions of the double electric layers surrounding each particle.

From this point of view the coagulation should obviously be related to the action of the van der Waals forces which are able to provide the aggregation of the particles and inevitably induce it in the absence of the repulsive forces. Thus the stability of the lyophobic sols is due to the repulsive electrostatic forces—this is the point of view which, by the way, was expressed considerably earlier than the note of H. Kallmann and M. Willstätter, e.g. by F. Donnan⁵.

In order to find the criterion for coagulation, which corresponds to the moment of time when the attractive forces overwhelm the repulsive forces, it is obviously necessary to obtain the expressions for both types of forces as a function of the interparticle distance. For the van der Waals interaction energy between two molecules, at a separation which is sufficiently large in comparison with their radii, there is an expression of the form

$$U = CR^{-6}, \quad (1)$$

where the constant C depends on the kinds of molecule involved. As H. Kallmann and M. Willstätter point out, it is possible, using this expression, to obtain the interaction energy (*in vacuo*) of two mycellae by a simple integration over their volumes. Thus the calculation of the energy and attractive forces of two mycellae is quite elementary. Exact formulae for the case of two spherical mycellae, even if they are of unequal radius, were later given by Hamaker†⁶.

The calculation of the repulsive electrostatic forces is a considerably more complicated and interesting problem. At the same time it is just a quantitative consideration of the repulsive forces which is the key to the solution both of

† It is necessary to note, in this connection, that the doubts expressed by A. I. Rabinovich, concerning the validity of the formula of Hamaker for attractive forces between spheres, are groundless. Since this formula is theoretically derived, the absence of its direct verification for spheres of the size of colloid particles (which is hardly possible) cannot be considered to be a defect.

the stability problem of lyophobic sols and of a series of other problems in the physics of colloids and surfaces.

The main difficulty in the calculation of these forces is contained in the need to determine, before summing the electrostatic interaction forces of the ions, their positions, i.e. the concentration of the various ions near the mycellae, where the ionic atmospheres of both mycellae, superposed on each other, are mutually deformed. For a solution of this question it is necessary to solve the Debye-Hückel equation of the theory of strong electrolytes. Any calculations which attempt to determine the electrostatic interaction forces without considering the question of changes in the localities of the ions which take place as a result of the mutual approach of two mycellae, risk leading to incorrect results even in a qualitative sense. The formulae thus derived are, at best, empirical. Similar formulae were used by Freundlich⁷, Hamaker⁸, Houwink and Burgers^{9,10} who applied them in order to obtain some conclusions of a qualitative character. Independently of these papers†, and mainly in connection with his experimental researches on the "disjoining pressure" of thin liquid films¹¹, a theoretical treatment of the repulsive forces acting between lyophobic mycellae and of the stability of lyophobic sols was carried out by one of us. In the paper¹² which appeared in 1937 there was presented for the first time a calculation of the iono-electrostatic repulsive forces based on a solution of the Debye-Hückel equation for the cases of both two spheres and two plates separated by a plane-parallel layer of electrolyte. The solution obtained is exact in the case of plates and also in the case of spheres having a radius large in comparison with the Debye thickness of the ionic atmospheres. Besides this, since the calculations were carried out on the basis of a simplified (linear) Debye-Hückel equation, the formulae obtained are valid only for sufficiently weakly charged sols. The attractive forces between mycellae were taken into account only crudely in this work; in analogy with the theory of rapid Smoluchowsky coagulation, to these forces was given a radius of action infinitely short in comparison with the thickness of the ionic atmosphere. As a result a stability criterion was obtained which led to a critical value of the ζ -potential having an order of magnitude which was near to that observed experimentally, but depending too strongly on the radius of the particles (inversely proportional to the square root of the radius).

2. A STABILITY CRITERION FOR WEAKLY CHARGED SOLS

In the later work of B. Deryagin³ the theory of stability was improved by the introduction of an expression for the van der Waals forces based on equation (1). The stability criterion thus obtained for weakly charged sols or

† The contrary statement by A. I. Rabinovich is obviously based on his ignorance of the contents both of a paper by one of us¹⁸ (not cited by Rabinovich) and a report of the same material which was given at the Karpov Institute in 1935.

emulsions has the form

$$\frac{\kappa A}{D\psi_0^2} = \text{const}, \quad (2)$$

where κ is the inverse of the thickness of the ionic atmosphere, A is van der Waals constant (see section 5), D is the dielectric constant and ψ_0 the surface potential of the particles. This formula provides a theoretical basis for the empirical rule found by Eilers and Korff¹³ according to which the coagulation of the same emulsions by different electrolytes takes place at constant values of the expression

$$\frac{\kappa}{\zeta^2} = \text{const}, \quad (2a)$$

where ζ is the electrokinetic potential. The criterion (2) applies only to weakly charged and sufficiently large particles, which drops of the usual emulsions exactly are, since it was derived on the basis of the simplified Debye-Hückel equation which is applicable when the electrical potential near the surface of the particles satisfies the inequality

$$\psi < \frac{50}{z'} \text{ m V}, \quad (3)$$

where z' is the electrovalence of the ion having the biggest charge.

The condition (3), however, at the threshold of coagulation, is probably quite seldom valid when $z' > 1$, especially for sol particles. Even when $z' = 1$ it is, perhaps, far from always fulfilled. Therefore it becomes necessary to change the basis of calculation of the theory, introducing into the calculation the full, unsimplified, Debye-Hückel equation (see equation (5)).

3. CONTENTS OF THE INVESTIGATION

Section 4 contains the application of equation (5) to an exact calculation of the force interaction between two charged, infinite, parallel planes, separated by an electrolytic solution. It becomes possible to make an extremely important simplification in the following calculations by assuming that the potential ψ_0 , to which the planes are charged, is sufficiently large, namely that

$$\psi_0 > \frac{100}{z_1} \text{ m V}, \quad (4)$$

where z_1 is the electrovalence of the dominating ion. Without doubt condition (4), which is to a certain degree contrary to condition (3), should be fulfilled, even when $z_1 = 1$, in a very large number of cases, so that the assumption made becomes quite natural.

In section 5 the formulae obtained are applied to working out the condition for the coalescence of plates in an electrolytic solution under the influence of van der Waals forces, or, as is the same, the conditions for the stability of the electrolytic film separating them. The calculations could, however, be

carried to their completion, i.e. to the point of obtaining the stability criterion in an explicit form, only for the case of a symmetric electrolyte and for the case of an electrolyte in which the ratio of the valence of the accessory ion to the valence of the dominant ion is equal to two. The results thus obtained can be considered as supporting the unimportance of the influence of the valence of the accessory ion on the influence of the electrolyte on the stability of the film.

The criterion obtained has nearly the same form (although it is far from identical) as the rule of Ostwald¹⁴, the applicability of which to the coalescence of quartz surfaces was proved by Buzágh¹⁵.

Section 6 contains calculations of the interaction and coalescence of strongly charged spherical particles in a solution of symmetric electrolyte. The criterion for coalescence obtained there coincides to within a numerical factor with the analogous criterion for parallel particles.

Section 7 contains calculations of the stability conditions for strongly charged sols, leading to a criterion of the same form. Lastly in section 8 a summary and discussion of the results obtained is given, and also their comparison with experiment.

4. THE INTERACTION BETWEEN PLANE SURFACES

The full Debye-Hückel equation for the electric potential, as applied to a binary electrolyte with two types of ions, has the form

$$\Delta\psi = \frac{a}{2} (e^{z_1 b \psi} - e^{-z_2 b \psi}),$$

where z_1, z_2 are the electrovalencies of the dominant and accessory ions,

$$\left. \begin{aligned} a &= \frac{8\pi}{D} e n_1 z_1 \gamma = \frac{8\pi}{D} e n_2 z_2 \gamma = \frac{8\pi}{D} e c, \\ c &= n_1 z_1 \gamma = n_2 z_2 \gamma, \quad b = \frac{e}{kT}. \end{aligned} \right\} \quad (5)$$

Here D is the dielectric constant of the solution, e the electronic charge, γ the concentration in moles per cm^3 , c the concentration in equivalents per cm^3 , n_1 and n_2 the numbers of anions and cations in the electrolyte molecule, k the Boltzmann constant and T the absolute temperature. Considering the interaction of two infinite parallel plates separated by a gap h (Fig. 1), which lie in an infinite volume of electrolytic solution, we put the origin of co-ordinates 0 in the middle of the gap between the plates and take the x -axis to be perpendicular to the surfaces of the plates. Considering ψ to be independent of y and z we can write instead of (5)

$$\frac{d^2 \psi}{dx^2} = \frac{a}{2} (e^{z_1 b \psi} - e^{-z_2 b \psi}) \quad (5a)$$

Multiplying both sides of (5a) by $2(\partial\psi/\partial x)dx = 2d\psi$ and integrating, we obtain

$$\left(\frac{d\psi}{dx}\right)^2 = \frac{a}{b} \left[\frac{e^{z_1 b \psi}}{z_1} + \frac{e^{-z_2 b \psi}}{z_2} - C \right], \quad (6)$$

where C is the constant of integration. Let us allow, as earlier^{3,12}, that at the surface of the plates there exists the boundary condition

$$\psi = \psi_0 = \text{const.} \quad (7)$$

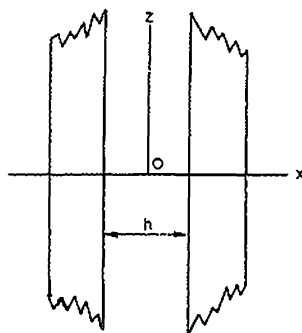


FIG. 1.

For the layer of electrolyte between the plates C has the value

$$C = C_1 = \frac{e^{z_1 b \psi_1}}{z_1} + \frac{e^{-z_2 b \psi_1}}{z_2}, \quad (8)$$

where ψ_1 is the value of ψ for $x = 0$, since the electric force $E = -\partial\psi/\partial x$ vanishes at $x = 0$ from considerations of symmetry. For the external volume of the electrolyte we have since at $x = \pm\infty$, $E = -\partial\psi/\partial x = 0$ and $\psi = 0$.

$$C = C_2 = \frac{1}{z_1} + \frac{1}{z_2}, \quad (8a)$$

Following the technique used earlier¹⁶ we can write, for the interaction (repulsion) of the plates $P(h)$ calculated per unit area:

$$\begin{aligned} P(h) &= \frac{D}{8\pi} [E_2^2 - E_1^2] = \frac{D}{8\pi} \frac{a}{b} (C_1 - C_2) \\ &= c k T \left[\frac{e^{z_1 b \psi_1} - 1}{z_1} + \frac{e^{-z_2 b \psi_1} - 1}{z_2} \right], \end{aligned} \quad (9)$$

where E_2 and E_1 are the values of the electric force at the outer (where it is large) and at the inner surfaces of the plates, respectively. In equation (9) the mechanical (ponderomotive) force acting on each plate is equated to the difference in the strains in the lines of force at both sides of the plate, whilst the osmotic pressures acting on these surfaces counterbalance each other

since they are functions only of the potential ψ at the two surfaces of each plate at which it has the same value $\psi = \psi_0$.

As is seen from (9), the same value of $P(h)$ can be obtained by determining the excess osmotic pressure of the ions in the plane $x = 0$, as was, e.g. done by I. Langmuir^{2,17}.

Thus the two ways of calculation are completely equivalent. This fact, noted in particular by A. N. Frumkin¹⁸, escaped, for an incomprehensible reason, the attention of S. Levine¹⁹, who came forward with criticisms of the papers of B. Deryagin and I. Langmuir and rejected them for various reasons. Incidentally, the main reason was a disagreement with the formulae obtained by S. Levine himself. The cause of this discrepancy was pointed out earlier by one of us²⁰ and consists of an erroneous expression for the free energy of the charged surfaces placed in the electrolyte. It was, in particular, shown that the expression for the free energy, found by S. Levine using an incorrectly applied statistical method, is in contradiction with thermodynamics. An expression equivalent to the expression of S. Levine, and hence also erroneous, is used by A. Corkill and L. Rosenhead²¹ in calculations of the interactions of parallel plates in an electrolyte. It is the presence of this mistake which explains the discrepancy between their results and ours, in particular, the attraction between similarly charged plates which was found by them†.

In order to extract from (9) the explicit expression for $P(h)$ in terms of h , it is necessary in (9) to express C_1 in terms of h .

The connection between h and C_1 we obtain after integrating equation (6), in the following form

$$\begin{aligned} \frac{1}{2}h &= \sqrt{\frac{b}{a}} \int_{\psi_1}^{\psi_0} \frac{d\psi}{\sqrt{\frac{e^{2z_1 b \psi}}{z_1} + \frac{e^{-2z_2 b \psi}}{z_2} - C_1}} \\ &= \sqrt{\frac{b}{a}} \int_{\psi_1}^{\psi_0} \frac{d\psi}{\sqrt{\frac{e^{2z_1 b \psi}}{z_1} + \frac{e^{-2z_2 b \psi}}{z_2} - \frac{e^{2z_1 b \psi_1}}{z_1} - \frac{e^{-2z_2 b \psi_1}}{z_2}}}. \end{aligned} \quad (10)$$

Here it has been taken into account that $\psi = \psi_0$ when $h = 0$. Let us introduce now an extremely important simplification. Putting

$$\psi_0 > 0,$$

let us assume that

$$z_1 b \psi_0 \gg 1, \quad (11)$$

which practically coincides with the condition (4).

Then the upper limit ψ_0 in the integral (10) may, without a noticeable loss of accuracy, be changed to ∞ since the expression under the integral will be

† The presence of a repulsion and not an attraction between the similarly charged surfaces of the division between a thin water film and air is in agreement with the observations of B. Deryagin and A. Titievskaya on the stability of such films.

extremely small when $\psi > \psi_0$, except in the case of extremely small h when the difference $\psi_0 - \psi_1$ would be too small.

Introducing besides this the notation

$$c k T = \gamma n_1 z_1 k T = \gamma n_2 z_2 k T = p, \quad (12)$$

$$\sqrt{ab} = \kappa' = \sqrt{\frac{8\pi e^2 c}{D k T}} = \sqrt{\frac{2}{z_1 + z_2}} \kappa, \quad (12a)$$

$$b\psi = \eta, \quad b\psi_1 = \eta_1, \quad \kappa = \frac{1}{d} \quad (12b)$$

where d is the Debye thickness of the ionic atmosphere, we can write equations (9) and (10) in the following form

$$P = p \left[\frac{e^{z_1 \eta_1} - 1}{z_1} + \frac{e^{-z_2 \eta_1} - 1}{z_2} \right], \quad (13)$$

$$\frac{\kappa' h}{2} = \int_{\eta_1}^{\infty} \frac{d\eta}{\sqrt{\frac{e^{z_1 \eta}}{z_1} + \frac{e^{-z_2 \eta}}{z_2} - \frac{e^{z_1 \eta_1}}{z_1} - \frac{e^{-z_2 \eta_1}}{z_2}}} \quad (14)$$

Equations (13) and (14) express the connexion between P and h in parametric form. In order to obtain this dependence in an explicit form it is necessary to eliminate η_1 from (13) and (14). It is easy to see that the dependence which we are seeking can be expressed with the help of elliptic integrals for the cases when

$$\beta = \frac{z_2}{z_1} = 1, 2, \quad \frac{1}{2} \quad (15)$$

By looking at (13) and (14) it is seen that as η_1 increases from 0 to ∞ P increases monotonically from 0 to ∞ whilst h , on the other hand, decreases monotonically from ∞ to 0. Hence with decreasing h , P continuously increases.

Let us consider now the approximate relations which exist when

$$z_1 \eta > z_1 \eta_1 \gg 1,$$

i.e., when because of the large value of η_1 (which happens at sufficiently small h) it is possible to retain, in equations (13) and (14), only those terms depending on the dominating ions.

Then instead of (13) and (14), introducing into the latter equation a new integration variable $\xi = z_1(\eta - \eta_1)$, we obtain approximately

$$P = \frac{p e^{z_1 \eta_1}}{z_1}, \quad (16)$$

$$\frac{\kappa' h}{2} = \frac{1}{\sqrt{z_1 e^{z_1 \eta_1}}} \int_0^{\infty} \frac{d\xi}{\sqrt{e^{\xi} - 1}} = \frac{\pi}{\sqrt{z_1 e^{z_1 \eta_1}}} \quad (16a)$$

From (16) and (16a) we find

$$P = p \frac{4\pi^2}{z_1^2 \kappa'^2} \frac{1}{h^2} = \frac{2p(z_1 + z_2) \pi^2}{z_1^2} \frac{1}{(\kappa h^2)} = \frac{\pi}{2} D \left(\frac{kT}{z_1 e} \right)^2 \frac{1}{h^2}. \quad (17)$$

This formula was found by I. Langmuir¹⁷ in the particular case $z_1 = z_2$. From a comparison of (13) and (14) with (16) and (16a) it follows that the approximation made increases the values of P and decreases the values of h corresponding to a given value of η and therefore, *a fortiori*, increases the values of P corresponding to given values of h , i.e. the formula (17) always gives exaggerated values of P †.

This is true not only with respect to the deviation of P from the formula (17) at large h but also with respect to the deviation from (17) which should occur for very small h at any finite, even if very high, value ψ_0 of the potential of the plate.

This point is connected with the following statement of a physical nature, which can be proved by an analogous consideration; allowing for the influence of the accessory ion (e.g. for the term $(e^{-z_2 \eta} - 1)/z_2$ in formula (13)) can only diminish the pressure P calculated at the given value of h .

Let us consider now the case of large values of $\kappa' h$ and small η and put in accordance with this

$$\eta_1 \ll 1.$$

Expanding the exponential function appearing in (13) in a series, we obtain for this case

$$P = p \frac{(z_1 + z_2)}{2} \eta_1^2. \quad (18)$$

Assume further that the electrical potential, at a point of the intermediate layer considerably removed from the surfaces of both plates, can be found by a simple addition of the potentials which would be induced at that point by each plate in the absence of the other, i.e. on the basis of the assumption of the additivity of these two actions‡. With such an assumption we can write

$$\eta_1 = 2\eta \left(\frac{h}{2} \right), \quad (18a)$$

where $\eta(h/2)$ is the value of the potential induced at a distance $(h/2)$ by one charged plate in the absence of the other.

Since for small $\eta(h/2)$, instead of (5a), one can use the equation

$$\frac{d^2 \psi}{dx^2} = \kappa^2 \psi, \quad (19)$$

† The contrary statement by I. Langmuir is, consequently, incorrect, just as the curve given by him of the dependence of $\ln P$ on $\ln h$ is, in this sense, erroneous.

‡ This assumption can be proved purely analytically.

or

$$\frac{d^2\eta}{dx^2} = \kappa^2 \eta, \quad (19a)$$

for large values of κh , $\eta = b \psi$ has the form

$$\eta = \eta_0 e^{-\kappa x}, \quad (20)$$

where η_0 is a constant depending only on z_1 and z_2 . Consequently:

$$\eta_1 = 2\eta_0 e^{-0.5\kappa h}, \quad (21)$$

$$P = 2p(z_1 + z_2) \eta_0^2 e^{-\kappa h} = P_0 e^{-\kappa h}, \quad (21a)$$

where P_0 is another constant.For a symmetrical electrolyte with $z_1 = z_2 = z$ and $n_1 = n_2 = n$

$$\eta_0 = \frac{4}{z} \quad (22)$$

and consequently

$$P = P_0 e^{-\kappa h} = \frac{64p}{z} e^{-\kappa h} = 64\gamma n kT e^{-\kappa h}. \quad (22a)$$

For an unsymmetrical electrolyte η_0 is calculated in a quite cumbersome manner from the relation

$$\begin{aligned} \ln(z_1 \eta_0) = F(\beta) = & \int_0^1 \left[\frac{\sqrt{\frac{1+\beta}{2}}}{\sqrt{e^t + \frac{e^{-\beta t}}{\beta} - 1 - \frac{1}{\beta}}} - \frac{1}{t} \right] dt \\ & + \int_1^\infty \frac{\sqrt{\frac{1+\beta}{2}} dt}{\sqrt{e^t + \frac{e^{-\beta t}}{\beta} - 1 - \frac{1}{\beta}}}, \end{aligned} \quad (22b)$$

where $\beta = z_2/z_1$. This relation is obtained as a result of a comparison of the value of κh obtained from the relation (20), with that which follows from the exact formula (14) after the introduction of the integration variable $t = z \eta$ and the parameter $\beta = z_2/z_1$ and allowing for (12). Let us consider now the detailed exact solution for the case of a symmetrical electrolyte, putting:

$$\left. \begin{aligned} z_1 &= z_2 = z, \\ n_1 &= n_2 = n. \end{aligned} \right\} \quad (23)$$

The relation (14) is transformed in this case, after introducing the new integration variable

where

$$\left. \begin{aligned} u &= \frac{1}{k \cosh\left(\frac{z\eta}{2}\right)}, \\ k &= \frac{1}{\cosh\frac{z\eta_1}{2}}, \end{aligned} \right\} \quad (23a)$$

in the following manner

$$\begin{aligned} \frac{zh}{2} &= \int_{\eta_1}^{\infty} \frac{z d\eta}{\sqrt{2(\cosh z\eta - \cosh z\eta_1)}} = \int_{\eta_1}^{\infty} \frac{z d\eta}{2\sqrt{\cosh^2 \frac{z\eta}{2} - \cosh^2 \frac{z\eta_1}{2}}} \\ &= k \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = kK(k) = \sin\alpha K(\sin\alpha), \end{aligned} \quad (24)$$

where $K(k)$ is the usual notation for the complete elliptical integral of the first kind with modulus $k = \sin\alpha$.

With the help of (12), (23) and (23a) we can write in place of (13)

$$P = 4\gamma n k T \left(\frac{1}{k^2} - 1 \right) = 4\gamma n k T \cosh^2\alpha. \quad (25)$$

Equations (24) and (25) define the dependence $P(h)$ in a parametric form which is a little bit more convenient than its representation given by I. Langmuir¹⁷. Using the asymptotic expressions for $K(k)$ as $k \rightarrow 1$, it is not difficult to derive the relations (22) and (22a).

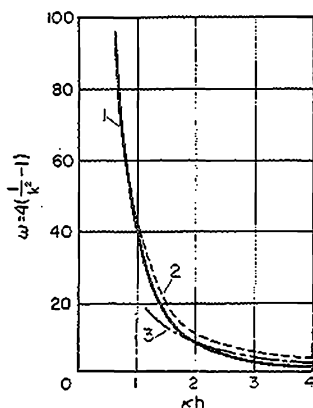


FIG. 2.

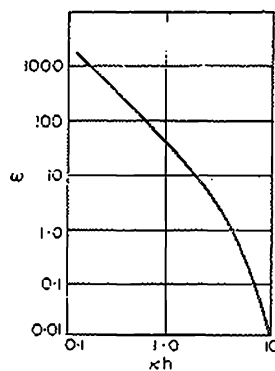


FIG. 3.

In Figs. 2 and 3 we give diagrams of the functional dependence of the quantity

$$\frac{P}{\gamma n k T} = \omega$$

on κh with both the usual and double logarithmic scales. The diagram in Fig. 3 corrects the inaccuracy which is present in the article of I. Langmuir¹⁷. The calculations have been performed with the help of the numerical values of the function $K(k)$. (In Fig. 2 curve 1 gives the exact values, curves 2 and 3 the values according to equations (17) and (22a).)

5. THE CONDITIONS FOR THE COALESCENCE, UNDER THE INFLUENCE OF VAN DER WAALS FORCES, OF SURFACES SEPARATED BY A LAYER OF ELECTROLYTE

Now take into consideration the van der Waals forces, which decrease inversely proportionally to the seventh power of the intermolecular distance, and which provide, per unit area of the film separating two parallel plates of considerable thickness, an attraction

$$Q = \frac{A}{6\pi h^3}, \quad (26)$$

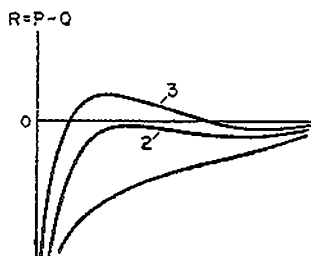


FIG. 4.

where A depends on the material both of the plates and of the medium between them. It follows from a comparison of the formulae (17), (21a) and (26) that at large and small distances the attractive forces dominate. Denoting by $R(h) = P - Q$ the resultant repulsion we see that the curve R, h can be of the following three principal forms (Fig. 4). Obviously, only when curve 3, with a maximum lying above the h axis, is realised can the film be stable and prevent the surfaces which it separates from coalescing.

The limit of stability will be observed when the vertex of this maximum will lie at the level of the h -axis, i.e. when the relations

$$\frac{\partial R}{\partial h} = 0, \quad (27)$$

$$R = 0, \quad (27')$$

are satisfied simultaneously, or according to (26)

$$\frac{\partial P}{\partial h} = - \frac{A}{2\pi h^4}, \quad (28)$$

$$P = \frac{A}{6\pi h^3}. \quad (28a)$$

From here we derive

$$\frac{d \ln P}{d \ln h} = -3. \quad (29)$$

Equation (29) will define, after the values of P and h from (13) and (14) as functions of the parameter $(z_1 \eta_1)$ have been substituted into it, the values of this parameter for a given value of the ratio $\beta = z_2/z_1$. After this the quantities

$$\left(\frac{zP}{p}\right)_c = \omega_c = u(\beta), \quad (30)$$

$$(\kappa h)_c = v(\beta) \quad (30a)$$

will be determined from the same relations (13) and (14).

Substituting these quantities into (28) we obtain the stability criterion for the film, or the criterion for the coalescence of the plates, in the form

$$\frac{\kappa^3 A}{\gamma n_1 k T} = C(\beta) \quad (31)$$

where $C(\beta)$ is a function only of β . The exact calculation of the quantity C for the various values of β is connected with cumbersome numerical calculations. Therefore we shall consider from the beginning only the simplest case for calculation, $\beta = 1$, corresponding to a symmetric electrolyte. In this case, using equations (24) and (25) we can write down instead of (21)

$$\frac{\frac{d \ln \left(\frac{1}{k^2} - 1 \right)}{d k}}{\frac{d \ln k K}{d k}} = -3. \quad (32)$$

Using the relation

$$\frac{d k K}{d k} = \frac{E}{1 - k^2}, \quad (33)$$

well known from the theory of elliptic functions²², where $E(k)$ is the complete elliptic integral of the second kind with modulus $k = \sin \alpha$:

$$E = \int_0^1 \frac{\sqrt{1 - k^2 u^2}}{\sqrt{1 - u^2}} du,$$

we obtain from (32) after simplifications

$$\frac{K}{E} = \frac{3}{2}. \quad (32a)$$

From here, using tables of the values of K and E we find the root of this equation

$$k = 0.773 \dots \quad (32b)$$

Using this value of the modulus k we find from (24) and (23) the "critical" values of the quantities (κh) and $(z \eta_1)$:

$$(\kappa h)_c = 3.014 \dots, \quad (33)$$

$$(z \eta_1)_c = 1.5 \dots, \quad (33a)$$

$$\omega_c = \frac{P_c}{\gamma n k T} = 2.68 \dots \quad (33b)$$

As a result the criterion (31) takes the form

$$\frac{\kappa^3 A}{\gamma n_1 k T} = B \cong 1390 \quad (31)$$

or

$$m = \frac{1}{\gamma n_1} = C \frac{A^2}{D^3} \frac{e^6 z_1^6}{(k T)^5}, \quad (31a)$$

where

$$C = \frac{(4\pi)^3}{\beta^3} = 0.819 \dots 10^{-2},$$

and m , which is the reciprocal of the coagulating concentration of the dominating ions, can be accepted as a measure of the coagulating concentration of the electrolyte.

The values (33) and (33a) show that the "critical point" lies at such thicknesses that, in the first approximation, one may try to use the formulae (21a) and (21). Indeed, if this attempt is made, substituting the value (21a) into (29) we obtain

$$(\kappa h)_c' = 3, \quad (34)$$

which is near to the value (33).

Substituting further the value just found, $\kappa h_c = 3$, into (22a) and also in (21) and (22) we obtain

$$(z \eta_1)_c' = 1.78 \dots, \quad (34a)$$

$$\omega_c' = \frac{P_c}{\gamma n k T} = 3.20 \dots, \quad (34b)$$

$$B' \cong 1630. \quad (34c)$$

Thus the inaccuracy which occurs here is not large.

The value C found in (31a) is only valid when $z_2 = z_1$. However, since in all the formulae giving $P(h)$ the terms depending on the accessory ion are of secondary importance, the influence of z_1 on the numerical value of C is not strong.

The calculations presented can be applied† to the case when $\beta = z_2/z_1 = 2$. It happens here that the constant decreases only by approximately 4 per cent. Thus the weak influence of the valence of the accessory ion is confirmed.

6. CONDITIONS FOR THE COALESCENCE OF CONVEX SURFACES SEPARATED BY A LAYER OF ELECTROLYTE

In the theory of coalescence developed earlier by one of us²³, it was shown that it is possible to pass from the interaction of plane surfaces to the interaction of convex surfaces of the same nature, in the same medium, by using the formula

$$N = G \int_H^{\infty} R(h) dh, \quad (35a)$$

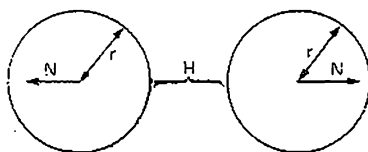


FIG. 5.

where $R(h)$ is the repulsion per unit area of the surfaces separated by a plane parallel layer of thickness h , N is the repulsive force between two spheres when the nearest distance between their surfaces is H (Fig. 5) and G is a quantity depending on the form (radii of curvature) and mutual orientation of the surfaces in the region of their closest approach.

For two spheres of radius r

$$G = \pi r. \quad (35)$$

Formula (35) is a good approximation in the case of sufficiently large r .

In the case when the interaction between the surfaces is of a completely iono-electrostatic origin the values of r will be "sufficiently large" if they satisfy the condition

$$r \gg 1. \quad (36)$$

In the case of forces obeying formula (1), instead of (36) we shall consider the condition

$$\frac{r}{H} \gg 1. \quad (36a)$$

† We omit these calculations, they are part of the calculations carried out by B. Deryagin and devoted to a generalisation of the theory as applied to asymmetric electrolytes.

In the case of the simultaneous action of both kinds of force, it is necessary to consider the conditions (36) and (36a) simultaneously. Since, however, the values of N when H is quite large in comparison with $d = 1/\kappa$ will only slightly affect the following calculations in comparison with the influence of the values of N for smaller H , it is practically sufficient to consider only condition (36), taking (36a) to be its consequence.

Assuming that the conditions for the validity of equation (35) are fulfilled, we obtain from (35), (28a), (26) and (26a)

$$\frac{N}{G} = I(H) - \frac{A}{12\pi} \frac{1}{H^2}, \quad (37)$$

where

$$I(H) = \int_H^\infty P(h) dh. \quad (37a)$$

Then, as a limiting condition for the coalescence of the particles, we obtain instead of (28) and (29)

$$\frac{d \ln I(H)}{d \ln H} = -2, \quad (38)$$

$$I(H) \cdot H^2 = \frac{A}{12\pi}. \quad (38a)$$

For the particular case of a symmetrical electrolyte we find from (24) and (25)

$$\begin{aligned} I(H) &= \int_H^\infty \frac{4\gamma n k T}{\kappa} \left(\frac{1}{k^2} - 1 \right) 2 \frac{d(kK)}{dH} dH = \frac{8\gamma n k T}{\kappa} \int_H^\infty \left(\frac{1}{k^2} - 1 \right) \frac{d(kK)}{dH} dH \\ &= \frac{8\gamma n k T}{\kappa} \left(\frac{1}{k^2} - 1 \right) kK + \frac{8\gamma n k T}{\kappa} \int_k^1 \frac{2K}{k^2} dk, \end{aligned}$$

or, using the easily proven property of the complete elliptic integrals

$$I(H) = \frac{8\gamma n k T}{\kappa} \left[\frac{2}{k} E - \left(\frac{1}{k} - k \right) K - 2 \right], \quad (39)$$

$$H = 2kK, \quad (39a)$$

we obtain from (38) and (39), after simple transformations, the equation

$$4E - 3(1 - k^2)K - 4k = 0,$$

the root of which is

$$k = 0.543 \dots \quad (40)$$

From (40) and (39) we obtain the "critical" thickness

$$\kappa H_c = 1.858 \dots \quad (40a)$$

and a criterion for coalescence

$$\frac{\kappa^3 A}{\gamma n k T} = B' = 1160, \quad (41)$$

or

$$m = \frac{1}{\gamma n_1} = C' \frac{A^2}{D^3} \frac{e^6 z_1^6}{(kT)^5},$$

where

$$C = \frac{(8\pi)^3}{(B')^2} = 1,174 \dots 10^{-2}.$$

We see that not only is the form of the coalescence criterion unchanged, but also the numerical value of its constant alters only slightly when one goes from plane to convex surfaces. This fact is quite important from the point of view of experimental verification of the theory or its applicability to systems containing particles of irregular form but sufficiently, large.

Let us notice that although the critical thickness has now moved, in comparison with the plane case, towards smaller values (cf. (33)), a calculation of the stability equation (22a), instead of the exact relations (24) and (25), leads to a practically identical result. Indeed, with the help of (37), (38) and (33a) we find, based on (22a):

$$(\kappa H)_c = 2, \quad (40b)$$

$$\frac{\kappa^3 A}{\gamma n k T} \cong 1300, \quad (41a)$$

so that in this case the error has no significance.

7. THE STABILITY CRITERION FOR STRONGLY CHARGED SOLS

The process of coagulation of a sol, when the coalescence takes place as a result of a preliminary approach of the particles during the process of Brownian motion, differs from the process of coalescence of two particles when considered statistically. As was shown earlier²⁴, the quantity L , inverse to the rate of coagulation v , considered as a measure of the stability of the sol, is proportional

$$L = \frac{1}{v} \sim \int_0^\infty e^{U(H)/kT} \frac{2r dH}{(2r + H)^2}, \quad (42)$$

to the expression where $U(H)$ is the interaction energy of the particles when the distance between their surfaces at the point of closest approach is H . It is seen from (42) that this interaction begins to play a positive role in the stability of the sol only when, at some values of H , $U(H)$ is positive, i.e. the maximum of the function $U(H)$ lies above the abscissa. In other words there exists an energy barrier.

The height of this barrier, as well as the strength of the repulsive force N , is proportional to the radius of the particles r , according to (35), (35a). It follows from here that the larger r , the sharper (all other conditions being equal) the transition between the unstable states of the sol and the stable ones which are realised in the presence of the barrier, i.e. the sharper the boundary between the two states. In the limit of extremely "large" particles, any spread of this boundary must disappear.

One should, generally speaking, differentiate between the stability criterion and its spread corresponding to the beginning of the coagulation (to the aggregation of the primary particles), and those corresponding to the coagulation after its completion.

If the primary particles have radii considerably larger than the thickness of the ionic atmospheres, the spread of the coagulation threshold will not be large. But at the same time this spread will not decrease with the formation (during the process of coagulation) of large aggregates, since the radii of curvature of the approaching surface regions of any two accessory aggregates will be equal to the radii of the primary particles. The situation changes if the primary particles have a radius comparable to, or especially small in comparison with, the thickness of the ionic atmosphere†. The coagulation threshold (at its beginning) will then be widely spread; however, the spread will diminish with the aggregation of the particles, since the surface granularity of the aggregates formed will have dimensions small in comparison with thickness of the diffuse ionic atmospheres. Because of this the interaction of colliding aggregates, and the energy barrier which it causes, will be the same as if the surface of the aggregates were smooth, with a radius of curvature which corresponds to the crude form of the aggregates (i.e. to the surface enveloping its micro-relief).

Thus the difference between the criteria at the beginning and at the end of the coagulation, which was pointed out above, will in this case be especially large, the spread of the latter criterion being less even than for the case of a sol with larger primary particles. It is therefore possible to adopt as the stability criterion the condition for the disappearance of the energy barrier which is expressed by the equations:

$$U = 0,$$

$$\frac{dU}{dH} = 0.$$

The stability criterion thus obtained for symmetric electrolytes is not different in form from (31a) and (41). As far as a numerical value of the criterion is concerned, its evaluation requires cumbersome calculations which have not yet been performed. But the exact calculation of the corresponding constant has hardly any importance because of a certain spread in the transition be-

† The study of this latter case was carried out and reported to the colloquium of the Colloid-Electrochemical Institute of the Academy of Sciences of the U.S.S.R. in 1940 in the paper by L. Landau.

tween stable and unstable states of the sol. For a rough calculation, however, one can use the following method. A stability criterion for weakly charged sols was found by one of us³ in the following form:

$$\frac{A \kappa}{D \psi_0^2} = C, \quad (43)$$

where C has an order of magnitude near to two and ψ_0 is the surface potential of the particles. An approximate stability criterion for strongly charged sols is obtained by changing ψ_0 , in (43), to the "apparent" potential of the surface of the particles† with the help of the formulae:

$$b \psi_0 = \eta_0 = \frac{4}{z}$$

(see formulae (22) and (12b)).

After simple transformations we obtain

$$\frac{\kappa^3 A}{\gamma n_1 k T} \cong 800 \quad (44)$$

and

$$m = \frac{1}{\gamma n_1} = C'' \frac{A^2 e^6 z_1^6}{D^3 (k T)^5}, \quad (44a)$$

where

$$C'' \sim 2.5 \times 10^{-2}. \quad (44b)$$

Because of the approximate character of the calculation it actually gives only a crude estimate of the value of the constant C'' .

As far as the influence of the valence of the accessory ion is concerned, it remains small.

According to an approximate evaluation (not pretending to be exact) of this influence, the constant C and the coagulative ability of the electrolyte in all cases, apparently, increases with increased valence of the accessory ion, but a little more slowly than is required from the proportionality to the quantity $(1 + \beta)$ which follows from the rule of V. Ostwald (see p. 350).

8. DISCUSSION OF THE RESULTS AND THEIR COMPARISON WITH EXPERIMENT

We have considered three problems connected with the stabilising influence of the ionic atmospheres on three phenomena: the coalescence of plates, the coalescence of particles with convex surfaces and the coagulation and

† By the "apparent" potential ψ_0 we mean the potential which must be prescribed on the plane surfaces in such a way that it is exponentially decreasing (as a function of the distance from the surface) in agreement with the approximate, and not the exact, Debye-Hückel equation (19), and asymptotically tends to the value of the potential which actually exists in the electrolyte at large distances from the surface (see formula (20)). The apparent potential of the charged surface is less than the actual one (which can be extremely large) because of that part of the screening cloud of the dominating ions which is not considered when one uses the simplified Debye-Hückel equation instead of the complete one.

aggregation of sols and suspensions. Exact stability criteria are easily calculable for the case of a symmetric electrolyte and have, in all three cases, the same form

$$\frac{\kappa^3 A}{\gamma n k T} = B,$$

with nearly the same values of the constant B in all cases.

Using the well-known formula for κ we can give to the stability criterion the following form:

$$m = \frac{1}{\gamma n} = \frac{(8\pi)^3}{B^2} \frac{A^2 e^6 z^6}{D^3 (kT)^5} \equiv C \frac{A^2 e^6 z^6}{D^3 (kT)^5}; \quad (45)$$

$m = 1/\gamma n$, which is a measure of the coagulative ability of the electrolyte, thus happens to be directly proportional to the sixth power of the ionic charge. Because of the dependence of A on the type of the particles, the quantity γ will be different for various sols, even for equal z .

In the case of a non-symmetric electrolyte the calculation becomes very complicated. The coagulative ability of the electrolyte depends mainly on the dominating ion. The influence of the accessory ion is relatively small. Thus the Hardy-Schulze rule is well founded. The stability criterion in the presence of a non-symmetric electrolyte, in all three cases has the form:

$$m = \frac{1}{\gamma n_1} = f(\beta) \cdot C \frac{A^2 e^6 z_1^6}{D^3 (kT)^5}, \quad (46)$$

where $f(1) = 1$. The exact calculation of the function $f(\beta)$ is quite a laborious matter, which requires cumbersome numerical and graphical calculations, but which is not difficult in principle. Let us compare the condition (46) with the empirical stability criterion of Ostwald¹⁴ for weakly ionic sols:

$$-\ln f_1 = \text{const}, \quad (46a)$$

where f_1 is the activity coefficient of the dominant ion. Substituting for $\ln f_1$ its limiting expression taken from the Debye-Hückel theory of strong electrolytes, we obtain instead of (46a)

$$m = \frac{1}{\gamma n_1} = \text{const}(1 + \beta) \frac{e^6 z_1^6}{D^3 (kT)^3}. \quad (46b)$$

It is known that the rule of Ostwald (46b) was supported by voluminous experimental data which is concerned both with the coagulation of sols¹⁴ and the coalescence of quartz particles¹⁵. As an illustration we give the following table taken from the work of Ostwald²⁵ (Table 1).

We see that in the whole of the large range of observed concentrations, there is agreement to within the experimental errors and the inevitable spread of the coagulation threshold. A comparison of (46) and (46b) shows that the coagulation criterion derived theoretically by us differs from that found empi-

TABLE I

Type of electrolyte				$m = \frac{1}{\gamma n_1}$ Theoretical according to (46b)	$m = \frac{1}{\gamma n_1}$ Experimental†
z_1	n_1	z_2	n_2	As unit is taken the m which corresponds to the case $z_1 = 1, z_2 = 1$	
1	1	1	1	1	1
1	2	2	1	1.5	1.65
1	3	3	1	2.0	2.5
1	4	4	1	2.5	2.8
2	1	1	2	48	51
2	1	2	1	64	47
3	1	1	3	486	573
3	2	2	3	608	938
4	1	1	4	2560	1720
6	1	1	6	27216	14050

† The figure given in this column are averages over many experimental values.

rically by Ostwald by the factor

$$\left(\frac{A}{kT}\right)^2$$

and the factor $f(\beta)$ instead of $(1 + \beta)$.

The factor A/kT does not depend on the type of electrolyte and the ratio of $f(\beta)$ to $(1 + \beta)$ for various β should remain of the order of unity.

Therefore the theory developed by us is important in two ways.

Firstly the theory just developed, which leads to the formula (46), clarifies the existence of the agreement between numerous experiments, in which A/kT did not vary (the coagulation of the same sol by different electrolytes was investigated), with the rule of Ostwald, whilst this agreement sometimes provoked previously only mistrustful surprise. The basic cause of such a view is, probably, the biased tendency to see the coagulation criterion inevitably expressed in terms of the ζ -potential, which is connected both with some experimental results (the experiments of Ellis and Powis²⁶ and others) and with theoretical calculations related to the case of weakly charged particles (e.g. oil emulsions in water)³.

The calculations contained in this work show with all possible clarity that for weakly charged sols the stability criterion ceases to contain the particle potential and must in the main be determined by the concentration and nature of the electrolyte. There can exist, as follows from (43), in the case when the ζ -potential happens to be near in magnitude to the "apparent" potential ψ_0 , a form

$$\frac{A\kappa}{D\zeta^2} = C,$$

for the stability criterion containing the ζ -potential. In this case, however (contrary to the case of weakly charged sols), the introduction of the ζ -potential is completely unnecessary, since the criterion (46), containing only concentrations, is incomparably more useful because it allows the possibility of predicting the start of coagulation without first making measurements of the ζ -potential, which are sometimes very difficult and inaccurate. In addition, the coincidence of ζ and ψ_0 happens rather exceptionally.

Thus the calculation performed gives a solid foundation to the concept of coagulative concentration and to the Schulze-Hardy rule. At the same time the contrast between the concept of the contraction of the ionic atmosphere and the proper concept of the dominant role of the dispersion medium in coagulation, which was drawn by Ostwald²⁷ is not correct.

From all the calculations given above, it is clear that coagulation begins when, because of the contraction of the diffuse layer of ions, the radius of action of the repulsive forces of ionic origin is contracted to such a degree, in comparison, with the radius of the attractive van der Waals forces, that the energy barrier (or force in the case of statistical coalescence) disappears and as an inevitable consequence a rapid coagulation of the system occurs.

Secondly, our theory corrects the Ostwald rule, giving a basis to the exact theoretical rule (46) which replaces that of Ostwald.

Such a change in the form of the rule leads one to believe that the connection between the coagulation criterion and the activity coefficient is accidental and not regular, since for a similar regular connection between the theoretical coagulative ability and the additivity coefficient to exist, it is necessary, as a comparison of the formulae (46), (46b) and (46a) shows, that $f(\beta)$ be exactly equal to $(1 + \beta)$, which does not happen in reality.

Further, the existence of the factor $(A/kT)^2$ allows one to point out those experiments which could finally settle the question in favour of our coagulation criterion and reject that of Ostwald. Indeed, by studying the temperature dependence of the coagulative concentration it would be possible to discover a divergence from the Ostwald rule only if A did not change directly proportionally to T , which is not very probable. However the poor accuracy of the measurements of the coagulative concentrations, makes such a solution of the question, at this time, not very reliable. The advantages of the theoretically based rule over the empirical rule are so clear, however, that this circumstance, alone is decisive.

It is necessary to note that not all observations of the coagulative ability of electrolytes can be fitted by the rule (46).

The dependence of the coagulative concentration on the concentration of the sole (the Burton rule²⁸) and on the type of dominating ion (for equal electrovalence) does not agree of course with (46). As far as the Burton rule is concerned, it is possible that its interpretation is only slightly connected with the theory presented, since it rests, in particular, on the difficulty of a correct determination of the coagulative concentration, especially at low concentrations. The use of a photoelectric cell for this purpose does not remove the doubts of the real comparability between the results of measurements at

different concentrations of the sol. It is perhaps possible to explain the influence of the type of the ion²⁹, in the case of large organic ions, by introducing into our theory the radius of the ion, which is, however, a problem for the future. Finally, the question of the role of the solvation remains open, understanding by this solvation, those effects which depend on repulsive forces of a non-electrostatic nature between the surfaces separated by the layers of the dispersion medium. As the next problem, it is necessary to turn to more exact calculations of the coagulation criterion for symmetric electrolytes.

9. CONCLUSIONS

1. The calculations which have been presented allowed us to find the interaction between strongly charged plane and convex surfaces, in particular spheres, placed in an electrolytic solution.

2. The interaction of similarly charged surfaces always reduces to a repulsion, in contradiction to the calculations of S. Levine, A. Corkill and L. Rosenhead, the incorrectness of which was proved earlier. The superposition, on the repulsive forces of an electrical nature, of the van der Waals attraction, leads to the curves for the force and energy as functions of the distance having one or two minima, separated (if there are two) by a maximum. The existence and the height of the latter determine, in the case of the force diagram, the stability of the film of electrolytic solution which separates the surfaces, and in the case of the energy diagram, the stability of the sol or suspension.

3. Allowing for van der Waals forces, it is possible to find the following criterion for the start of the coalescence of the surfaces mentioned:

$$m = \frac{1}{\gamma n_1} \approx C \cdot f(\beta) \frac{A^2 e^6 z_1^6}{D^3 (kT)^5}.$$

Here A is the van der Waals constant, e the electronic charge, γ the electrolyte concentration in moles per cm^3 , n_1 the number of dominating ions in one molecule, D the dielectric constant, k the Boltzmann constant, T the absolute temperature, β the ratio of the electro-valence of the accessory ion z_2 to that of the counter ion z_1 , C a constant, and $f(\beta)$ a function of "the asymmetry of the electrolyte"; $f(1) = 1$ and has the same order of magnitude for other values of β .

4. We have obtained the stability criterion for strongly charged lyophobic sols and suspensions, which defines the boundary between quick and slow coagulations, in the same form as (1) and having approximately the same numerical value of the constant in its right-hand part.

5. The stability criterion obtained differs by a factor $(A/kT)^2$ from the well-known empirical rule of Ostwald, which, on the one hand, allows one to give a foundation to this rule, explaining its good agreement with large amounts of experimental data, and, on the other hand, points out the necessity of its being changed.

6. Above all, the rule of Hardy-Schulze concerning the influence of the valencies of the dominant and accessory ions on the coagulative concentration, becomes firmly based.

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48. DRAGGING OF A LIQUID BY A MOVING PLATE

ONE of the methods of depositing a thin layer of liquid upon a surface of a solid, which is wetted by it, consists of pulling out an infinite film with constant speed v_0 .

This method is applied, in particular, in the cinefilm manufacture with the purpose of depositing a uniform layer of photosensitive emulsion upon the cinefilm base.

From the trough containing the dissolved photosensitive emulsion an infinite film, wetted by the solution, is pulled out.

After the solvent has evaporated, a uniform layer of emulsion is left deposited upon the surface of the film base.

The problem of determining the thickness of the dragged layer as a function of the speed of the motion of the film and of parameters characteristic of the properties of the fluid (its viscosity η , its surface tension σ and its density ρ) is of essential interest for practice.

Numerous attempts at evaluating the thickness of the dragged layer of fluid found in the literature contain some incorrect assumptions in the very basis of the method of computation, thus leading to erroneous formulae for the value of this thickness.

In the present paper the thickness of the layer and the quantity of fluid carried along when pulling an infinite plate out of a vessel, which is sufficiently large to permit the neglecting of the effect of its walls and of the edges of the plate, is evaluated.

Let us choose a set of co-ordinates in such a way as to make the plate—a plane $x = 0$ and the surface of the liquid undisturbed by the capillary meniscus (i.e. sufficiently far from the plate)—the plane $z = 0$, with the z axis directed upward along the plate.

First of all, let us consider the case of low velocity of motion of the plate (it will be stated below which velocities can still be considered low).

In this case all the surface of the liquid may be separated into two independent regions; the region of the surface situated high above the meniscus and directly dragged by the plate, where the surface of liquid may be taken to be nearly parallel to the plate surface, and the region of the meniscus of liquid, which will be slightly deformed by the motion of the plate, hence the shape of the surface will nearly coincide with the shape of the static meniscus.

Below we shall write down the solutions of the hydrodynamical equations in both independent regions and then connect both of the solutions found.

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Let us denote by h the thickness of a layer of liquid, when measured from the plate. We shall look for $h = h(z)$ in both of the independent regions. First of all, let us write down the equations for the thickness of the liquid film carried along by the moving plate, i.e. the equations for h in the first region.

Since the surface of the liquid in the first region is nearly flat, it is clear that the motion of the moving liquid in this region will also be nearly flat. In other words, the main component of the motion of the liquid in the first region will be the flowing down of it nearly parallel to the plate surface.

This peculiarity may be used in order to simplify the equations of hydrodynamics suitably for this case.

In fact, it is evident that the only component of the velocity of fluid which plays an essential part is the vertical (along the z axis) component u .

It is evident as well that the gradients of the velocity along the direction normal to the plate $\partial u_z / \partial x$ are large as compared with the gradients of velocity along the plate $\partial u_z / \partial z$.

Therefore the motion of a liquid carried along with the plate is described by the equations of the Prandtl boundary layer, which in the stationary case have the form:

$$v \frac{d^2 u}{dz^2} = \frac{1}{\rho} \frac{\partial p}{\partial z} + g, \quad (1)$$

$$\frac{\partial p}{\partial x} = 0. \quad (2)$$

Here, for the sake of brevity, the z component of the velocity u_z is designated simply by u . Other symbols have their usual meaning.

As the boundary conditions for equations (1) and (2) the following conditions will serve: on the surface of the plate, no slip between the liquid and the plate occurs, hence

$$u = v_0 \quad \text{at} \quad x = 0. \quad (3)$$

Here v_0 is the velocity of motion of the plate.

On the free surface of the liquid, at $x = h(z)$, where h is the thickness of the liquid layer, the pressure inside the liquid must be equal to the capillary pressure p_σ and, moreover, the tangent stress must be absent, so that:

$$\left. \begin{aligned} p &= p_\sigma, \\ \eta \frac{\partial u}{\partial x} &= 0, \end{aligned} \right\} x = h. \quad (4)$$

The capillary pressure is known to equal

$$p_\sigma = \frac{\sigma}{R},$$

σ being the surface tension, R —the radius of curvature of the surface.

Substituting the well-known expression for the radius of curvature R , we get:

$$p_\sigma = -\sigma \frac{\frac{d^2 h}{dz^2}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{3/2}}. \quad (5)$$

As long as the thickness of the layer of the liquid carried along is very small, it is evident that the curvature of the surface of the liquid in the vicinity of the plate also will be very small. Therefore, the square of dh/dz in the denominator of equation (5) may be neglected, and the following equation for the capillary pressure in the region of the liquid carried along may be finally written down:

$$p_\sigma \approx -\sigma \frac{d^2 h}{dz^2}. \quad (5a)$$

Therefore, the first of the boundary conditions (4) may be re-written as follows:

$$p = -\sigma \frac{d^2 h}{dz^2} \quad \text{at} \quad z = h.$$

However, it follows from equation (2) that the pressure is constant along the thickness of the liquid layer carried along. Thus, not only on the free surface, but throughout inside the liquid pressure is also given by

$$p = -\sigma \frac{d^2 h}{dz^2}. \quad (6)$$

The solution of equation (1) satisfying the boundary conditions (3) and (4) is:

$$u = v_0 + \left(\frac{1}{\eta} \frac{dp}{dz} + \frac{\rho g}{\eta}\right) \left(\frac{x^2}{2} - hx\right) = v_0 + \left(\frac{\rho g}{\eta} - \frac{\sigma}{\eta} \frac{d^3 h}{dz^3}\right) \left(\frac{x^2}{2} - hx\right), \quad (7)$$

where the value of p is substituted from equation (6).

Let us finally make use of the continuity equation, in order to connect the thickness of the liquid layer with the flow of the fluid carried along by the plate.

For steady motion of the fluid, keeping in mind the incompressibility of fluid, we may, evidently, write down the continuity equation in the form:

$$j = \int_0^h u \, dh = \text{const},$$

where j is the flux of the fluid per unit of width of the plate. Substituting the value of u from equation (7), we obtain:

$$v_0 h - \left(\rho g - \sigma \frac{d^3 h}{dz^3}\right) \frac{h^3}{3\eta} = j. \quad (8)$$

Equation (8) defines the thickness of liquid layer $h(z)$ far from the surface of the fluid (in the first region†).

In the second region, near the fluid surface, the thickness $h(z)$, as has been said above, is determined by the equation for the static meniscus.

Let us re-write equation (8) in a more suitable form:

$$\frac{d^3 h}{dz^3} = \frac{3\eta}{\sigma} \frac{(j - v_0 h)}{h^3} - \frac{\rho g}{\sigma}. \quad (8a)$$

Let us introduce a new dimensionless co-ordinate λ determined by the equation

$$\lambda = \left(\frac{\sigma}{3\eta} \right)^{1/3} \frac{j}{v_0^{4/3}} z, \quad (9)$$

and a dimensionless expression for the thickness of the liquid layer

$$\mu(\lambda) = \frac{v_0 h(z)}{j}. \quad (10)$$

Then, introducing λ and $\mu(\lambda)$ into equation (8a), we find:

$$\frac{d^3 \mu}{d\lambda^3} - \frac{1 - \mu}{\mu^3} - \frac{\rho g j^2}{3\eta v_0^3} = 0. \quad (11)$$

The order of magnitude of the last term in equation (11) which contains all the dimensional quantities involved in the problem, is determined by the kind of dependence of the flux j on the velocity of the plate v_0 . If the flux j is proportional to v_0 in the power higher than $2/3$, then the last term in equation (11) will be simply proportional to v_0 , and for sufficiently small values of velocity of elevation v_0 will be small as compared, with unity.

We assume the flux j to depend on v_0 in the way mentioned above. Then the last term in equation (11) is in fact small as compared with both the first ones and may therefore be neglected. It will be shown later that this assumption turns out to hold within some regions of velocities of elevation of the plate. Thus, the region of validity of solutions of equation (11) obtained on the basis of this assumption will be defined.

Neglecting the last term in equation (11), we get finally:

$$\frac{d^3 \mu}{d\lambda^3} = \frac{1 - \mu}{\mu^3}. \quad (12)$$

The following conditions serve as the boundary conditions for equation (12): on the upper boundary of the region, for very large values of z (far from the surface of the liquid) the thickness of the liquid layer h must tend to a constant

† One of us (B. Levich) has been kindly informed by B. V. Deryagin that he was first to obtain this equation. However, B. V. Deryagin has failed to derive from it any particular conclusions concerning the thickness of the film carried along.

limit h_0 , which evidently equals:

$$h_0 \rightarrow \frac{f}{v_0} \quad \text{for } z \rightarrow \infty.$$

The derivatives dh/dz and d^2h/dz^2 must in this case tend to zero.

Therefore, in terms of dimensionless quantities λ and μ , the boundary conditions may on the upper boundary of the region be written down in the form:

$$\left. \begin{aligned} \mu &\rightarrow 1 \\ \frac{d\mu}{d\lambda} &\rightarrow 0 \\ \frac{d^2\mu}{d\lambda^2} &\rightarrow 0 \end{aligned} \right\} \quad \text{for } h \rightarrow \infty. \quad (13)$$

In order to determine the boundary conditions on the lower boundary of the region, at small values of z (near the meniscus), let us turn to the equations for the static meniscus, which determines the shape of the surface of the layer in the second region.

The equations for the static meniscus have, as is known, the following form:

$$\frac{\frac{d^2h}{dz^2}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{3/2}} = \frac{\rho g z}{\sigma}. \quad (14)$$

Integrating equation (14), we find

$$\frac{\frac{dh}{dz}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{1/2}} = \frac{\rho g z^2}{2\sigma} + c.$$

The constant of integration may be determined from the boundary conditions far away from the plate. In this case precisely, $z \rightarrow 0$ and $dh/dz \rightarrow 0$, i.e. the surface of the fluid is horizontal. Therefore $c = -1$ and

$$\frac{\frac{dh}{dz}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{1/2}} = \frac{\rho g z^2}{2\sigma} - 1. \quad (15)$$

The quantity $(\rho g/c)^{1/2}$ entering the equations (14) and (15) is the Laplace capillary constant, having the dimensionality of a length. The capillary constant is the characteristic length of the problem of the static meniscus. If we

designate it by a , then equation (14) and (15) may be written down in the form:

$$\frac{\frac{d^2 h}{dz^2}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{3/2}} = \frac{z}{a^2}, \quad (14a)$$

$$\frac{\frac{dh}{dz}}{\left[1 + \left(\frac{dh}{dz}\right)^2\right]^{1/2}} = \frac{z^2}{2a^2} - 1. \quad (15a)$$

Equations (14) and (15) determine the thickness of the layer of liquid $h(z)$ in the second region close to the meniscus.

At small values of $h(z)$, small as compared with the capillary constant, and large values of z , the solution of the equality (14) must evidently go over into the solution of the equation (12) for the thickness of the film carried along. We must, therefore, chain together the solution of equation (12) with that of equation (11). The conditions of chaining together of both the equations will at the same time serve as the sought-for boundary conditions of equation (11) at the lower boundary of the first region.

The conditions of chaining together the two solutions shall be obtained from equation (15a) with the aid of the transition to the limit of the small thicknesses $h \rightarrow 0$.

It is clear that with h tending to zero the quantity dh/dz in the formula (15a) will tend to zero as well. Since the surface of the liquid wetting the film in the vicinity of the film itself would be almost vertical, we find at the same time from equation (15) that to the thickness, tending to zero and to the almost vertical surface of the liquid, the finite altitude, tending to the limit value

$$z \rightarrow 2a \quad (16)$$

corresponds.

Next, with the aid of equation (14) we find at the same time that the second derivative of the thickness $d^2 h/dz^2$ in the static solution tends to the limit

$$\frac{d^2 h}{dz^2} \rightarrow \frac{2}{a}. \quad (17)$$

Going over to the dimensionless co-ordinate λ and to the thickness μ , we find, with the aid of equations (19), (10), (16), (17):

$$\mu \rightarrow 0 \left\{ \begin{array}{l} \lambda \rightarrow \left(\frac{\sigma}{3\eta}\right)^{1/3} \frac{2j a}{v_0^{4/3}}, \\ \frac{d^2 \mu}{d\lambda^2} \rightarrow \frac{\sqrt{2} a \sigma^{2/3} j}{v_0^{5/3} (3\eta)^{2/3}}. \end{array} \right. \quad (18)$$

Thus, we see that the boundary with the first region, $d^2\mu/d\lambda^2$, tends to a constant limit, determined by equation (18).

Let us turn now to the boundary conditions of equations (11) at the lower boundary of the region. Here the thickness of the film of the liquid carried along will be large as compared with the limit thickness $\mu = 1$ at the upper boundary of the region. In other words, the value of μ tending to infinity correspond to the lower boundary of the region. Therefore, we must find the boundary conditions of equation (11) with $\mu \rightarrow \infty$. Thus, the boundaries of both independent regions overlap. $\mu \rightarrow \infty$ corresponds to the lower boundary of the first region, $\mu \rightarrow 0$ — to the upper boundary of the second region.

We shall require, as a condition for chaining together the solutions of both regions, the continuity of the second derivative $d^2\mu/d\lambda^2$.

From the point of view of geometry, the continuity of $d^2\mu/d\lambda^2$ expresses the continuity of the curvature of the surface in the region of small curvatures. If we designate by α the limit $(d^2\mu/d\lambda^2)_{\mu \rightarrow \infty}$, where μ is the solution of equation (11), then with the aid of equation (18) the condition of chaining together the solutions for both regions may be written in the form:

$$\left(\frac{d^2\mu}{d\lambda^2}\right)_{\mu \rightarrow 0} = \frac{\sqrt{2} a \sigma^{2/3} j}{v_0^{5/3} (3\eta)^{2/3}} = \alpha. \quad (19)$$

Inasmuch as no dimensional quantities enter equation (11), α is a pure number. The quantity α may be found by means of the numerical integration of equation (11), which will be performed below.

Equation (19) gives us

$$j = \frac{\alpha}{\sqrt{2}} \frac{v_0^{5/3} (3\eta)^{2/3}}{\sigma^{2/3} a}. \quad (20)$$

The condition (20) will be discussed later on. In order to perform the numerical integration (11), it is necessary to investigate in more detail the character of the tending of the derivatives $d\mu/d\lambda$ and $d^2\mu/d\lambda^2$ to zero with λ increasing infinitely, i.e. at the upper boundary of the region (see equation (13)). It may be established from the behaviour of the asymptotic solutions of equation (11).

Namely, for sufficiently large values of λ , the thickness μ may, evidently, be represented in the form:

$$\mu(\lambda) = 1 + \mu_1(\lambda) \quad (21)$$

with $\mu_1(\lambda) \ll 1$.

Then substituting the value $\mu(\lambda)$ from equation (21) into equation (11), we find, after neglecting the squares of small quantities, the linear equation for μ_1 :

$$\frac{d^3\mu_1}{d\lambda^3} = -\mu_1. \quad (22)$$

As the boundary conditions for equation (22) serves:

$$\mu_1 \rightarrow 0 \quad \text{for} \quad \lambda \rightarrow \infty.$$

As a particular solution of equation (22), satisfying this boundary condition, may serve:

$$\mu_1 = \text{const} \cdot e^{-\lambda}$$

It is evident at the same time that

$$\frac{d\mu_1}{d\lambda} = -\text{const} \cdot e^{-\lambda} = -\mu_1,$$

$$\frac{d^2\mu_1}{d\lambda^2} = \text{const} \cdot e^{-\lambda} = \mu_1.$$

Thus, at large values of λ we have the following asymptotic equations for μ and for its derivatives:

$$\mu = 1 + \text{const} \cdot e^{-\lambda}, \quad (23)$$

$$\frac{d\mu}{d\lambda} = 1 - \mu, \quad (23a)$$

$$\frac{d^2\mu}{d\lambda^2} = \mu - 1. \quad (23b)$$

The values of the constant figuring in equation (23) turn out to be inessential for our purpose. For the direct performance of the numerical integration of equation (11), with boundary conditions (23), (23a) and (23b) kept in mind, it would be convenient to lower the order of the equation, choosing μ as the new variable quantity, and $(d\mu/d\lambda)^2$ as the new unknown function. If we designate $(d\mu/d\lambda)^2$ by ξ , then, after simple transformations, we find the following equation for ξ :

$$\frac{d^2\xi}{d\mu^2} = \frac{2(1-\mu)}{\mu^3\sqrt{\xi}}. \quad (24)$$

The boundary conditions (23-23b) may be now written down as

$$\left. \begin{aligned} \xi &\rightarrow (1-\mu)^2, \\ \frac{d\xi}{d\mu} &\rightarrow 2(\mu-1) \end{aligned} \right\} \quad (25)$$

for $\mu \rightarrow 1$.

We are directly interested in the quantity α equal in the new designations to

$$\alpha = \lim_{\mu \rightarrow \infty} \left(\frac{d^2\mu}{d\lambda^2} \right) = \frac{1}{2} \lim_{\mu \rightarrow \infty} \left(\frac{d\xi}{d\mu} \right)$$

The numerical analysis of equations (18) with boundary condition (27) gives for α the value

$$\alpha = 0.63 \dots$$

Substituting this value of α into equation (20) for the flux of the fluid, we get finally

$$j = 2.29 \cdot \frac{\eta^{2/3} v_0^{5/3}}{\sigma^{1/6} (\rho g)^{1/2}}. \quad (20a)$$

We see thus that the conditions of chaining together of the solutions in both regions allow to express the flux j by the characteristic quantities of our problem. It follows from equation (20) that the quantity of the fluid carried along by a slowly moving plate is directly proportional to the velocity of the movement v_0 to the power $5/3$, and to the viscosity of the fluid to the power $2/3$, and is inversely proportional to the tension to the power $1/6$ (i.e. it shows a very slight dependence on the surface tension).

The limit thickness of the layer of the liquid carried along by the plate far away from the meniscus of the fluid, i.e. at $\mu = 1$, would be (cf. (10)):

$$h_0 = 2.29 \frac{(v_0 \eta)^{2/3}}{\sigma^{1/2} \sqrt{\rho g}}. \quad (26)$$

We see, therefore, that the limit thickness of the layer of liquid carried along is proportional to the velocity of the elevation of the plate and to its viscosity to the power $2/3$, and shows a rather slight dependence on the surface tension, being inversely proportional to it to the power $1/6$.

Let us ascertain now the conditions of the applicability of the formulae received (equations (20) and (21)) for the consumption and for the thickness of the layer carried along.

In going over from (15) to (16), we omit the last term of equation (15), on the assumption that the quantity $\rho g j^{2/3} \eta v_0^5$ is small as compared with unity.

Substituting for j its value obtained from formula (22), we find that this quantity is really small as compared with unity, and our calculation is legitimate if the following inequality is fulfilled:

$$\left(\frac{\eta v_0}{\sigma} \right)^{1/3} \ll 1, \quad \text{i.e.} \quad v_0 \ll \frac{\sigma}{\eta}, \quad (27)$$

i.e. at sufficiently small values of the velocity of the plate.

The expressions obtained for j and h seem to be in good accordance with experiment. Indeed, experiment shows that the thickness of the layer carried along is proportional at small velocities to v_0 to the power 0.6, which agrees with the power index obtained by us.

In the opposite limiting case, when the velocity v_0 turns out to be greater than σ/η , the calculation given becomes inapplicable. Namely, the supposition that all the surface of the fluid may be split into two independent regions, which has led to all the expressions written above, does not hold here any longer. It is impossible to obtain in this case the exact expressions for j and h . However, on the basis of dimensionality considerations, the general character of the dependence of these quantities on the fundamental parameters v_0 , η , ρ , and σ at sufficiently large velocity may be pointed out.

Namely, it is clear that at high velocities the consumption j and the thickness h of the layer carried along should not depend on the surface tension. From the viewpoint of physics, this may be seen from the fact that at sufficiently high velocities of the plate, the shape of the entire surface of the fluid would be determined by the character of the process of carrying along, and not by the static properties of the surface of the fluid. Therefore, at high velocities of the plate, the thickness of the layer carried along must depend only on the quantities v_0 , η , ϱ and g . The only quantity of the dimensionality of length, which may be obtained from these quantities, is the quantity $(\eta v/\varrho g)^{1/2}$.

Therefore, at sufficiently high velocities of the elevation of the plate, the thickness of the layer of liquid carried along must have the form:

$$h \approx A \left(\frac{\eta v_0}{\varrho g} \right) \quad (28)$$

and the consumption

$$j \approx A \left(\frac{\eta}{\varrho g} \right)^{1/2} v_0^{2/3}.$$

The numerical value of the constant A may be found only by means of experiment.

Finally, in the intermediate region of velocities, it may be seen from dimensionality considerations that the thickness of the layer carried along must have the following form:

$$h = \left(\frac{\eta v_0}{\varrho g} \right)^{1/2} f \left(\frac{v_0 \eta}{\sigma} \right), \quad (29)$$

where $f(v_0 \eta/\sigma)$ is some function of the dimensionless parameter $v_0 \eta/\sigma$, the form of which must be found by experiment.

In both the limit cases of large and small values of $v_0 \eta/\sigma$ the function $f(v_0 \eta/\sigma)$ tends accordingly to unity and to $(v_0 \eta/\sigma)^{1/6}$, respectively, so that equation (29) goes over into equations (28) or (26).

In conclusion we should like to express our gratitude to M. M. Kussakov and B. V. Derjaguin, who have called our attention to the experimental and the technical interest of the problem treated here.

49. ON THE THEORY OF THE INTERMEDIATE STATE OF SUPERCONDUCTORS

A quantitative analysis of the laminar structure of the superconductor in the intermediate state is given. The behaviour of the superconductor with a transversal slit in the magnetic field is investigated. An explanation of the hysteresis effect in the transition to the intermediate state is advanced.

It is well known that at a strength of the external magnetic field lying within a definite range (depending upon the shape of the specimen) the superconductor goes over into a so-called intermediate state. Formally this state can be described macroscopically as a state in which the magnetic induction within the body, B , has values intermediate between zero and the critical value of the field. However, in reality, as it has been already pointed out by the author^{1,2}, the body of a superconductor in the intermediate state should consist of regions, each of them being either in the superconducting, or in the normal state. The usual formal description of the intermediate state follows from considering physical quantities averaged over all of these regions. Here the question of the size and shape of superconducting (s) and normal (n) regions arises.

First of all, we notice that the boundaries between different regions should be formed by the lines of forces of the magnetic field. In fact if in an n -region the magnetic induction B would have a non-zero component B_n normal to the boundary surface, then in virtue of the continuity of B_n , in the neighbouring s -region there would be an non-zero induction, while in the superconducting state B should always vanish. As to the absolute values of the field, at the boundary the s - and the n -regions, it evidently should be equal exactly to the critical field H_c ; the field being greater than H_c , within the n -region. In the opposite case we would not obtain a thermodynamically stable state corresponding to the minimum of free energy.

It is most natural to suppose that in the intermediate state the body splits up into laminae, alternately superconducting and normal, arranged parallel to the external magnetic field. However, simple argumentation convinces that these laminae cannot have a constant thickness along their length, "curving" only at their ends. Indeed, let us suppose at first that the superconducting laminae, near their ends, are bounded by convex surfaces (Fig. 1a). Since in the absence of the current, the magnetic field possesses a potential, we can draw the equipotential surfaces of the field. These surfaces, in particular,

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should be normal to the surfaces separating from each other the s - and n -laminae which, as it has been pointed out above, are formed by lines of force. In the convex parts of the boundary the equipotential surfaces will be arranged as shown by the dotted lines in Fig. 1a, i.e. they will diverge into the n -lamina. But the absolute value of the field decreases in the direction of the divergence of the equipotential surfaces. Therefore the field in the n -lamina would turn out to be smaller than H_c , i.e. the whole configuration would be thermodynamically unstable†. Let us suppose now that the s -laminae are bounded at their

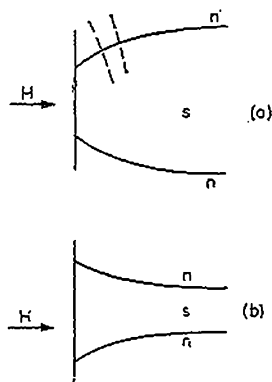


FIG. 1.

ends by concave surfaces. Then on the lines of force enveloping the s -lamina (of course, the lines of force do not enter this lamina) in the points in which the edge is sharpened the field would be infinite, which is in contradiction with the boundary condition, $H = H_c$; in other words, the solution of the field equations with such a shape of the laminae does not exist at all.

Thus we see that any simple shape of the laminae turns out to be impossible. However, all the necessary conditions can be satisfied if we assume the following, more complicated picture of the structure of a superconductor in the intermediate state. In approaching the surface of the specimen, at some definite distance from this surface each of the n -laminae splits up into two laminae of smaller thickness. When approaching still nearer to the surface, the laminae formed split up again, etc. In such a way, the laminae are branched formally an unlimited number of times so that the thickness of the laminae tends to zero, as the distance to the surface of the specimen decreases. In reality, of course, we can speak of separate n - and s -laminae in the above sense i.e. considering $B = 0$ in the s -laminae) only in so far as the order of magnitude of their thickness does not approach the depth of penetration of the magnetic field into a superconductor. Further on we shall return to this question.

For the sake of deducing quantitative relationships let us consider a plane parallel plate in a transverse magnetic field. For a superconductor of this shape the range of the intermediate state with respect to the values of the external field

† This circumstance was not taken into account in ref. 1.

extends from zero up to H_c . In Fig. 2 a section of the plate in the direction parallel to the external field is given; we have shaded the n -laminae.

In each section between two consecutive branchings we can consider the n -laminae as having a constant thickness and as bounded by parallel planes. The field within the n -laminae has a constant magnitude (equal to H_c) and is directed parallel to the boundaries. It is easy to see that at each branching the thickness of the laminae is halved. This follows directly from the require-

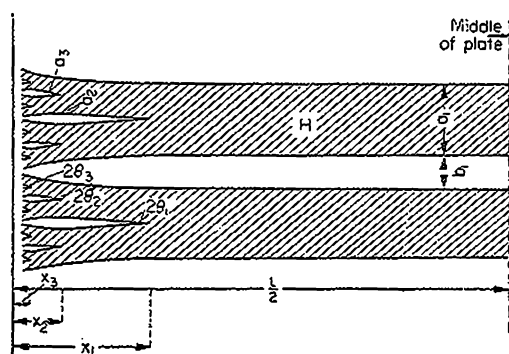


FIG. 2.

ment that the full flux of the induction must remain constant along the thickness of the plate. Since in the s -laminae $B = 0$, while in the n -laminae $B = \text{const} = H_c$, the induction flux is simply proportional to the sum of the cross-sections of the n -laminae. Therefore this sum must remain constant in the course of the consecutive branchings. (It must be emphasised that we are concerned here not with the area of the sections of laminae by planes parallel to the surfaces of the plate, but with the areas of sections, transverse to the n -laminae).

The shape and sizes of the laminae are determined by the minimisation of the total free energy. The total free energy consists of the sum of the volume energies of the superconducting and normal regions and of the surface energy which is connected with the existence of the surfaces of separation between these regions. As it is known, in the absence of field the free energy in the normal state (per unit volume) exceeds the free energy in the superconducting state by $H_c^2/8\pi$. In the presence of a magnetic field to the surplus energy of the substance in the n -laminae (where $B = H_c$), in comparison to the substance in the s -laminae (where $B = 0$), the energy of the field is added which is also equal to $H_c^2/8\pi$. Therefore, the total surplus volume energy depending on the presence of n -laminae in a superconductor is equal to the product of their total volume by $H_c^2/4\pi$. As to the surface energy, it is proportional to the area S of the boundary surfaces between n - and s -laminae and is equal to αS , where α is the "surface tension" between the normal and superconducting phases. Instead of α we shall use the quantity Δ , determined by the equation Δ has the dimension

$$\alpha = \frac{H_c^2}{8\pi} \Delta \quad (1)$$

of length.

One can expect (assuming that superconductivity is characterised by a single quantity having the dimension of length) that the length l and the depth of penetration of the external magnetic field into the superconductor are of the same order of magnitude. It is however to be emphasised that there are no grounds for considering these quantities simply proportional to each other; in particular, the laws of their temperature dependence may turn out to be entirely different.

Consider, first, any one of the consecutive branchings of an n -lamina (Fig. 3) disregarding its further and previous splittings. Let a be thickness of the lamina before its splitting. After the branching we get two laminae whose thickness equals $a/2$. Let 2θ be the angle between these two laminae ("the angle of branching"). Between a and θ there exists a dependence which can be determined in the following way.

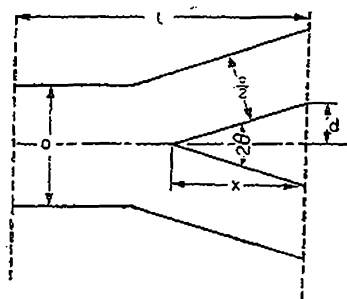


FIG. 3.

Let us imagine that all the 3 ends of the investigated lamina are fastened on 2 planes (shown in Fig. 3 by dotted lines), located at a given distance from each other. Since the total surplus free energy connected with the presence of n -laminae should be a minimum with respect to all possible variations in their shape, then, in particular it should be minimal with respect to a displacement of the branching point (i.e., to the change of the distance x —see Fig. 3), the location of the ends of the lamina being kept constant (i.e., at l and d fixed). The total volume of the branching n -lamina, per unit of length in the direction, perpendicular to the plane of drawing, is equal to

$$(l - x) a + 2 \frac{a}{2} \sqrt{x^2 + d^2}$$

and its surface

$$2[(l - x) + 2\sqrt{x^2 + d^2}]$$

Therefore the full surplus free energy is

$$2[(l - x) + 2\sqrt{x^2 + d^2}] \frac{H_c^2}{8\pi} \Delta + [(l - x) + \sqrt{x^2 + d^2}] \frac{H_c^2}{4\pi} a$$

In performing the calculations it should be kept in mind that the angle of branching θ is very small (as will be verified by further calculations). Introducing the small angle $\theta \cong d/x$ instead of x , we get the surplus energy in the form (dropping the constant terms which do not depend upon θ and are of no interest to us, and neglecting the terms known to be of a higher order of smallness; it is supposed that $\Delta \ll a$):

$$\frac{H_c^2}{4\pi} \frac{d\Delta}{\theta} + \frac{H_c^2}{4\pi} \frac{d\theta}{2} a.$$

This expression has a minimum when

$$\theta = \sqrt{\frac{2\Delta}{a}}, \quad (2)$$

by which equation the connection between the thickness of the lamina and the branching angle is determined.

We note that when relation (2) is fulfilled, the surface and the volume parts of the surplus energy are equal to each other.

After these preliminary computations we can determine completely the shape of the n - and s -laminae into which the plate splits up in the intermediate state. Let a_1 be the thickness of the n -laminae in the middle of the plate, i.e. before the beginning of their branching, a_2 —the thickness of the laminae formed as a result of the first branching etc. The angles of the first, second and etc. branching let us designate by $2\theta_1, 2\theta_2$. Finally, let x_1, x_2, \dots be the distance from the first, second, third, etc. branching point to the nearest surface of the plate. First of all, since at every branching the thickness of the n -lamina decreases twice we have

$$a_n = \frac{a_1}{2^{n-1}}. \quad (3)$$

For each pair a_n, θ_n we have the relation (2)

$$\theta_n = \sqrt{\frac{2\Delta}{a_n}} = 2^{(n-1)/2} \theta_1. \quad (4)$$

Before the beginning of the branching the laminae are arranged perpendicular to the surface of the plate (parallel to the external field). After the first branching the laminae formed are arranged at angles $\pm \theta_1$ to the initial direction. Together with them all the magnetic lines of force passing through n -laminae are deflected through the same angle. If no further branching would take place, the ends of the lines of force (in points of their intersection with the surface of the plate) would deflect through distances $\pm \theta_1 x_1$ from the points in which they would get if no branching would take place at all. The additional deflection caused by the second branching is equal to $\pm \theta_2 x_2$, etc. (all angles $\theta_1, \theta_2, \dots$ are small). Therefore, as a result of repeated branching of the initial

lamina, the full deflections of the ends of the lines of force are determined by the infinite sums

$$\pm x_1 \theta_1 \pm x_2 \theta_2 \pm x_3 \theta_3 \pm \dots \quad (5)$$

with all the possible combinations of the \pm signs.

The lines of force, passing on the very boundary of the initial (not branched) n -laminae should deflect at a distance equal to a half of the distance b_1 (Fig. 2) between the neighbouring n -laminae, i.e. of the width of the s -laminae. This means that we should have

$$x_1 \theta_1 + x_2 \theta_2 + x_3 \theta_3 + \dots = \frac{b_1}{2}. \quad (6)$$

Further, the distances between the displacements of the ends of the various lines of force passing originally in the same initial n -lamina, are equal to the differences of the quantities (5), i.e. are expressed by the sums

$$2(e_1 x_1 \theta_1 + e_2 x_2 \theta_2 + \dots),$$

where each e_1, e_2, \dots can have the value either 0 or 1. Since the lines of force cannot intersect, all the quantities received in this way should fill up, continuously, the interval between zero and b_1 . It is known that the interval between 0 and 1 can be filled up by sums of powers of $1/2$ of the type

$$e_1 \frac{1}{2} + e_2 \frac{1}{4} + e_3 \frac{1}{8} + \dots$$

with all the possible combinations of values e_1, e_2, \dots equal to 0 or 1 each (this being a binary analogue of decimals). Keeping this in mind, we conclude that the sequence of the quantities $x_1 \theta_1, x_2 \theta_2, \dots$ should form a geometrical progression with a denominator equal to $\frac{1}{2}$. In this way, we get the relation

$$x_n \theta_n = \frac{x_1 \theta_1}{2^{n-1}}. \quad (7)$$

From equation (6) we find now

$$2x_1 \theta_1 = \frac{b_1}{2}. \quad (8)$$

Combining equations (3), (4) and (7) we find that

$$\frac{x_{n-1}}{x_n} = 2\sqrt{2}. \quad (9)$$

Thus, each of the subsequent branchings takes place at a distance $2\sqrt{2}$ times nearer to the surface of the plate, than the preceding one.

Further, between a_1 and b_1 there exists a relation which follows directly out of the fact that the mean (averaged over the laminae) induction in the plate

should have a given value B (in the case of a plate, equal to the external field). Since, on the other hand, the induction in the n -laminae equals H_c , and in the s -laminae equals 0, it is clear that it should be $(a_1 + b_1) B = a_1 H_c$ or

$$\frac{a_1}{a_1 + b_1} = \frac{B}{H_c}. \quad (10)$$

Finally, let us make use of the condition of the minimum of the free energy. If the n -laminae would not branch at all, i.e. they would have a constant thickness a_1 , the surplus free energy, caused by the presence of the substance in the normal state, would be equal (per unit area of the surface of the plate):

$$\frac{1}{a_1 + b_1} \left(\frac{H_c^2}{4\pi} L a_1 + \frac{H_c^2 \Delta}{8\pi} 2L \right)$$

(L —the width of the plate). The first term in the brackets gives the volume energy, and the second the surface energy of a single lamina, the factor before the brackets being the number of n -laminae, per unit of length of the plate. With the aid of equation (10) we can re-write this equation in the form

$$\frac{H_c B}{4\pi} L + \frac{H_c^2}{4\pi} \Delta L \frac{1}{a_1 + b_1}.$$

We see that the first term does not depend upon the width of the lamina (for fixed B) and therefore is of no interest to us. Further, owing to the first branching, the surface of the sections of the length x_1 from both sides of the n -lamina is doubled, i.e. a new surface energy appears equal to

$$\frac{H_c^2 \Delta}{8\pi} 4x_1.$$

The additional volume energy, connected with the branching of the lamina, is, as was shown above, equal to the surface energy. Therefore, the first branching leads to an additional energy (again taken per unit area of the surface of the plate):

$$\frac{1}{a_1 + b_1} \frac{H_c^2}{4\pi} 4x_1.$$

In like manner we find that the second branching gives an energy equal to $\{1/(a_1 + b_1)\} \times (H_c^2/4\pi) 8x_2$, etc.

Thus, the additional free energy f connected with the presence of branched n -laminae is equal to

$$f = \frac{H_c^2}{4\pi} \frac{\Delta}{a_1 + b_1} (L + 4x_1 + 8x_2 + 16x_3 + \dots) + \frac{H_c B}{4\pi} L.$$

Summing the geometrical progression, the denominator of which, according to equation (9), is equal to $1/\sqrt{2}$, we get

$$f = \frac{H_c^2}{4\pi} \frac{\Delta}{a_1 + b_1} \left(L + \frac{4x_1}{1 - \frac{1}{\sqrt{2}}} \right) + \frac{H_c B}{4\pi} L.$$

Combining equations (4), (8) and (10), the relation

$$x_1 = \frac{a_1^{3/2}}{4\sqrt{2}\Delta} \frac{H_c - B}{B} \quad (11)$$

is readily obtained.

Expressing all the quantities in f through a_1 , we get

$$f = \frac{H_c B}{4\pi} \Delta \left[\frac{L}{a_1} + \frac{\sqrt{a_1}}{\sqrt{\Delta}(\sqrt{2} - 1)} \right] \left(\frac{H_c - B}{B} \right) + \frac{H_c B}{4\pi} L. \quad (12)$$

This expression considered as a function of a_1 , has a minimum at

$$a_1 = 2^{2/3} (\sqrt{2} - 1)^{2/3} \Delta^{1/3} L^{2/3} \left(\frac{B}{H_c - B} \right)^{1/3} \quad (13)$$

the thickness of n -laminae being thus determined. For the thickness of the s -laminae, we get, according to equation (10), $b_1 = a_1(H_c - B)/B$, so that

$$b_1 = 2^{2/3} (\sqrt{2} - 1)^{2/3} \Delta^{1/3} L^{2/3} \left(\frac{H_c - B}{B} \right)^{1/3}. \quad (14)$$

Finally

$$a_1 + b_1 = 2^{2/3} (\sqrt{2} - 1)^{2/3} \Delta^{1/3} L^{2/3} \frac{H_c}{B^{1/3}(H_c - B)^{2/3}}. \quad (15)$$

Thus, the thickness of the laminae is proportional to the $2/3$ power of the width of the plate L . When B increases, the thickness of the n -laminae a_1 increases and the thickness of the s -laminae b_1 diminishes, just as it should.

The functions of B which enter equations (13–15) may be considered practically constant almost in the whole range of variation of B with the only exception of regions in the immediate vicinity of its ends (i.e. near zero and H_c). In a practically sufficient approximation we can write

$$a_1 + b_1 \cong 2\Delta^{1/3} L^{2/3}. \quad (16)$$

If we take $\Delta \cong 3 \times 10^{-5}$ cm, then for $L = 1$ cm we get $a_1 + b_1 = 0.06$. Hence, within the length of 1 cm of the plate there will be about 20 layers. It should be noted that the laminae turn out to be comparatively very thick†.

† Let us note that the angles θ are of the order $\theta \sim \sqrt{\frac{\Delta}{a}} \approx \left(\frac{\Delta}{L} \right)^{1/3}$, i.e. that they are really small, as it was supposed above.

Substituting (13) into equation (11) we get

$$x_1 = L \frac{\sqrt{2} - 1}{2\sqrt{2}} = 0.14 L. \quad (17)$$

Thus, the length of the unbranched section of the laminae ($L - 2x_1$) is 0.71 of their full length (the width of the plate).

Let us, moreover, write down the expression for the surplus energy f which may be obtained by substituting (13) into equation (12):

$$f = \frac{3L^{1/3} \Delta^{2/3} H_c B^{1/3} (H_c - B)^{2/3}}{2^{2/3} (\sqrt{2} - 1)^{2/3} \pi} + \frac{H_c B}{4\pi} L. \quad (18)$$

All the formulae obtained can be applied not only to plates but to bodies of a different shape, provided the average field within them is uniform (it is known that ellipsoids, in particular a sphere and cylinder, possess this property). In this case under L the length of lines of force passing along the considered n -lamina should be understood. Of course L is different for different laminae. If we have bodies of another shape, which do not satisfy this condition, the results received can be applied only qualitatively; in particular we can apply, as previously, the approximate formula (16).

As far as the shape of the laminae as a whole is concerned (in the plate—they are the parallel, plane laminae), for instance, in the case of a spherical specimen, one can presume that the most advantageous energetically are the laminae in the form of coaxial cylinders whose axis is the diameter of the sphere, parallel to the applied field. For each of the cylindrical laminae L and, consequently, its thickness is constant. In a cylindrical specimen placed in a transverse field the laminae should be plane, parallel to the axis of the cylinder and directed along the external field; each one of such laminae should have a constant thickness.

Approaching the surface of the body, as a result of successive branchings the thickness of the laminae will decrease indefinitely, until a macroscopic description of a superconducting state as the state with $B = 0$ will lose its meaning. When this happens, the concept of an "intermediate state" will also lose its meaning and the substance will acquire some singular, new state, which can be called "mixed". As to the properties of this state one can make no conclusions on the basis of the usual macroscopic theory of the superconducting state; the study of its properties would require a more detailed investigation.

As to the shape of the s - and n -regions the following note is to be added. We supposed that they have the appearance of alternating laminae. When the field is weak enough so that most of the substance is in a superconducting state, the laminae should be very thin. In these conditions, the disintegration of the n -laminae into separate thread-like regions, arranged parallel to the field, may turn out to be energetically more advantageous. The branching of these threads should be of the same type as in the case of laminae. In like manner, when the fields are sufficiently near to the critical, there should form,

instead of very narrow *s*-laminae, thread-like *s*-regions, arranged in the bulk mass of the substance which is in the normal state.

The branching of the laminae which leads to the formation of a "mixed" state on the surface of the specimen does not allow to observe directly in the experiment the existence of laminae. This difficulty was avoided in experiments proposed and begun before the breaking out of the present war by Prof. A. I. Shalnikov†. Basing on the supposition that at the sides of a narrow slit cut in the specimen, in the direction normal to the field, the laminae do not branch, he performed the measurement of field within a slit cut in the equatorial plane of a superconducting sphere. The theory developed here shows that for a sufficiently narrow slit such an effect actually should exist. The reasons of this are as follows.

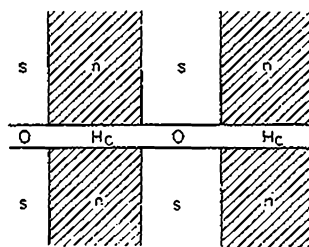


FIG. 4.

We can be convinced of the energetic expedience of branched laminae over unbranched ones also by the following considerations. If the layers would reach the surface of the specimen unbranched, then the magnetic field in the outer space near this surface would be inhomogeneous (at the surface of the *s*- and *n*-laminae the field would be correspondingly equal to 0 and H_c). But if the laminae branch themselves, forming a mixed state on the surface, then the external field can be considered practically uniform and consequently its energy will be less. It is nevertheless easy to ascertain that, having a slit narrow enough, it is possible, that energetically the most expedient may turn out to be the laminae unbranched at its surface.

Consider a slit of the width d , cut in a direction perpendicular to the direction of the field. If the laminae do not branch off, as though passing directly from one side of the slit to the other, one can consider the field in the laminae as alternately equal to 0 and H_c over the sections of the space which are a continuation of the laminae (Fig. 4). The energy of the field in the slit, per unit area of its surface, will then equal

$$\frac{1}{a_1 + b_1} d a_1 \frac{H_c^2}{8\pi} = \frac{B H_c}{8\pi} d.$$

† Cf. *J. Phys. U.S.S.R.* 6, 226 (1942), where preliminary results are given.

The energy of a uniform field B in the space of the slit would be equal to $(B^2/8\pi) d$. Therefore, the surplus energy due to the presence of laminae unbranched at the slit is

$$\frac{B(H_c - B)}{8\pi} d.$$

If, on the other hand, the laminae branch at the surface of the slit, then an additional energy arises which is equal to

$$[f(L_1) + f(L_2)] - f(L_1 + L_2)$$

where $f(L)$ is determined by the equation (18), and L_1 and L_2 are the lengths of the two parts into which the lamina is divided by the slit. The first term gives the additional energy of the laminae, branching both at the surface of the slit and at the two outer surfaces of the body. From this energy we have to subtract the energy given by the second term of the laminae branching only at the outer surfaces, so as it would have been in the absence of the slit.

The "critical width" of the slit d_0 is determined from the condition of equality of both additional energies written above.

$$\frac{B(H_c - B)}{8\pi} d_0 = [f(L_1) + f(L_2)] - f(L_1 + L_2); \quad (19)$$

when $d > d_0$ the laminae branch at the surface of the slit, while when $d < d_0$ the laminae come up to the surface of the slit unbranched.

For the sake of brevity let us assume that the body is divided by the slit into two equal parts; then $L_1 = L_2 = L/2$, L being the length of the body in the direction of the external field. From equations (18) and (19) we get

$$d_0 = \frac{3(2 - 2^{1/3})}{(\sqrt{2} - 1)^{2/3}} \Delta^{2/3} L^{1/3} \frac{H_c}{B^{2/3}(H_c - B)^{1/3}}. \quad (20)$$

Almost in the whole range of the variation of B (with the exception of the regions near 0 and H_c) we can write, in a practically sufficient approximation:

$$d_0 \cong 8 \Delta^{2/3} L^{1/3}. \quad (21)$$

Thus if we take $\Delta \cong 3 \times 10^{-5}$, then for $L = 1$ cm we will get $d_0 \cong 0.01$. Such a width is possible to achieve practically.

A. I. Shalnikov in his very ingenious experiments succeeded in discovering the effect described, thus confirming experimentally the existence of laminae. Unfortunately, these experiments were interrupted by the war, and the quantitative comparison with theory cannot be carried out now.

Another effect which is connected with the "lamina structure" of the intermediate state and which can be observed experimentally is the well-known "lagging behind" of the transition from the superconducting to the inter-

mediate state when the applied magnetic field is increased. The transition sets in at higher values of the field than may have been expected on the basis of the usual macroscopic theory (for instance, for a cylindrical conductor in a transverse field the transition point turns out to be lying somewhat above $\frac{1}{2}H_c$). From the point of view of the theory described here the reason for this phenomenon is the following. The transition into the intermediate state takes place when this state becomes energetically expedient. But the lamina structure of the intermediate state, besides the purely "volume" free energy ($H_c B/4\pi$ per unit volume) ascribed to the intermediate state in the usual macroscopic theory, is connected with additional energy, caused by the presence of the boundary surfaces between s - and n -regions, and by the branching of laminae; this causes a displacement of the transition point towards greater fields.

A precise theoretical determination of the transition point would demand a full determination of the distribution of the mean field within the specimen at the instant of its transition into the intermediate state. This distribution is determined by the field equations in which the connection between the two vectors H and B entering into these equations is given by the well-known thermodynamic relation

$$H = \frac{\partial \Phi}{\partial B}, \quad (22)$$

$\Phi/4\pi$ being the free energy per unit volume of the body.

In the intermediate state, when account is taken of the presence of branching laminae, the function Φ is equal to $(4\pi/L)f$ where f (18) is the energy per unit area of the surface of the body. Φ can be written down in the form

$$\Phi = H_c B + \varphi \quad (23)$$

where φ corresponds to the first term in equation (18), i.e. to the energy due to the presence of boundary surfaces between s - and n -laminae and to the branching of the laminae. It is essential that φ contains the length of the line of force L which depends upon the shape and size of the body and is different in its various points; L consequently will enter into the relation between H and B . Since L is different in various points of the body, it will follow that even within an ellipsoid B and H will not be uniform. In these conditions the problem of determining the field is very complicated from the mathematical point of view.

However, we can presume that we can get a result for the transition point in the ellipsoid practically precise enough by starting out from the rough supposition that the mean induction B is uniform in the ellipsoid (in reality, B will be uniform only if φ is neglected). Let us perform the necessary calculations.

Preliminarily let us remember some general relations which are known from the ordinary field theory for macroscopic material bodies. Let $(1/4\pi)\Phi(B)$ be the free energy of a unit volume of a body, the magnetic induction within

which is B . The total energy of the body, however, is not equal simply to the integral $(1/4\pi) \int \Phi \, dV$ over the volume of the body. The presence of the body causes a change in the distribution of the field in the outer space as well; the corresponding change in the energy (as compared with the energy in the absence of the body) should be included into the total energy store of the body. Let $F(B)$ be the total free energy, formally taken per unit volume of the body, i.e. a quantity, determined in such a way that the integral $\int F \, dV$ over the volume of the body is equal to the total energy. Then we have the following expression†.

$$F = \frac{1}{4\pi} \left[\Phi(B) - \frac{HB}{2} - \frac{h(B-H)}{2} \right]. \quad (24)$$

Here H and B are the absolute magnitudes of the vectors \mathbf{H} and \mathbf{B} within the body and h is the external field (to be more exact, the value it would have in the absence of the body). The quantities B , H and h are connected with one another by equation (22) and (for ellipsoids) by the relation

$$nB + (1-n)H = h, \quad (25)$$

where $4\pi n$ is the demagnetising factor (i.e. $n = \frac{1}{2}$ for a cylinder in a transverse field, $n = \frac{2}{3}$ for a sphere).

In the intermediate state we write Φ in the form of (23), so that equation (24) gives

$$F_i = \frac{1}{4\pi} \left[\varphi(B) + H_c B - \frac{HB}{2} - \frac{h(B-H)}{2} \right]. \quad (26)$$

In the superconducting state we have $B = 0$, and also $\Phi = 0$ since in reality we measure the energy from the value which it has when no substance in the normal state is present. From equations (24) and (25) we have the well known formula

$$F_s = \frac{h^2}{8\pi(1-n)}. \quad (27)$$

The transition point (h_t) from the superconducting state to the intermediate state is determined from the condition $F_s = F_i$, i.e.

$$\varphi + H_c B - \frac{HB}{2} + \frac{h_t(B-H)}{2} - \frac{h_t^2}{2(1-n)} = 0. \quad (28)$$

In order to determine from the above the value of h_t , it is necessary to combine this equation with the relation (25), which can be written down in the

† On can check this formula for instance by determining the differential dF , i.e. the work performed during an isothermic change of the field. Simple computations (with the aid of equations (22), (24), (25)) give $dF = -\frac{B-H}{4\pi} dh = -M dh$ (where M is the magnetic moment of a unit volume of the body) just as it should have been.

following form

$$h_t = (1 - n) H_c + nB + (1 - n) \frac{d\varphi}{dB} \quad (29)$$

$$\left(\text{we substitute } H = \frac{d\varphi}{dB} = H_c + \frac{d\varphi}{dB} \text{ for } H \right).$$

By substituting equation (29) into equation (28) we obtain

$$\varphi - B \frac{d\varphi}{dB} = \frac{n}{2(1 - n)} B^2. \quad (30)$$

Equations (29–30) are suitable for computing h_t ; knowing the function $\varphi(B)$, we compute from equation (30) the value of B and substituting it into equation (29) we will get the value of the external field h_t in the transition point.

As far as the function $\varphi(B)$ is concerned, we must note that when the external fields are very close to the transition point, the superconducting regions have most probably, as was shown above, the shape not of laminae but of threads. Therefore, generally speaking, we cannot make use here of expression (18) for the energy which was obtained for a laminar structure of the intermediate state (even though the difference between the two formulae is not great in this region). Here we shall not perform the corresponding computation for the energy of the thread-like structure, but shall give only the result obtained. For a transition point of a cylindrical superconductor in a transverse field we get

$$h_t = H_c \left[\frac{1}{2} + 0.4 \sqrt{\frac{\Delta}{r}} \right]. \quad (31)$$

For a sphere of radius r the transition point is equal to

$$h_t = H_c \left[\frac{2}{3} + 0.5 \sqrt{\frac{\Delta}{r}} \right]. \quad (32)$$

The second terms in the parentheses are the looked for deviations from the usually expected values of h_t . We note that they are inversely proportional to the square root of the size of the body, and their dependence upon temperature is connected with the temperature dependence of Δ .

Unfortunately, no measurements for the dependence of h_t/H_c upon r were performed. Systematic measurements of the dependence of this ratio upon temperature for tin, indium and lead were performed by Misener³. Unfortunately, he does not give the exact diameters of the wires used by him but only shows the order of their magnitude. Besides that, it is not clear to what extent an equilibrium state was reached and whether any hysteresis phenomena were present. In any case, we can conclude that the quantity Δ decreases when the temperature increases, vanishing when $T = T_0$ (T_0 is the point of superconducting transition in the absence of the field), i.e. it behaves inversely as the depth of penetration of magnetic field into the superconductor. As to

the order of magnitude of the resulting values of Δ is concerned, it depends strongly upon the temperature.

We emphasise that the whole theory developed here relates to "static" properties of the intermediate state. As to the questions, connected with the conductivity of superconductors in the intermediate state, we can say that they demand a more detailed investigation of the properties of the "mixed" state near the surface of the specimen.

In conclusion, I wish to thank sincerely Prof. A. I. Shalnikov for the communication of the results of his experiments.

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50. ON THE RELATION BETWEEN THE LIQUID AND THE GASEOUS STATES OF METALS

A METAL sharply differs from a dielectric with respect to its spectrum of electron energy levels at absolute zero temperature. The fundamental state of the metal borders upon a continuous spectrum of states: this explains the fact that even the weakest electrical field gives rise in a metal to an electrical current, due to a transition of the system to adjacent levels. On the contrary, the electron energy spectrum of a dielectric is characterised by the existence of a finite "gap", i.e. of a definite energy difference between the fundamental state with the lowest energy (corresponding to the absence of a current) and the nearest excited states, in which one of the electrons of the dielectric becomes free and the electric conductivity appears.

It should be mentioned that a metal cannot be defined as a body with a continuum of levels adjacent to the fundamental one without accessory conditions: as a matter of fact, any paramagnetic substance such as liquid oxygen or gadolinium sulphate has a continuous spectrum of states, corresponding to different values of the magnetic moment: this explains the variation of the magnetic moment in a weak magnetic field, which is characteristic of a paramagnetic substance. Nevertheless liquid oxygen or gadolinium sulphate are not metals and have no electrical conductivity. The existence of a continuous spectrum is, thus, necessary but not sufficient for the metallic state: the excited levels adjacent to the fundamental one must have the property of transferring the electrical charge in order to ensure the conductivity.

It has often been surmised (without a general proof) that under sufficiently high pressure every substance must be transformed into a metal. This is illustrated by the transition of phosphorus into a conducting modification at high pressures (Bridgman's black phosphorus). At the absolute zero temperature a metal and a dielectric are qualitatively different, so that it is always possible to say with what kind of substance we have to do, there existing a definite transition pressure†. A dielectric differs from a metal by the presence of an energy gap in the electronic spectrum. Can however this gap tend to zero

Я. Зельдович и Л. Ландау, О соотношении между жидким и газообразным состоянием у металлов, *Журнал Экспериментальной и Теоретической Физики* 14, 32 (1944).

L. Landau and G. Zeldovich, On the relation between the liquid and gaseous states of metals, *Acta Phys.-chim. USSR*, 18, 194 (1943).

† At a temperature different from zero any dielectric must be slightly excited, a certain minute fraction of the electrons being in an excited state, corresponding to a non-vanishing electrical conductivity, so that the system as a whole is in a state belonging to a continuous spectrum. A dielectric can therefore be sharply distinguished from a metal at the temperature of absolute zero only.

when the transition point into a metal is approached (on the side of the dielectric)? In this case we should have to do with a transition without latent heat, without change of volume and of other properties. Peierls has pointed out that a continuous transition—in this sense—is impossible. Let us consider the excited state of the dielectric in which it is capable of conducting an electric current: an electron has left its place, leaving a positive charge in a certain place of the lattice and is moving throughout the latter. At large distances from the positive charge, the electron must certainly suffer a Coulomb attraction tending to bring it back. In a Coulomb attraction field there always exist discrete levels of negative energy, corresponding to a binding of the electron; the excited conducting state of the dielectric must therefore always be separated from the fundamental one, in which the electron is bound by a gap of a finite width.

If at 0°K the transition of a metal into a dielectric is a phase transition of the first kind (i.e. with a latent heat and a discontinuous change of properties) the transition will clearly be of the first kind also at a low temperature different from zero. A continuous transition is possible only at a high temperature, when the excitation, and conductivity of the dielectric are high. Since the excitation energy is of the order of the ionisation energy, i.e. of the order of a few eV, one eV at least, the line of thermodynamical equilibrium between the metallic and the dielectric phase can end with a critical point only at a very high temperature of the order of one eV, i.e. 10^4 degrees, and accordingly at an enormous pressure. At a high temperature both phases are non-crystalline (the melting of a metal does not deprive it of metallic properties). There arises the question as to the relation between the transition line from the metallic into a dielectric state and the transition line liquid-gas in the case of metals. It is quite clear that at a low pressure the substance with a small density (in the limit—an ideal gas) is a non-conductor. In the case of mercury the energy spectrum of the gas is discrete; in the case of paramagnetic sodium vapours we have to do with a continuous spectrum which however (just as in the case of liquid or solid oxygen) bears no direct relation to conductivity and metallic state.

Three cases are theoretically possible.† (1) the transition from the metallic into the dielectric state is always accompanied by a transition from the liquid into the gaseous state; there exists a single common curve, a single critical point, corresponding to very high temperature. This relation can be expected in the case of non-volatile metals.

In the case of metals with a low evaporation heat (mercury, for example), the critical point liquid-gas (*LG* point) must be expected to be much lower than the critical point of the transition metal-dielectric (*MD* point).

The cases 2 and 3 are here possible (Figs. 1 and 2).

(2) the heating of the liquid metal at a high pressure causes a discontinuous change of the density on the line gas-liquid *TLG* line; the phase with the

† We do not consider crystalline phases existing at low temperatures; the corresponding transitions, as noted above, have no connection with our topic.

smaller density remains, however, metallic ("metallic gas"). The transition into an ordinary gas takes place on the line TMD . This case is very improbable.

(3) the rise of temperature within a certain pressure range must be expected to be accompanied by the transition of the liquid metal into a liquid non-conducting phase (on the line TMD), which thereafter on the line TLG is transformed into a gas. The loss of metallic properties takes place as a phase transition metal-gas also at value of T and p much larger than those which correspond to the critical point liquid-gas. In the two latter cases a triple point T appears corresponding to the co-existence of two metallic and one dielectric phase in case 2 and one metal and two dielectric phases (liquid and gaseous) in the third case.

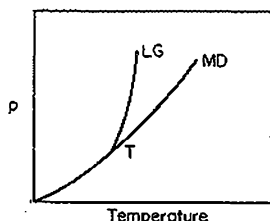


FIG 1.

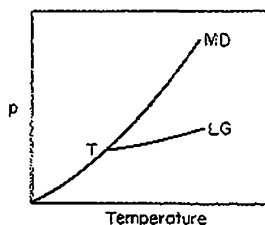


FIG. 2.

In the case of mercury the relatively small evaporation heat indicates that LG point is relatively low (1000–1500° K according to different estimates), whereas the MD point is probably inaccessible experimentally at the present time. There follows from our considerations that here our third case is to be expected. Our physical predictions thus are as follows (1) there exists a non-conducting liquid phase and (2) at a temperature and pressure lying above the critical values a phase transition with a discontinuous change of the electrical conductivity, volume and other properties must take place.

51. A NEW EXACT SOLUTION OF THE NAVIER-STOKES EQUATIONS

THE problem of fluid motion in a laminar jet discharged into a still fluid has been solved by Schlichting for large Reynolds numbers. In the present work the author shows that the motion in an axially symmetrical jet spreading in an unbounded space can be determined accurately for arbitrary Reynolds numbers by a rigorous solution of Navier-Stokes equations.

Let us consider a jet discharged, say, from a thin pipe into an unbounded space. We choose spherical co-ordinates r, θ, φ with polar axis directed along the velocity of the jet at its exit point, which we define to be located at the origin. The motion is axially symmetrical around the polar axis, so that $v_\varphi = 0$ and v_r, v_θ are functions of r and θ only. Any closed surface around the origin (including the particular case of a surface removed to infinity) should be traversed by the same full flow momentum ("momentum of the jet"). This requires that the velocity should vary inversely as the distance r from the origin, so that

$$v_r = \frac{1}{r} F(\theta), \quad v_\theta = \frac{1}{r} f(\theta), \quad (1)$$

where F, f are functions of θ only. The continuity equation runs thus

$$\frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cdot v_\theta) = 0.$$

By means of it we find that

$$F(\theta) = -\frac{d\theta}{df} - f \cot \theta. \quad (2)$$

We introduce the tensor Π_{ik} of the density of the momentum flow according to

$$\Pi_{ik} = p \delta_{ik} + \rho v_i v_k - \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right).$$

It permits the equation of motion of an incompressible viscous liquid (Navier-Stokes equations) to be written in the form

$$\sum_{k=1}^3 \frac{\partial \Pi_{ik}}{\partial x_k} = 0.$$

Л. Д. Ландау, Об одном точном новом решении уравнений навье-стокса, *Доклады Академии Наук СССР*, 43, 299 (1944).

L. Landau, A new exact solution of the Navier-Stokes equations, *C. R. Acad. Sci. URSS*, 43, 286 (1944).

In spherical co-ordinates the components $\Pi_{r\varphi}$, $\Pi_{\theta\varphi}$ of the tensor of momentum flow density in the jet vanish identically, as is plain enough from considerations of symmetry. Let us make the assumption that the components $\Pi_{\theta\theta}$ and $\Pi_{\varphi\varphi}$ are likewise zero (the justification for it lies in the fact that we thereby obtain a solution satisfying all necessary conditions). With the aid of the well-known expressions for the components $\partial v_i / \partial x_k + \partial v_k / \partial x_i$ in spherical co-ordinates and of the formulae (1-2) one easily realises that the following relation connects the components $\Pi_{\theta\theta}$, $\Pi_{\varphi\varphi}$ and $\Pi_{r\theta}$ of the tensor Π_{ik} in the jet:

$$\sin^2 \theta \cdot \Pi_{r\theta} = \frac{1}{r} \frac{\partial}{\partial \theta} [\sin^2 \theta (\Pi_{\varphi\varphi} - \Pi_{\theta\theta})].$$

We have therefore to infer from $\Pi_{\varphi\varphi}$ and $\Pi_{\theta\theta}$, being equal to zero, that $\Pi_{r\theta} = 0$ also. Thus, of all of the components of Π_{ik} only Π_{rr} , which depends on both r and $1/r^2$, is distinct from zero. This being so, it may easily be seen that the equations of motion $\sum_k \partial \Pi_{ik} / \partial x_k = 0$ are satisfied automatically.

We then write

$$\frac{1}{\varrho} (\Pi_{\theta\theta} - \Pi_{\varphi\varphi}) = \frac{1}{r^2} (f^2 + 2\nu f \cot \theta - 2\nu f') = 0,$$

$$\left(\nu = \frac{\eta}{\varrho} \right) \text{ or}$$

$$\frac{\partial}{\partial \theta} \left(\frac{1}{f} \right) + \cot \theta \frac{1}{f} + \frac{1}{2\nu} = 0.$$

The solution of this equation is

$$f = - \frac{2\nu \sin \theta}{A - \cos \theta} \quad (3)$$

(where A is constant), and from (2) we now get for F the expression

$$F = 2\nu \left\{ \frac{A^2 - 1}{(A - \cos \theta)^2} - 1 \right\}. \quad (4)$$

The distribution of pressure is determined from the equation

$$\frac{1}{\varrho} \Pi_{\theta\theta} = \frac{p}{\varrho} + \frac{f}{r^2} (f + 2\nu \cot \theta) = 0.$$

We obtain

$$p = - \frac{4\nu^2}{r^2} \frac{(A \cos \theta - 1)}{(A - \cos \theta)^3}. \quad (5)$$

The constant A can be connected with the "momentum of the jet" P , that is with the full flow of momentum in it. This flow is equal to the integral

over the spherical surface

$$P = \oint \Pi_{rr} \cos \theta \, df = 2\pi \int_0^\pi r^2 \Pi_{rr} \cos \theta \sin \theta \, d\theta.$$

The quantity Π_{rr} is equal to

$$\frac{1}{\rho} \Pi_{rr} = \frac{4\nu^2}{r^2} \left\{ \frac{(A^2 - 1)^2}{(A - \cos \theta)^4} - \frac{A}{A - \cos \theta} \right\},$$

and the computation of the integral gives

$$P = 16\pi \nu^2 \rho A \left\{ 1 + \frac{4}{3(A^2 - 1)} - \frac{A}{2} \ln \frac{A+1}{A-1} \right\}. \quad (6)$$

Formulae (1-6) solve the problem.

The streamlines are determined by the equation $dr/v_r = r \, d\theta/v_\theta$; integrating it, one obtains

$$\frac{r \sin^2 \theta}{A - \cos \theta} = \text{const.} \quad (7)$$

Let us consider two limiting cases: a weak jet (momentum P small) and a strong jet (P large). With $P \rightarrow 0$ the constant A tends to infinity; from (6) we have

$$P = \frac{16\pi \nu^2 \rho}{A}. \quad (8)$$

For the velocity we obtain in this case

$$v_\theta = -\frac{P}{8\pi \nu \rho} \frac{\sin \theta}{r}; \quad v_r = \frac{P}{4\pi \nu \rho} \frac{\cos \theta}{r}. \quad (9)$$

With $P \rightarrow \infty$ (strong jet) A tends to one; here (6) gives

$$A = 1 + \frac{\alpha^2}{2}, \quad \text{where} \quad \alpha = \frac{32\pi \nu^2 \rho}{3P}. \quad (10)$$

For large angles ($\theta \sim 1$) the velocity is determined by the formula

$$v_\theta = -2\nu \frac{\cot \theta/2}{r}; \quad v_r = -\frac{2\nu}{r}. \quad (11)$$

For small angles ($\theta \sim \alpha$) we have

$$v_\theta = -\frac{4\nu \theta}{\alpha^2 + \theta^2}; \quad v_r = 8\nu \frac{\alpha^2}{(\alpha^2 + \theta^2)^2}. \quad (12)$$

in accordance with the results obtained by Schlichting (the role of the Reynolds number is played in this problem by the quantity $[1/\nu]\sqrt{P/\rho}$).

It may be remarked in conclusion that the solution as given here is rigorous for a jet supposed to be discharged from a point source. But if the finite dimensions of the orifice should be taken into account, the solution obtained would present the first term of the expansion in powers of the ratio between the dimensions of the orifice and the distance r from it. With this circumstance is connected the fact that the total flow of the fluid passing through a closed surface around the origin will be found, if calculated by means of the obtained solution, to be equal to zero. A total flow different from zero is obtained if the next terms of the above-mentioned expansion are taken into account.

52. ON THE PROBLEM OF TURBULENCE

ALTHOUGH the turbulent motion has been extensively discussed in literature from different points of view, the very essence of this phenomenon is still lacking sufficient clearness. To the author's opinion, the problem may appear in a new light if the process of initiation of turbulence is examined thoroughly.

In the case of incompressible fluids the unsteadiness of the laminar motion is known to be determined as follows. Upon the principal motion with a velocity distribution $v_0(x, y, z)$ there is superimposed a small disturbance $v_1(x, y, z, t)$; the substitution of $v = v_0 + v_1$ in the equation of motion of a viscous fluid and the neglect of terms of the second order of smallness lead to a linear differential equation for the perturbation v_1 . Further, v_1 is sought in the form

$$v_1 = A(t) f(x, y, z), \quad (1)$$

where the time function $A(t)$ may be represented as

$$A(t) = \text{const} \cdot e^{-i\Omega t}. \quad (2)$$

The problem of determining the possible values of the "frequencies" Ω where the boundary conditions of motion are given, is an "Eigenwert" problem. By solving it one will obtain a spectrum of proper frequencies Ω (which are complex values in the general case). This spectrum, generally speaking, contains separate, isolated, values ("discrete spectrum") and also contains frequencies continuously filling whole intervals of values ("continuous spectrum"). It may be supposed that the frequencies of the continuous spectrum correspond to such motions v_1 as are not damped at infinity, while the frequencies of the discrete spectrum correspond to motions which are damped at infinity rather rapidly (as is the case in many other Eigenwert problems).

For the problem of steadiness of the principal motion those of the frequencies $\Omega = \omega + i\gamma$ (ω, γ are real) are relevant in which the imaginary part is negative ($\gamma < 0$). The presence of such proper frequencies in the spectrum indicates the unsteadiness of the principal motion with respect to infinitely small perturbations. Such values of Ω are only possible among the frequencies of the discrete spectrum. In fact, the principal motion presents at infinity a plane-parallel homogeneous flow (we mean a flow past a body of finite dimensions. In so far as a plane-parallel flow is in no case steady, it will be evident that any perturbation that fails to disappear at infinity must necessarily be damped in time, or, in other words, correspond to frequencies Ω with $\gamma > 0$). Accordingly, only the Ω frequencies of the discrete spectrum can be considered below.

Л. Ландау, К проблеме турбулентности, Доклады Академии Наук СССР, 44, 339 (1944).

L. Landau, On the problem of turbulence, C. R. Acad. Sci. URSS, 44, 311 (1944).

In the case of sufficiently small velocities the principal motion is a steady one (inasmuch as a resting fluid is in any case steady). On the other hand, with sufficiently large Reynolds numbers the laminar flow past a body is unsteady at any rate. In fact, with large Reynolds numbers the motion far away from the body is not appreciably different from a plane-parallel flow unless in the region of the narrow "track". Now it follows from Lord Rayleigh's work that no motion with a two-dimensional velocity distribution of such a type is steady, and one may expect that the same is true of the three-dimensional track.

If the values of the proper frequencies Ω are taken to be functions of the Reynolds number of the principal motion, then the critical value Re_{cr} is determined by the fact that for $\text{Re} = \text{Re}_{\text{cr}}$ the imaginary part of one of the frequencies Ω will vanish; suppose this frequency to be $\Omega_1 = \omega_1 + i\gamma_1$. For $\text{Re} > \text{Re}_{\text{cr}}$ we have $\gamma_1 > 0$; for such Reynolds numbers as are near to the critical value Re_{cr} , γ_1 is small in comparison with ω_1 . However, the expression (1-2) for the respective function $v_1(x, y, z, t)$ (with $\Omega = \Omega_1$) is only true for a very brief interval of time, as measured from the instant at which the stationary regime is broken. This is owing to the fact that the factor $e^{\gamma_1 t}$ grows rapidly with time. As a matter of fact, the modulus $|A|$ of the amplitude of non-stationary motion does not increase infinitely, but rather tends to a certain limit. With Re near to Re_{cr} (Re is always supposed to be greater than Re_{cr}), this limit is yet very small, too, and for determining it one may proceed as follows.

For very small times, when (2) is still applicable, we have

$$\frac{d|A|^2}{dt} = 2\gamma_1 |A|^2.$$

In substance this expression is but the first term of a series of powers of A and A^* . With the increase of the modulus $|A|$ the subsequent three terms of this series must be taken into account. The next terms are terms of the third order. We, however, are interested not in the exact value of the differential quotient $d|A|^2/dt$, but in its mean value with respect to time, the averaging being made over time intervals that are large in comparison with the period $2\pi/\omega_1$ of the periodic spectrum $e^{-i\omega_1 t}$ (as $\omega_1 \gg \gamma_1$, this period is small compared to the time $1/\gamma_1$ during which the modulus $|A|$ changes appreciably). But the terms of the third order involve a periodic spectrum, and so they are eliminated upon averaging. (Strictly speaking, they do not vanish altogether, but yield quantities of order four; these quantities are supposed to be included into the terms of the fourth order). The terms of the fourth order include a term proportional to $A^2 A^{*2} = |A|^4$; this term is not eliminated by averaging. Thus, up to terms of the fourth order we have

$$\overline{\frac{d|A|^2}{dt}} = 2\gamma_1 |A|^2 - \alpha |A|^4. \quad (3)$$

Here α is a positive constant (the case of negative α is considered below).

There are no signs of averaging over $|A|^2$ and $|A|^4$, because this operation is carried out over such time intervals as are small in comparison with $1/\gamma_1$. For the same reason in solving this equation we must disregard the bar over the derivative in the left hand member. The solution of the equation (3) has the form

$$\frac{1}{|A|^2} = \frac{\alpha}{2\gamma_1} + \text{const} \cdot e^{-2\gamma_1 t},$$

i.e. $|A|^2$ tends asymptotically to a limit

$$|A|_{\text{max}}^2 = 2\gamma_1/\alpha; \quad (4)$$

γ_1 is a function of Reynolds' number; it vanishes with $\text{Re} = \text{Re}_{\text{cr}}$. Therefore for small $\text{Re} - \text{Re}_{\text{cr}}$ we have $\gamma_1 = \text{const} \cdot (\text{Re} - \text{Re}_{\text{cr}})$. Substituting this in (4) we shall see that

$$|A|_{\text{max}} \sim \sqrt{\text{Re} - \text{Re}_{\text{cr}}}. \quad (5)$$

Thus, the unsteadiness of the laminar motion for $\text{Re} > \text{Re}_{\text{cr}}$ leads to the appearance of a non-stationary periodic motion. When Re is close to Re_{cr} , this motion can be represented as a superposition of a stationary motion $v_0(x, y, z)$ over a periodic motion $v_1(x, y, z, t)$, having a small but finite amplitude which varies with Re directly as $\sqrt{\text{Re} - \text{Re}_{\text{cr}}}$. The velocity distribution in this motion has therefore the form

$$v_1 = f(x, y, z) e^{-i(\omega_1 t + \beta_1)}, \quad (6)$$

where β_1 is a constant initial phase. When the differences $\text{Re} - \text{Re}_{\text{cr}}$ are large, there is no longer any sense in separating the velocities into two parts v_0 and v_1 . Here we have to deal simply with a periodic motion of frequency ω_1 . If instead of the time the phase $\varphi_1 \equiv \omega_1 t + \beta_1$ is used as the independent variable, the function $v(x, y, z, \varphi_1)$ may be said to be a periodic function of φ_1 with a period 2π , but no simple trigonometric function. It can be represented as a Fourier series

$$v = \sum A_p(x, y, z) e^{-i\varphi_1 p} \quad (7)$$

(the summation is carried out over all positive and negative integers p).

The essential fact is that only the absolute value of the factor, but not its phase are determined by the equation (3). The phase φ_1 remains in substance indefinite and depends upon the initial conditions which are a matter of change and may cause β_1 to take any value. It will be obvious that the periodic motion under consideration is not determined uniquely by the given stationary boundary conditions of motion; one quantity, the phase, remains arbitrary. This motion may be said to have one degree of freedom, whereas stationary motion is completely determined by the given boundary conditions, and enjoys not a single degree of freedom.

As Re is further increased, this periodic motion, too, eventually becomes unsteady. The investigation of its unsteadiness should be conducted in a

manner analogous to that described above. The role of the principal motion is now played by the periodic motion $v_0(x, y, z, t)$ of frequency ω_1 . Substituting $v = v_0 + v_2$ with small v_2 into the equation of motion, we shall again obtain for v_2 a linear equation, but this time the coefficients of this equation are not only functions of the co-ordinates, but of time also; with respect to time, they are periodic functions with a period $2\pi/\omega_1$. The solution of such an equation should be sought in the form $v_2 = \Pi(x, y, z, t)e^{-i\Omega t}$ where $\Pi(x, y, z, t)$ is a periodical function of time (with a period $2\pi/\omega_1$). Unsteadiness sets in again when the frequency $\Omega_2 = \omega_2 + i\gamma_2$ turns up whose imaginary part γ_2 is positive and the corresponding real part ω_2 determines then the newly appearing frequency.

The result is a quasi-periodic motion characterised by two different periods. It involves two arbitrary quantities (phases), i.e. has two degrees of freedom.

In the course of a further increase of the Reynolds number more and more new periods appear in succession, and the motion assumes an involved character typical of a developed turbulence. For every value of Re the motion has a definite number of degrees of freedom; in the limit as Re tends to infinity, the number of degrees of freedom becomes likewise infinitely large. With n degrees of freedom the velocity distribution is described by an expression of the type

$$v(x, y, z, t) = \sum_{p_1, p_2, \dots, p_n} A_{p_1, \dots, p_n}(x, y, z) e^{-i \sum_{i=1}^n p_i \varphi_i} \quad (8)$$

(summation over all integral numbers p_1, p_2, \dots, p_n) where the phases are $\varphi_i = \omega_i t + \beta_i$; it contains n arbitrary initial phases β_i . The frequencies ω_i being incommensurable, it will be apparent that during a sufficiently long interval of time the fluid will pass through the states which are as close as we will it to a state set beforehand by choosing freely a set of simultaneous values for the phases φ_i . It should, of course, be borne in mind that the states whose phases differ only by a multiple of 2π are identical physically. So a turbulent motion is to a certain extent a quasi-periodical motion.

The setting-up of a turbulent regime has a somewhat different character in those exceptional cases (the Poiseuille motion and others) where the laminar motion remains stable with respect to infinitesimal perturbations, no matter how large are the Reynolds numbers. If the latter are sufficiently small, no non-stationary motion is possible here at all; a steady non-stationary motion becomes possible only after a certain value of Re is reached, which is here in the nature of a critical value. With very large Reynolds numbers, the stationary motion may, notoriously, be materialised only if one is careful enough in eliminating the perturbations superimposed upon the motion. Contrary to this, if Re is close to Re_{cr} , the non-stationary motion is difficult to materialise. It may be thought therefore that the true value of Re_{cr} , say, in the case of Poiseuille motion, lies in any case below the value generally adopted at present. As for the properties of the turbulent motion that appears here with $Re > Re_{cr}$, it should, contrary to the preceding case, enjoy from the outset a large number of degrees of freedom.

Finally, in principle, there is one more possible type of the loss of steadiness by the laminar motion; this corresponds to the case where the coefficient before $|A|^4$ in (3) is positive, so that

$$\frac{d|A|^2}{dt} = 2\gamma |A|^2 + \alpha |A|^4$$

with positive α . If Re is somewhat smaller than Re_{cr} , the term of the second order is negative (since $\gamma_1 < 0$ for $Re < Re_{cr}$). But, the term of the fourth order being positive, the derivative $d|A|^2/dt$ will become positive when the amplitude of $|A|$ is sufficiently large. This means that the motion becomes steady with respect to sufficiently large perturbations even for $Re > Re_{cr}$. Thus, this type of unsteadiness is characterised by the fact that for a certain value, Re_{cr} , of the Reynolds number the motion becomes unsteady with respect to infinitesimal disturbances, but even with $Re > Re_{cr}$ there is unsteadiness in response to perturbations of a finite magnitude. In this case along with the above-mentioned critical Reynolds number there should exist another, "lower" number which determines the instant of appearance of stable non-stationary solutions of the equations of motion.

53. ON THE HYDRODYNAMICS OF HELIUM II

A method is developed for solving problems of the hydrodynamical motion of helium II by considering it as an incompressible liquid.

On the basis of a microscopic mechanism of superfluidity suggested by the author it was shown in the preceding paper¹ that a complete system of hydrodynamical equations can be established which describe the movement of helium II macroscopically.

In this hydrodynamics the motion of helium is described simultaneously by two velocities—that of the “superfluid” movement v_s and that of the “normal” movement v_n . Superfluid motion is always potential, i.e.

$$\text{curl } v_s = 0. \quad (1)$$

The density ρ of the liquid can be divided into a superfluid component ρ_s and a normal component ρ_n ; we therefore have for ρ and for the current of the liquid mass j

$$\rho = \rho_s + \rho_n, \quad j = \rho_s v_s + \rho_n v_n. \quad (2)$$

These quantities must satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div } j = 0. \quad (3)$$

The law of conservation of momentum is expressed by the equation

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \quad (4)$$

where Π_{ik} is the tensor of the momentum current. In the general case when the viscosity of the normal liquid is taken into account the tensor is equal to

$$\begin{aligned} \Pi_{ik} = & p \delta_{ik} + \rho_n v_i^{(n)} v_k^{(n)} + \rho_s v_i^{(s)} v_k^{(s)} \\ & - \eta \left(\frac{\partial v_i^{(n)}}{\partial x_k} + \frac{\partial v_k^{(n)}}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_i^{(n)}}{\partial x_i} \right) - \zeta \delta_{ik} \frac{\partial v_i^{(n)}}{\partial x_i} \end{aligned} \quad (5)$$

(summation is extended over indices which occur twice). η and ζ are two viscosity coefficients for normal helium II. Further, if viscosity is neglected, conservation of entropy will be fulfilled and is expressed by the equation

$$\frac{\partial (\rho S)}{\partial t} + \text{div} (\rho S v_n) = 0 \quad (6)$$

Л. Ландау, К гидродинамике гелия II, *Журнал Экспериментальной и Теоретической Физики* 14, 112 (1944).

L. Landau, On the hydrodynamics of helium II, *J. Phys. U.S.S.R.* 8, 1 (1944).

(S is the entropy referred to a unit mass of the liquid); $\rho S v_n$ is the "entropy current" due exclusively to the normal component of the liquid. If the viscosity is taken into account additional terms must be included in the right-hand side of (6); these terms express the increase of entropy resulting from the irreversibility of viscous friction. However, it appears that these terms are of a higher order of smallness and may therefore be neglected even when the viscosity terms are retained in equations (4) and (5). Finally, as was shown in ref. 1, the last equation in the complete system of hydrodynamic equations of motion takes the form

$$\frac{\partial v_s}{\partial t} = -\text{grad} \left\{ \Phi + \frac{v_s^2}{2} - \frac{\rho_n (v_n - v_s)^2}{2\rho} \right\}. \quad (7)$$

The boundary conditions for these equations consist in the following. First of all, obviously the component of the current of mass \dot{m} normal to the surface of any (stationary) solid body must vanish on that surface. Further, one must keep in mind that in reality the "normal part" of the liquid is an aggregate of thermal perturbations in the liquid—phonons and rotons (at sufficiently low temperatures we may speak of a "phonon" and "roton gas"). During movement along a rigid surface the phonons and rotons interact with the wall, and the normal part of the liquid adheres to it in the same way as any usual viscous fluid. In accordance with this the tangential component of the velocity v_n must vanish on the solid wall. As to the perpendicular component, it should be remembered that phonons and rotons can be absorbed by rigid bodies and this simply corresponds to heat transfer from the liquid to the body. The density of this heat flow is $\rho ST v_n$ as a unit of mass of the liquid transfers the quantity of heat ST with a velocity v_n (see ref. 1). The component of this flow normal to the body's surface does not necessarily vanish; the boundary conditions require only continuity on the surface. Moreover, the temperature itself must be continuous. Thus the boundary conditions on the surface of a rigid body may be written as follows (choosing the co-ordinate system with the x axis along the normal to the surface at the given point):

$$\begin{aligned} \rho_s v_{sx} + \rho_n v_{nx} &= 0, \quad v_{ny} = v_{nz} = 0, \\ \rho ST v_{ns} &= -\kappa \left(\frac{\partial T}{\partial x} \right)_{\text{sol}}, \quad T = T_{\text{sol}} \end{aligned} \quad (8)$$

where κ is the coefficient of heat conduction of the solid body.

As a matter of fact, however, down to the lowest temperatures, heat transfer in the solid body is extremely slow as compared to heat transfer in helium II, and effects due to heat transfer in the body turn out to be very weak. In this case κ may simply be considered equal to zero and the boundary conditions will then take the form

$$v_{sx} = 0, \quad v_n = 0, \quad (9)$$

or, in other words, the usual boundary conditions for ideal and viscous liquids are also true for v_s and v_n .

We shall now study the motion of helium II when the liquid may be considered to be incompressible (as is usually the case). Equations (3-7) may then be simplified. Considering ρ_s , ρ_n , S as constants we obtain from (3) and (6)

$$\operatorname{div} \mathbf{v}_s = 0, \quad \operatorname{div} \mathbf{v}_n = 0. \quad (10)$$

Equation (4) now gives

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_s (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \rho_n (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\nabla p + \eta \nabla^2 \mathbf{v}_n, \quad (11)$$

and equation (7) retains its original form.

As the superfluid motion is potential it will be possible to introduce a velocity potential φ_s

$$\mathbf{v}_s = \operatorname{grad} \varphi_s.$$

Inasmuch as $\operatorname{div} \mathbf{v}_s = 0$, φ_s must satisfy the usual Laplace equation

$$\nabla^2 \varphi_s = 0. \quad (12)$$

Inserting φ_s in equations (11) and writing $(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = \nabla \frac{\mathbf{v}_s^2}{2}$ we obtain

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n + \rho_s \operatorname{grad} \frac{\partial \varphi_s}{\partial t} + \rho_s \operatorname{grad} \frac{\mathbf{v}_s^2}{2} = -\nabla p + \eta \nabla^2 \mathbf{v}_n.$$

We shall introduce as two auxiliary quantities the pressures of the superfluid and normal currents p_s and p_n according to the equation

$$p = p_0 + p_s + p_n, \quad (13)$$

where p_0 is the pressure at infinity and p_s is defined by the usual formula for an ideal liquid

$$p_s = -\rho_s \frac{\partial \varphi_s}{\partial t} - \frac{\rho_s \mathbf{v}_s^2}{2}. \quad (14)$$

The equation of motion for \mathbf{v}_n then becomes

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\frac{1}{\rho_n} \nabla p_n + \frac{\eta}{\rho_n} \nabla^2 \mathbf{v}_n. \quad (15)$$

This equation is of the same form as the usual Navier-Stokes equation for a liquid of density ρ_n and viscosity η (and corresponding kinematic viscosity η/ρ_n).

Thus the problem of the motion of incompressible helium II is reduced to the solution of two problems of classical hydrodynamics—for an ideal liquid and for a viscous liquid. Namely, the velocity distribution for \mathbf{v}_s is given by Laplace's equation (12) and the boundary conditions for $\partial \varphi_s / \partial n$ as in the usual problem on the potential flow of an ideal liquid. Further, the distribution of the velocities \mathbf{v}_n is given by the solution of the Navier-Stokes equation (15) and by the boundary conditions for velocity \mathbf{v}_n as in the usual case of flow of a viscous liquid. The distribution of the pressure is then determined from (13).

Finally, equation (7) permits us to determine the distribution of the temperature. Writing $v_s = \nabla \varphi_s$ and integrating we obtain

$$\Phi + \frac{v_s^2}{2} - \frac{\varrho_n(v_n - v_s)^2}{2\varrho} + \frac{\partial \varphi_s}{\partial t} = \text{const.} \quad (16)$$

Variation of temperature and pressure in an incompressible liquid is small and the thermodynamic potential Φ may therefore be expanded in powers of $T - T_0$, $p - p_0$. With an accuracy to terms of the first order

$$\Phi - \Phi_0 = -S(T - T_0) + \frac{p - p_0}{\varrho}.$$

Substituting into (16) we get:

$$-S(T - T_0) + \frac{p - p_0}{\varrho} + \frac{v_s^2}{2} - \frac{\varrho_n(v_n - v_s)^2}{2\varrho} + \frac{\partial \varphi_s}{\partial t} = 0.$$

Introducing p_n and p_s we obtain finally

$$-T_0 = \frac{\varrho_n}{\varrho S} \left\{ \frac{p_n}{\varrho_n} - \frac{p_s}{\varrho_s} - \frac{(v_n - v_s)^2}{2} \right\}. \quad (17)$$

It should be emphasized that the quantity η/ϱ_n enters equation (15) as the kinematic viscosity. Keesom and MacWood² measured the viscosity η of helium II by employing an oscillating disk apparatus. In the calculations the true density ϱ of helium II was substituted in the formula defining the torque applied to the disk due to the force of friction.

From the preceding it will be clear that the correct value for the density to be inserted in this formula is ϱ_n . Using the values for ϱ_n obtained in ref. 1 and re-evaluating η with the aid of Keesom and MacWood's data we find that in the temperature range from 1.5°K up to the transition point, η is approximately constant and is close to 2×10^{-5} poise. A direct re-evaluation is not possible for lower temperatures owing to the fact that the "depth of penetration" $\delta \sim \sqrt{\eta/\varrho_n \omega}$ (ω —frequency of the oscillations) which is characteristic of the viscosity, due to the small magnitude of ϱ_n is of the same order of magnitude as the gap between the disk and the walls of the apparatus; this means that the formula derived for a disk rotating in an unlimited mass of liquid is not valid in this case.

REFERENCES

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2. W. KEESOM and G. MACWOOD, *Physica*, 5, 737 (1938).

54. ON THE THEORY OF SLOW COMBUSTION

PHYSICAL theories of slow combustion are usually based on the conception that the transfer of heat to the unburned gaseous mixture from the products of combustion, heated owing to the reaction, is a result of simple thermal conduction. The thickness of the "combustion layer" separating the region of the initial mixture from that of the products of combustion is then determined completely by thermal conduction in the gas and by purely chemical properties of the given reaction. It is essential that this thickness is independent of the characteristic dimensions of the problem (dimensions of bodies in contact with the gas as, e.g. the diameter of the tube if combustion occurs in the latter).

We shall assume the characteristic dimensions of the problem to be large in comparison with the thickness of the combustion layer. Then in determining the hydrodynamical movement of the gas accompanying the combustion process, we may consider the whole transition layer as a surface separating the burned and the unburned gas. In the considered "thermal conduction" regime of slow combustion the rate of propagation of the combustion is certainly small in comparison with the velocity of sound.

A necessary condition for the existence of the combustion regime just described is its stability. The analysis of this problem is the purpose of the computations presented below.

Let us choose a small portion of the combustion layer (considered as a discontinuity surface), which may be regarded as plane, and consider the motion of the gas near it. We shall select a co-ordinate system with respect to which this is at rest (the plane of the latter will be taken as the yz plane). The unperturbed movement of the gas is stationary and the velocity of the gas is directed normally to the discontinuity surface (the positive direction of the x axis is chosen so as to coincide with that of the motion of the gases). The unperturbed velocity of the gas before the discontinuity (unburned gas, $x < 0$) will be denoted by v_1 , that of the gas behind the discontinuity (burned gas, $x > 0$) by v_2 . Inasmuch as v_1 and v_2 are small compared with the velocity of sound, we may regard the gas as incompressible. Suppose now that a small perturbation v' , periodical with respect to time and to the co-ordinate y , is superimposed on the unperturbed motion with constant velocities v_1, v_2 . This perturbation may be determined from the equation of continuity:

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} = 0, \quad (1)$$

Л. Ландау, К теории медленного горения, *Журнал Экспериментальной и Теоретической Физики*, 14, 240 (1944).

L. Landau, On the theory of slow combustion, *Acta Phys.-chim. URSS*, 19, 77 (1944).

and Euler's equations

$$\frac{\partial v'_x}{\partial t} + v \frac{\partial v'_x}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial x}; \quad \frac{\partial v'_y}{\partial t} + v \frac{\partial v'_y}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial y}, \quad (2)$$

v and ρ designating v_1, ρ_1 , or v_2, ρ_2 according to the gas for which the equations are written. Differentiating the first equation with respect to x , the second with respect to y and adding we obtain

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = 0. \quad (3)$$

The boundary conditions to be satisfied on the discontinuity surface are the following. Firstly, the tangential velocity must be continuous. Let $\zeta(y, t)$ be a small displacement of the discontinuity surface along the x axis during the perturbation; $d\zeta/dy$ will be its angle of inclination to the y axis. The component of the gas velocity tangent to the discontinuity surface consist of two parts—the projection of the velocity v' on this surface, which, in first approximation, equals simply v'_y , and the projection of the unperturbed velocity $v_x = v$, which in the same approximation equals $v(d\zeta/dx)$. Thus the following condition must be fulfilled:

$$v'_{1y} + v_1 \frac{\partial \zeta}{\partial y} = v'_{2y} + v_2 \frac{\partial \zeta}{\partial y} \quad (4)$$

(for $x = 0$).

Further, incompressibility of the gas means, in particular, that those changes in pressure which arise during motion are so small that the variations in the rate of combustion due to them can be neglected. In other words, the velocity of propagation of the discontinuity should remain constant during the perturbation. In our system of co-ordinates this means that variations in the relative velocities of the gas and the discontinuity surface must be zero. We obtain, therefore, $v'_x - d\zeta/dt = 0$ and arrive at the condition (for $x = 0$)

$$v'_{1x} = v'_{2x} = \frac{\partial \zeta}{\partial t}. \quad (5)$$

Finally, in the same approximation the pressure on both sides of the discontinuity surface should be identical and this yields the last condition (for $x = 0$):

$$p'_1 = p'_2. \quad (6)$$

We shall look for a solution of equations (1–3) in which p' , v'_x , v'_y are proportional to the factor $e^{iky + \Omega t}$. From (3) we find that in the zone of gas 1 ($x < 0$)

$$p'_1 = \text{const } e^{iky + kx} e^{\Omega t}.$$

Writing v'_x and v'_y in the same form and choosing the coefficients so that equations (1-3) be satisfied, we obtain

$$\begin{aligned} v'_{1x} &= A e^{iky + kx + \Omega t}, & v'_{1y} &= iA e^{iky + kx + \Omega t}, \\ p'_1 &= -A \varrho_1 \left(\frac{\Omega}{k} + v_1 \right) e^{iky + kx + \Omega t}. \end{aligned} \quad (7)$$

In gas 2 ($x > 0$), besides the solution in the form $\text{const } e^{iky - kx + \Omega t}$, another particular solution of equations (1-3), in which the dependence on y and t is determined by the same factor $e^{iky + \Omega t}$, should also be considered. This solution can be obtained by putting $p' = 0$, then the right-hand sides of equations (2) disappear and the remaining homogeneous equations have solutions for which v'_x and v'_y are proportional to $e^{iky + \Omega t - \frac{\Omega}{v} x}$. The reason for this solution being considered only for gas 2 consists in the following. Our ultimate purpose is to determine the possibility of existence of such values of Ω for which the real parts are positive. The existence of such values for Ω would signify the instability of the basic movement. For such values of Ω the factor $e^{-(\Omega/v)x}$ would indefinitely increase with increasing $|x|$ if $x < 0$ and, therefore, this solution must be rejected in gas 1. Correspondingly choosing the constant coefficients we shall look for the solution for $x < 0$ in the form

$$\begin{aligned} v'_{2x} &= B e^{iky - kx + \Omega t} + C e^{iky + \Omega t - \frac{\Omega}{v_2} x}, \\ v'_{2y} &= -iB e^{iky - kx + \Omega t} - \frac{i\Omega}{kv_2} e^{iky + \Omega t - \frac{\Omega}{v_2} x}, \\ p'_2 &= -B \varrho_2 \left(v_2 - \frac{\Omega}{k} \right) e^{iky - kx + \Omega t}. \end{aligned} \quad (8)$$

Putting

$$\zeta = D e^{iky + \Omega t} \quad (9)$$

and substituting these expressions in (4-6) we obtain four homogeneous equations for the coefficients A, B, C, D . A simple calculation leads to the following condition for the consistency of these equations

$$\Omega^2(v_1 + v_2) + 2\Omega v_1 v_2 k + k^2 v_1 v_2 (v_1 - v_2) = 0 \quad (10)$$

(in computing it should be remembered that $\varrho_1 v_1 = \varrho_2 v_2$). If $v_1 > v_2$ this equation will have either two negative real roots or two complex conjugated roots with negative real parts. If $v_1 < v_2$ both roots will be real and have different signs. Thus for $v_1 > v_2$ we shall always have $\text{Re}\{\Omega\} < 0$ and the motion is stable. However, if $v_1 < v_2$ there will be such values of Ω for which $\text{Re}\{\Omega\} > 0$, i.e. the primary motion will unstable. Since $\varrho_1 v_1 = \varrho_2 v_2$ it then follows from $v_1 < v_2$ that $\varrho_1 < \varrho_2$. We thus arrive at the conclusion that the discontinuity surface under consideration is unstable if

$$\varrho_1 > \varrho_2. \quad (11)$$

Owing to considerable heating during combustion the density of the combustion products (ρ_2) is practically always less than that of the initial gas mixture (ρ_1), so that condition (11) is practically always fulfilled and the combustion regime considered will be unstable.

Inasmuch as the viscosity of the gas is neglected in the computations, the result just obtained will be valid in those cases for which the Reynolds numbers av_1/ν_1 , av_2/ν_2 are large compared to unity (a is characteristic for the dimensions of the system; $1/a$ determines the minimum value of the wave vector k ; ν_1 , ν_2 are the kinematic viscosities of the initial gas mixture and combustion products, respectively. Thus a stationary regime of slow combustion can exist only under such conditions of motion which essentially depend on the viscosity and, consequently, also on the heat conduction at the gases. Generally speaking, the "combustion layer" cannot in this case be considered thin. Our results are thus in conflict with the usual point of view.

Instability of the "discontinuity surface" should lead to turbulence and instead of a narrow transition layer there will be a diffuse combustion zone in which the movement of the gas will be turbulent. As usual, turbulence will cause an intense convective mixing of the gas. This intermixing causes a transfer of heat, the velocity of which considerably exceeds that due to true thermal conduction. The propagation of combustion, is therefore, determined not by thermal conduction in the gas, as was assumed above, but by turbulent convection processes; one may speak of a "convective" regime of combustion.

It would now be incorrect to assert that the velocity of propagation of combustion and the width of the combustion zone are independent of the characteristic dimensions of the problem since turbulent motion depends on these dimensions in an essential manner.

Let us examine the burning of a gas in a long tube. The burned gas is separated from the unburned gas by a turbulent "combustion zone", which gradually moves forward in the tube. Let l be the order of magnitude of the combustion zone thickness and u —the combustion propagation velocity. The dependence of l and u on the diameter of the tube d can easily be determined directly from dimensional considerations. Inasmuch as thermal conduction and viscosity are not important for turbulent motion (assuming the Reynolds numbers to be large) the coefficients of viscosity and heat conduction cannot enter l or u . Further, the purely chemical kinetics of the reaction are characterised by a certain quantity of the dimension of time, which determines the velocity of the reaction. The only parameter of the dimension of length is the diameter of the tube d . From considerations of dimensions it is thus evident that

$$l \sim d, \quad (12)$$

i.e. the width of the combustion zone should be proportional to the diameter of the tube. The following relation then exists for the combustion velocity

$$u = \frac{d}{\tau} \quad (13)$$

where τ is the time characteristic for the given reaction. The velocity of propagation of the combustion along the tube is thus proportional to the diameter of the latter. The following point should be noted. Since u is now dependent on the diameter of the tube, which is an arbitrary quantity, it cannot be asserted that the combustion velocity is always small as compared with the velocity of sound c . However it must be remembered that the relations (12) and (13) are valid only if $u \ll c$. Otherwise the gas could not be considered incompressible and there would be no reasons to assume that l and u cannot depend on the compressibility of the gas (as it was tacitly assumed above). Moreover, these results are true for tubes that are not very narrow, viz. the Reynolds numbers $Re \sim ul/\nu$ must be sufficiently large.

For the further investigation of the properties of slow combustion it will be more convenient to make use of a graphical representation with the aid of

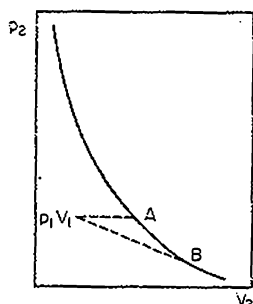


FIG. 1.

Hugoniot's adiabetic curve. The latter, drawn for a definite combustion process is shown in Fig. 1 (p_2 , V_2 are the pressure and specific volume of the combustion products; p_1 , V_1 —the corresponding quantities for the initial gas mixture). Only the part AB of the curve which actually correspond to slow combustion will be of interest to us (the straight line drawn from the point p_1 , V_1 to A is parallel to the axis of abscissae and that drawn to point B is tangent to the adiabetic curve). Well-known properties of Hugoniot's adiabetic curve easily yield the result that on AB the propagation velocities of the combustion layer (considered as a discontinuity surface) relative to the burned and unburned gases, v_1 and v_2 respectively, are less than the corresponding velocities of sound c_1 and c_2 ($v_1 < c_1$, $v_2 < c_2$).

The combustion velocity can be described by the "material flow" $j = \rho_1 v_1 = \rho_2 v_2$ through the discontinuity surface or, in other words, by the amount of matter burned in one second per unit area of the surface. In point A the flow j and the velocities v_1 , v_2 are zero; j increases along the direction from A to B . For a given reaction the material flow j depends also on the diameter of the tube (to be more definite, let us consider the combustion in a tube), increasing with increasing diameter, which corresponds to displacement along the adiabetic curve from A towards B .

It can be shown in these conditions no combustion regime will correspond to the points on the adiabatic curve lying below point *B*. The discontinuity surface may possess a peculiar type of instability as a result of which sound waves will be radiated spontaneously; obviously, no such discontinuity can exist in reality. The "number" of various sound waves which could be radiated from the discontinuity is mainly limited by purely geometrical circumstances, viz. by the relative magnitudes of velocities v_1 , v_2 , and the velocities of sound c_1 , c_2 ; this relation should be such that the discontinuity could not during its motion "overtake" the radiated wave. For $v_1 < c_1$, $v_2 < c_2$ (which corresponds to the section *AB* on the adiabatic curve) two waves could be radiated—one of them from the discontinuity towards gas 1 and the other in the direction of gas 2. For $v_1 < c_1$, $v_2 > c_2$ (corresponding to the part of the adiabatic curve lying below point *B*) three waves are possible: one moving from the discontinuity into gas 1 and two moving in gas 2, one of them towards and the other away from the discontinuity surface. Besides its purely geometrical possibility, another necessary condition for the radiation to exist in fact is that the corresponding boundary conditions on the discontinuity surface itself may be satisfied. Computations show that two waves are not sufficient for this purpose; three waves are, however, enough to satisfy these boundary conditions. Thus the discontinuities corresponding to points lying below point *B* are unstable and therefore this part of the adiabatic curve does not in general correspond to any real discontinuities whatsoever.

Point *B* corresponds to a certain "limiting" regime of slow combustion. As to the combustion of gases in tubes with diameters larger than that corresponding to point *B*, the only possible stable regime will be detonation.

The usual equation of the Hugoniot adiabatic curve and the fact that the straight line drawn from p_1 , v_1 to point *B* is tangent to the curve enable one to obtain easily an expression for the limiting velocity of propagation of "slow combustion". Calculations which will not be given here yield the following result

$$v_1 = \sqrt{\frac{\gamma_2 - 1}{2} [(\gamma_2 + 1)q + (\gamma_1 + \gamma_2)c_{v1}T_1]} - \sqrt{\frac{\gamma_2 + 1}{2} [(\gamma_2 - 1)q + (\gamma_2 - \gamma_1)c_{v1}T_1]}, \quad (14)$$

where q is the heat of reaction per unit mass; c_v —the heat capacity at constant volume, $\gamma = c_p/c_v$, T —the temperature. If the heat of reaction is great ($q \gg c_{v1}T_1$), as is usually the case, this formula can be simplified to

$$v_1 = \frac{\gamma_2(\gamma_1 - 1)c_{v1}T_1}{\sqrt{2(\gamma_2^2 - 1)q}}. \quad (15)$$

The pressure and temperature of the gaseous product of combustion will be

$$p_2 = p_1 \frac{1}{\gamma_2 + 1}, \quad T_2 = \frac{2q}{\gamma_2(\gamma_2 + 1)c_{v2}}. \quad (16)$$

Let us now discuss the combustion of liquids. Only reactions of spontaneous decomposition will be considered, i.e. reactions which take place without the interference of a foreign substance as, for instance, of oxygen from air. Combustion takes place in the layer of saturated vapour over the surface of the liquid. If the whole combustion zone be considered as a discontinuity surface, separating the initial liquid (density ϱ_1) from the gaseous products of combustion (density ϱ_2), a surface tension coefficient α must be ascribed to this surface, which should equal the coefficient of surface tension at the boundary between the liquid and its saturated vapour. When investigating the stability of such a combustion regime the gravitation should also be considered.

All the computations for this case are exactly similar to those carried out for combustion in gaseous mixtures, except of condition (6) (equality of pressures) which must be replaced by the condition

$$p'_2 - p'_1 = -g(\varrho_2 - \varrho_1)\zeta - \alpha \frac{\partial^2 \zeta}{\partial y^2}. \quad (17)$$

Substituting expressions (7-9) into conditions (4), (5) and (17) we again obtain four equations for the coefficients A, B, C, D . Instead of (10) the condition of consistency of these equations will be given by the following quadratic equation:

$$\Omega^2(v_1 + v_2) + 2\Omega k v_1 v_2 + k^2 v_1 v_2(v_1 - v_2) - k g(v_1 - v_2) - \alpha k^3 \frac{v_1 v_2}{j} = 0.$$

For stability of the motion it is necessary that $\text{Re}(\Omega) < 0$, i.e. the equation must have either two negative real roots or two complex-conjugated roots with negative real parts. This will be the case if the free term in equation (16) is positive:

$$\alpha k^2 \frac{v_1 v_2}{j} - k v_1 v_2(v_2 - v_1) + g(v_2 - v_1) > 0. \quad (18)$$

On the other hand, this inequality will be true for any positive k if

$$v_1 v_2(v_1 - v_2)^2 - 4\alpha \frac{v_1 v_2}{j} g(v_1 - v_2) < 0.$$

Inasmuch as $v_2 > v_1$ (which follows from $j \equiv \varrho_1 v_1 = \varrho_2 v_2$ since $\varrho_1 > \varrho_2$, i.e. the density of the liquid is greater than that of the vapour) the inequality above will be equivalent to the following one

$$\frac{4\alpha g}{j} - v_1 v_2(v_2 - v_1) > 0.$$

Finally, expressing v_1 and v_2 in terms of the material flow j (amount of matter burned per unit time) we obtain the following inequality for the condition of stability of the combustion regime in the liquid

$$j^4 > \frac{4\alpha g \varrho_1^3 \varrho_2^2}{\varrho_1 - \varrho_2} \cong 4\alpha g \varrho_1 \varrho_2^2. \quad (19)$$

In this form this inequality is referred to an infinite liquid surface for which vibrations of any wave length are possible. For liquids contained in small vessels (19) is not a necessary condition for the stability of slow combustion. A sufficient condition is that for all k greater than a certain value of the order of $1/a$, where a is the dimension of the vessel, expression (18) could not be negative. Accordingly, if condition (19) is not fulfilled a necessary condition for stability will be the inequality $a < a_0$ where

$$a_0 \sim \frac{\alpha}{j(v_2 - v_1)} \sim \frac{\alpha}{j v_2} = \frac{\alpha \varrho_2}{j^2}. \quad (20)$$

55. ON THE THEORY OF SCATTERING OF PROTONS BY PROTONS

A formula is derived for the scattering of protons by protons without any assumption on the actual shape of the potential energy curve between two protons. From an analysis of the experimental data it is shown that the system proton-proton possesses no stable level. It is shown further, that the conclusion as to the approximate equality of the forces between a proton and a proton, on the one hand, and between a proton and a neutron, on the other, can be drawn without calculations with a rectangular well.

1. INTRODUCTION

The investigation of the scattering of elementary particles—protons and neutrons—plays an important role in the study of nuclear forces. A large number of theoretical papers have been devoted to this question. They are mostly based on certain assumptions as to the function $U(r)$ representing the potential energy between two particles and depending usually on two parameters (a rectangular well or error function). Such assumptions are, however, wholly unfounded, and it remains quite unclear, what actually follows from the experimental data and is not depending on the arbitrary assumption.

Bethe and Peierls^{1, 2} have drawn attention to the fact that in the case of the scattering of protons by neutrons use can be made of the circumstance that the energies of the scattered particles are small compared with the interaction energies, while the corresponding wavelengths are much larger than the range of the nuclear forces. The exact knowledge of the shape of the function $U(r)$ is therefore immaterial and the influence of the nuclear forces can be accounted for in this case by a variation of the boundary conditions which must be satisfied by the wave function at the origin of the co-ordinates. The effective cross section obtained in this way contains a certain constant which must be determined from experiments.

The scattering of protons by protons has been investigated much more carefully than the scattering of protons by neutrons. The existing theory of this scattering is, however, unsatisfactory.

A number of papers by Breit and his co-workers[†] are devoted to a theoretical investigation of this problem. In all these papers, however, the calculations are based on certain assumptions as to the character of the function $U(r)$.

И. Павдай и Я. Смородинский, Рассеяние протонов протонами, *Журнал Экспериментальной и Теоретической Физики*, 14, 269 (1944).

L. Landau and J. Smorodinsky, On the theory of scattering of protons by protons, *J. Phys. U.S.S.R.* 8, 154 (1944).

† A review of the theoretical and experimental work on the scattering of protons by protons is given in a paper by Breit³ where a bibliography on the question is given.

The fundamental conclusion about the approximate identity of the specific forces acting between a proton and a neutron, on the one hand, and a proton and a proton, on the other, is based on the application of rectangular wells.

It can be shown that in reality the main results of these investigations can be obtained, without these unnecessary assumptions, on the basis of ideas similar to those developed in the above mentioned paper by Bethe and Peierls. This is quite sufficient for clearing up a number of questions, in particular the comparison of the proton-proton and proton-neutron forces.

Similar investigations could be adopted for the scattering of deuterons and α -particles. But so far the results of such investigations have been rather fruitless owing to the lack of experimental data and also to the participation in the scattering of higher momenta.

2. METHOD

The wave function of a proton in a Coulomb field satisfies the following equation (in the system of the centre of inertia)

$$\left(-\frac{\hbar^2}{M}\nabla^2 + \frac{Ze^2}{r}\right)\psi = \frac{E}{2}\psi \quad (2.1)$$

(Z —nuclear charge; in the case of the scattering by proton $Z = 1$; M —mass of a proton E —energy of the scattered protons in the laboratory's coordinate system).

If there are no other forces, beside Coulomb ones, equation (2.1) holds throughout the whole space, and the function must satisfy the boundary conditions

$$\psi(0) \neq \infty. \quad (2.2)$$

If, however near the origin of the co-ordinate system there exists some field of forces different from the Coulomb ones, the wave function in this region proves to be different from (2.1). Since, however, the potential of these forces is unknown the equation for the correct function cannot be written at all.

If the energy of the scattered particle is much smaller than the interaction energy, the solution of this wave equation can be assumed to depend on the energy of the protons to a small extent only.

Instead of solving the equation in the region near the origin of the co-ordinates we can assume that the new wave function also satisfies (2.1), the presence of nuclear forces being revealed by the fact that the wave function must satisfy a certain condition different from (2.2). The boundary condition at the origin of the co-ordinates will be written down in the form of Bethe and Peierls

$$\left(\frac{f'}{f}\right)_{r=0} = c, \quad (2.3)$$

where $f = r\psi$.

Let us now consider the problem of the scattering of protons by protons for the special case $l = 0$ (S -scattering). The case $l > 0$ will be discussed below (section 5).

Introducing instead of ψ the function $f = r\psi$ we shall re-write equation (2.1) in the form.

$$\frac{\hbar^2}{M} \frac{d^2 f}{dr^2} + \left(\frac{E}{2} - \frac{Z e^2}{r} \right) f = 0. \quad (2.4)$$

Let us introduce the notations:

$$\left. \begin{aligned} \varrho &= kr, \\ k &= \frac{M v}{2\hbar}, \\ \eta &= \frac{1}{k a}, \\ a &= \frac{\hbar^2}{Z M e^2} = \frac{5.87 \times 10^{-12}}{Z} \text{ cm.} \end{aligned} \right\} \quad (2.5)$$

In these notations (2.1) assumes the form

$$\frac{d^2 f}{d\varrho^2} + \left(1 - \frac{\eta}{\varrho} \right) f = 0. \quad (2.6)$$

This equation is satisfied by Whittaker's functions⁴ $W_{i\eta\frac{1}{2}}(2i\varrho)$ and $W_{-i\eta\frac{1}{2}}(-2i\varrho)$.

With the help of the linear combination of these functions two solutions can be obtained, which at large distances have the form⁵

$$F \sim \sin(\varrho - \eta \ln 2\varrho + \sigma_0), \quad (2.7)$$

$$G \sim \cos(\varrho - \eta \ln 2\varrho + \sigma_0), \quad (2.8)$$

where

$$\sigma_0 = \arg \Gamma(1 + i\eta). \quad (2.9)$$

At small distances these functions can be expanded into a series

$$F = C e^{-i\varrho} \varrho \left\{ 1 + \frac{1-i\eta}{1!2!} 2i\varrho + \frac{(1-i\eta)(2-i\eta)}{2!3!} (2i\varrho)^2 + \dots \right\} \quad (2.10)$$

$$\begin{aligned} G &= \frac{1}{C} \cos \varrho + \frac{2\eta}{C} \left[-1 + \ln 2\varrho + 2\gamma + \operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} \right] \frac{F}{G} \\ &- \frac{2\eta}{C} \operatorname{Re} e^{-i\varrho} \varrho \sum_{s=1}^{\infty} \frac{\Gamma(1+s-i\eta) (2i\varrho)^s}{(1+s)! s! \Gamma(1-i\eta)} \sum_{t=1}^s \left(\frac{1}{t} + \frac{1}{t+1} - \frac{1}{t-i\eta} \right). \end{aligned} \quad (2.11)$$

Here

$$C^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}, \quad \gamma = 0.577 \dots$$

It should be noted that the complex nature of the expression for F is only apparent. If the exponential function is replaced by its expansion and the two series are multiplied, the product is found to be real.

As can be seen from (2.6) the function F vanishes at the origin of the co-ordinates, while the function G , owing to the presence of the term $\ln \varrho$, has a singularity at this point.

In the case of the scattering in a purely Coulomb field the boundary conditions are satisfied by the function F . In our case, however, after the boundary conditions have been altered the behaviour of the function at the origin becomes immaterial and the conditions of the problem are satisfied by a linear combination of the two solutions. Denoting the constant appearing in this new function by $\cot K_0$ we get for the solution

$$f = F \cot K_0 + G. \quad (2.12)$$

This choice has been made in order to reduce the difference of the asymptotic expression of the wave function from the wave function in the absence of nuclear forces to a constant phase K_0 only, as is usually done in the theory of scattering

$$f \sim \sin(\varrho - \eta \ln 2\varrho + \sigma_0 + K_0). \quad (2.13)$$

In equations (2.12) and (2.13) the constant factors (different in both cases), which are determined by the normalisation of the wave function and are immaterial for our purpose, are dropped.

In order to find the connection between K_0 and the constant c let us substitute into the boundary condition (2.4) the expression for the function (2.12). This condition will be referred not to the origin but to a point lying arbitrarily near to zero; but nevertheless at a given finite distance from it (this must be done because the function is singular in the origin[†]).

Limiting ourselves to the first term of the expansion we get:

$$\frac{f'}{f} = k \left[C^2 \cot K_0 + 2\eta \left(2\gamma + \ln 2\varrho + \operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} \right) \right] = c. \quad (2.14)$$

It should be remembered that the derivative on the left-hand side is taken with respect to r and not with respect to the variable $\varrho = kr$. Since the product $k\eta = 1/a$ is independent of the energy, the expression in the brackets contains a number of constant terms. By altering the notations they can be included into the constant c . The same can be done with the term $\ln(2r/a)$ [‡]. The constant obtained in this way will be denoted by α_{pp} . Thereafter in all the remaining terms we can put $r = 0$.

After simple transformations we then arrive at the final formula, connecting the phase K_0 with the constant α_{pp} :

$$\cot K_0 = \frac{1}{\pi} \left(e^{2\pi\eta} - 1 \right) \left[\ln \eta - \operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} + \frac{a}{2} \alpha_{pp} \right]. \quad (2.15)$$

[†] In the case of scattering of protons by neutrons the corresponding solutions were $\sin kr$ and $\cos kr$. They are both regular in the origin, so that in this case the above operation is superfluous and the boundary condition can be referred directly to the origin.

[‡] Which is obtained from $\ln 2\varrho = -\ln \eta + \ln(2r/a)$.

3. ANALYSIS OF THE EXPERIMENTAL DATA

The scattering of protons were most carefully investigated by Herb, Kerst, Parkinson and Plain⁷ for various energies of the protons in the range between 860 and 2392 keV, and also by Heydenburg, Hafstad and Tuve⁸ for protons with energies 670, 776 and 867 keV.

From these data Breit, Thaxton and Eisenbud⁹ have calculated the corresponding phases.

Ragan, Kanne and Tashek¹⁰ have also studied protons with energies of 250 and 300 keV. The calculation of the phases for this case has been carried out by us.

As the result of this analysis Breit, Thaxton and Eisenbud have found that in the energies investigated only the wave with $l = 0$ † participates (we mean, of course, only the additional scattering due to the presence of the nuclear forces, the scattering in the Coulomb field contains waves with all the values of the angular momentum). As regards the waves with higher values of the angular momentum, the corresponding phases are so small that they can with a great accuracy be neglected. When all the phases except K_0 are neglected the angular distribution of the protons in the laboratory system of co-ordinates is described by the formula¹²:

$$\sigma = 4 \cos \frac{\theta}{2} \cdot P, \quad (3.1)$$

where

$$P = \left(\frac{e^2}{M v^2} \right)^2 \left\{ \sin^{-4} \frac{\theta}{2} + \cos^{-4} \frac{\theta}{2} - \cos \left(2\eta \ln \tan \frac{\theta}{2} \right) \sin^{-2} \frac{\theta}{2} \right. \\ \left. - \frac{2}{\eta} \left[\cos \left(K_0 + 2\eta \ln \sin \frac{\theta}{2} \right) \sin^{-2} \frac{\theta}{2} + \cos \left(K_0 + 2\eta \ln \cos \frac{\theta}{2} \right) \cos^{-2} \frac{\theta}{2} \right] \sin K_0 \right. \\ \left. + \frac{4}{\eta^2} \sin K_0 \right\}. \quad (3.2)$$

If vice versa the angular distribution is known, this formula can be used for the calculation of K_0 as has actually been done‡.

The calculated values are given in Table 1.

TABLE 1. Values of the phase K_0

E , keV	249.5	298	670	776	860	867	1200	1390	1830	2105	2392
K_0 (exp)	9°	10°	24°8'	27°4'	29°3'	29°4'	35°9'	38°8'	44°	46°1'	48°1'
$\cot K_0$	6.0	5.0	2.16	1.93	1.78	1.78	1.38	1.25	1.03	0.96	0.90
$\cot K_0$ (theor)	6.32	5.15	2.24	1.92	1.78	1.78	1.38	1.25	1.03	0.95	0.90

† See also the paper by Creutz¹¹.

‡ The calculations are simplified if tables from refs. 9 and 12 are used. Additional values for these tables for large energies are given in ref. 13.

TABLE 2. Value of the constant $(a/2)\alpha_{pp}$

E , keV	249.5	298	670	776	860	867	1200	1390	1830	2105	2392
$\frac{a}{2}\alpha_{pp}$	3.8	3.8	4.00	4.16	4.19	4.19	4.34	4.45	4.61	4.73	4.85

Table 1 also contains the values of the function $\cot K_0$ appearing in equation (2.12). The data in the last line will be discussed later.

The phases determined in this way can be used for the calculation of the constant α . We shall perform this calculation for each value of the energy separately. The result of the calculations is given in Table 2.

We see that this quantity is in reality not exactly constant but slowly increases with the energy.

Such a variation of α_{pp} is not unexpected. In fact the value of α_{pp} corresponding to $E = 0$, is equal to $1.33 \times 10^{12} \text{ cm}^{-1}$ whereas an estimate of the approximate value of α_{pp} from the magnitude of the nuclear forces gives $\alpha_{pp} \sim \sqrt{MU}/\hbar$ where U is the energy of nuclear interaction of the order of 15 MeV, i.e. $\alpha_{pp} \sim 6 \times 10^{12} \text{ cm}^{-1}$. Hence it is seen that the values of α_{pp} are abnormally small compared with the expected values, which, just as in the case of the interaction proton-neutron, corresponds to a sharp resonance.

On the other hand, we have treated α_{pp} as a constant, assuming that in a series of powers of the ratio of the energy of the incident particle to the energy of the nuclear interaction one can limit oneself to the first term only.

In reality this cannot be allowed, since in the neighbourhood of the resonance the first term is small and the second term is therefore comparable with the first one.

In fact, all the values obtained for the constant $(a/2)\alpha_{pp}$ fit fairly well on a straight line

$$\frac{a}{2}\alpha_{pp} = 3.82 + 0.43 E \text{ (MeV)}, \quad (3.3)$$

where E denotes the energy of the proton in the laboratory system of co-ordinates expressed in MeV.

For the sequel it will be convenient to have also the value of this constant in cm^{-1} :

$$\alpha_{pp} = [1.33 + 0.15 E \text{ (MeV)}] \times 10^{12} \text{ cm}^{-1}. \quad (3.4)$$

For checking the calculations it is possible, with the help of the values of α_{pp} obtained from (3.3), to calculate the corresponding phases. The result of these calculations is given in Table 1 in the last line.

It is quite clear from the aforesaid that from all the experimental data existing at the present time only two constants can be determined, two coefficients in the expansion (3.3) and (3.4). If a certain form for the potential depending exactly on two parameters is chosen *a priori* these parameters can, of course, be determined.

In reality, however, if such assumptions are not introduced it is impossible to draw any quantitative conclusions about the function $U(r)$ from the existing experimental data.

Greater accuracy can only be obtained by the study of the scattering of faster protons. It should be taken into account that with increase of the energy the phase K_1 (and with a further increase of the energy K_2 , etc.) becomes important, which will lead to the appearance of additional constants. Such investigations must, accordingly, be very precise.

In the second part of this work it will be shown that a similar behaviour of the corresponding constants α_{pn} takes place in the case of the scattering of protons by neutrons. In the case of a scattering with antiparallel spins this constant proves to be

$$\alpha_{pn} = [0.45 + 0.10 E \text{ (MeV)}] \times 10^{12} \text{ cm}^{-1}. \quad (3.5)$$

From a comparison of (3.4) and (3.5) it can be seen that the second terms of these expressions are close to each other. This shows that the nuclear forces between a proton and a neutron, on the one hand, and between two protons, on the other, are approximately of the same magnitude. This is not contradicted by the fact that the first terms of (3.4) and (3.5) differ from each other. We have seen that these terms are abnormally small, therefore it would be of some value to compare, not their absolute values, but the differences between them and their normal values, which have been shown to be of the order $6 \times 10^{-12} \text{ cm}^{-1}$. A detailed discussion of these equations will be given in the next paper.

4. ON THE EXISTENCE OF A STABLE LEVEL OF DI-PROTON

On the basis of the data obtained the question of the existence of a stable level in the system consisting of two protons is clarified. This problem must be formulated in the following way.

The existence of such a level means that there exists a state with a wave function corresponding to a certain negative value of the energy which decays exponentially at an infinite distance. The corresponding energy is then the binding energy of this state. It is possible that such a solution does not exist. It does not mean, however, that in this case resonance phenomena cannot take place. Resonance can be present in the absence of a real level as, for example, in the case of the scattering of protons by neutrons with anti-parallel spins. The resonance corresponds here to the fact that already small variations of the parameters of the system can lead to the appearance of a real level (to the change of the sign of the constant α_{pn} and to a somewhat more complicated condition for the constant α_{pn}).

The solution of the wave equation, corresponding to a negative value of the energy $E = -\varepsilon$, can be obtained from the solution written down by us for a positive energy (2.10) and (2.11), if one puts $k = i\varepsilon$ (in order to have $k^2 < 0$). The asymptotic expansions for the functions obtained in this way

will be:

$$F \sim \sin(i\kappa r + \dots), \quad (4.1)$$

$$G \sim \cos(i\kappa r + \dots). \quad (4.2)$$

In order to obtain a function which behaves at infinity as an exponential one, it is obvious to use the linear combination

$$iF + G \sim \exp(-\kappa r + \dots). \quad (4.3)$$

For the sake of simplification of the calculations attention must be drawn to the fact that (4.3) is obtained from the corresponding function, used by us in the problem of the scattering, $\cot K_0 \cdot F + G$ by a formal replacement

$$\cot K_0 \rightarrow i, \quad k \rightarrow i\kappa. \quad (4.4)$$

Further we must satisfy the same boundary conditions as in the problem of the scattering, since they were derived by us under the assumption that it depends but slightly on the energy.

Since all these operations coincide with those carried out by us in the problem of the scattering, the final result can be obtained from (2.15) by the replacement of (4.4). We thus get the equation for the determination of κ †:

$$\zeta(x) = \frac{i\pi}{e^{-2\pi i x} - 1} - \ln x + i\frac{\pi}{2} + \frac{1}{2} \left[\frac{\Gamma'(-x)}{\Gamma(-x)} + \frac{\Gamma'(x)}{\Gamma(x)} \right] = \frac{a}{2} \alpha_{pp}. \quad (4.5)$$

For the sake of brevity the notation

$$x = \frac{1}{\kappa a} \quad (4.6)$$

is here introduced. The question of the character of the level is obviously solved by the sign of the root of equation (4.5). If it proves to be positive, then, as can be seen from (4.3), the solution at infinity must decay and, according to the aforesaid, this value of κ would correspond to a real level. In reality, however, this equation cannot have positive roots, which proves the absence of a real level in the case of a diproton.

Let us now prove this assertion.

It can easily be shown that

$$\frac{i\pi}{e^{-2\pi i x} - 1} = -\frac{i\pi}{2} - \frac{\pi}{2} \cot \pi x = -\frac{i\pi}{2} - \frac{1}{2} \frac{d}{dx} \ln(x \sin \pi x) + \frac{1}{2x}. \quad (4.7)$$

Let us use further the well-known relations

$$\Gamma(x) \Gamma(-x) = -\frac{\pi}{x \sin \pi x}. \quad (4.8)$$

† Let us note that the expression $\operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)}$ replace in the old formulae the sum $\frac{1}{2} \left[\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} + \frac{\Gamma'(i\eta)}{\Gamma(i\eta)} \right]$ which is equal to the preceding one for real η only. For imaginary the η latter expression must be used.

Then

$$-\frac{1}{2} \frac{d}{dx} \ln(x \sin \pi x) = \frac{1}{2} \left[\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(-x)}{\Gamma(-x)} \right]. \quad (4.9)$$

Substituting in (4.5) we get finally:

$$\zeta(x) = -\ln x + \frac{1}{2x} + \psi(x) = \frac{\alpha}{2} \alpha_{pp} \quad (4.10)$$

where $\psi(x)$ is the logarithmic derivative of the Γ -function.

It can easily be seen that the left part of (4.10) for all positive values of x is negative.

In fact, for large values of x we get:

$$\psi(x) \sim \ln x - \frac{1}{2x}, \quad (4.11)$$

hence

$$\zeta(\infty) = 0. \quad (4.12)$$

For small x , $\psi(x)$ behaves as $-1/x$ which can be seen from

$$\psi(x) = \psi(x+1) - \frac{1}{x}. \quad (4.13)$$

Therefore

$$\zeta(0) = -\infty. \quad (4.14)$$

It can be shown (for instance, graphically) that $\zeta(x)$ is a monotonic function. Therefore

$$\zeta(x) < 0 \text{ for } x > 0. \quad (4.15)$$

Noting that the right-hand side (4.10) contains a positive quantity we see that this equation cannot hold for any positive x , as was to be proved†.

The inequality (4.15) can be proved more strictly in the following way.

In the theory of Γ -functions (ref. 4 (12,32)) the relation is derived

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \ln x - \frac{1}{2x} - 2 \int_0^{\infty} \frac{t dt}{(t^2 + x^2)(e^{2\pi t} - 1)}. \quad (4.16)$$

Substituting this expression in (4.10) we get

$$\zeta(x) = -2 \int_0^{\infty} \frac{t dt}{(t^2 + x^2)(e^{2\pi t} - 1)}. \quad (4.17)$$

Hence the inequality (4.15) is obtained.

† It can easily be seen that, owing to the fact that α_{pp} depends but slightly on the energy, our conclusion is not altered if for α_{pp} the expression (3.3) with $E/2 = -\varepsilon$ is taken.

5. CASE $l > 0$

All the aforesaid referred to the problem of the scattering of protons by protons with a momentum $l = 0$ (S -scattering).

In the case of the scattering of heavier particles, states with angular momenta different from zero may also come into play. We shall therefore derive a formula similar to (3.12), which holds for any l †.

In the case $l \neq 0$ the term $l(l+1)/r^2$ is added to the wave equation for the function $f_l = \psi_l r$. In the notation of section 3 this equation has the form:

$$\frac{d^2 f_l}{dr^2} + \left(1 - \frac{\eta}{r} - \frac{l(l+1)}{r^2}\right) f_l = 0. \quad (5.1)$$

This equation is satisfied by the functions

$$W_{i\eta, l+\frac{1}{2}}(2i\varrho) \quad \text{and} \quad W_{-i\eta, l+\frac{1}{2}}(-2i\varrho).$$

As before we can take linear combinations, having at infinity the following form:

$$F_l \sim \sin\left(\varrho - \frac{l\pi}{2} - \eta \ln \varrho + \sigma_l\right), \quad (5.2)$$

$$G_l \sim \cos\left(\varrho - \frac{l\pi}{2} - \eta \ln \varrho + \sigma_l\right), \quad (5.3)$$

where

$$\sigma_l = \arg \Gamma(l+1-i\eta). \quad (5.4)$$

At small distances these functions can be expanded into a series

$$F_l = C_l e^{-i\varrho} \varrho^{l+1} \left\{ 1 + \frac{l+1-i\eta}{1!(2l+2)!} (2i\varrho) + \frac{(l+1-i\eta)(l+2-i\eta)}{(2l+1)(2l+3)} \frac{(2i\varrho)^2}{2!} + \dots \right\}, \quad (5.5)$$

$$\begin{aligned} G_l &= \frac{1}{C_l (2l+1)} \operatorname{Re} e^{-i\varrho} \varrho^{-l} \\ &\times \left\{ 1 - \frac{l+i\eta}{2l} \cdot \frac{2i\varrho}{1!} + \dots + \frac{(l+i\eta) \dots (-l+1+i\eta)}{(2l)!} \frac{(2i\varrho)^{2l}}{(2l)!} \right\} \\ &+ \frac{e^{2\pi\eta} - 1}{\pi} \left[\ln 2\varrho + 2\gamma - \sum_{s=1}^{2l+1} s^{-1} + \sum_{s=1}^l \frac{s}{s^2 + \eta^2} + \operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} \right] F_l \\ &- \frac{e^{2\pi\eta} - 1}{\pi} C_l \operatorname{Re} e^{-i\varrho} \varrho^{l+1} \sum_{s=1}^{\infty} \frac{(2l+1) \Gamma(l+1+s-i\eta) (2i\varrho)^s}{(2l+1+s)! s! \Gamma(l+1-i\eta)} \\ &\times \sum \left(\frac{1}{t} + \frac{1}{2l+1+t} - \frac{1}{l+t-i\eta} \right) \end{aligned} \quad (5.6)$$

† The functions required for this case are also given in the paper by Jost, Wheeler and Breit⁵.

where

$$C_l^2 = \frac{2^{2l}}{[(2l+1)!]^2} (l^2 + \eta^2) [(l-1)^2 + \eta^2] \dots (1 + \eta^2) \frac{2\pi\eta}{e^{2\pi\eta} - 1} \quad (5.7)$$

As before, we shall consider the expression f'/f . In the present case this expression is infinite at the origin of the co-ordinates. One must, therefore, equate to a constant not f'/f , but the expression $(r^l f')/r^l f$ which does not alter our argument in other respects

$$\frac{(r^l f)'}{r^l f} = c. \quad (5.8)$$

It can easily be seen that the last term vanishes for $r = 0$, since it has no singularities for any values of η . This clearly follows from the fact that since

$$\Gamma(l+1+s-i\eta) = (l+1-i\eta)(l+2-i\eta) \dots (l+s-i\eta) \Gamma(l+1-i\eta)$$

the terms in the second sum containing η in the denominator necessarily cancel with one of the summands of the first sum.

Including all the terms independent of η (including $\ln r$) in the constant c_l and then again putting $r = 0$ we get, after simple transformations, the final formula, representing a generalisation of (2.15) for the case of any l :

$$\cot K_l = \frac{1}{\pi} (e^{2\pi\eta} - 1) \left[\ln \eta - \operatorname{Re} \frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)} - \sum_{s=1}^l \frac{s}{s^2 + \eta^2} + \frac{\alpha_{lp}^l}{\prod_{p=i}^l \left(1 + \frac{p^2}{\eta^2} \right)} \right]. \quad (5.9)$$

6. SCATTERING OF LIGHT NUCLEI

An attempt might also be made to make an analysis similar to that of the scattering of protons by protons for the case of scattering of light nuclei.

The method of investigation remains the same as before. The data referring to the angular distribution must be used for the calculation of the phases K_l , states with different values of the total momentum J being characterized by their own phase K_{lj} . Thereafter the corresponding constant (or the function of E) can be calculated.

We thus exclude the influence of the Coulomb field and reduce the problem to the determination of the magnitude of α_j .

Since, however, the data on the angular distribution are much less complete and, on the other hand, the number of constants is larger (because of the participation of the P -state in the scattering, and also owing to a large spin in the case of the scattering of deuterons), we were not able to obtain such results as in the first case. We shall therefore briefly communicate the conclusions which could be drawn.

A. Scattering of Protons by Deuterons

The experimental investigation of this case was carried out by Tuve, Heydenburg and Hafstad¹⁴ for protons with an energy 0.83 MeV. As has been noted, however¹⁵, their data, especially for large angles, were apparently erroneous,

for they obtained for the ratio of the scattering cross section to the Rutherford one—2275 (for a scattering angle of 126° in the laboratory system). Such a magnitude could not possibly have been obtained with the assumption that only the first few angular momenta participate in the scattering.

The scattering of protons with energies between 200 and 300 keV has been investigated by Taschek¹⁶. Assuming, to begin with, a purely *S*-scattering a description might be attempted by means of two constants (corresponding to the total spin of the system $\frac{1}{2}$ and $\frac{3}{2}$). This is, however, insufficient. On the other hand, taking into account the *P*-state we introduce five more constants (according to the number of different values, which can be assumed by the quantum sum of three vectors $l = 1$, $s_1 = 1$ and $s_2 = \frac{1}{2}$). The existing data do not enable one, however, to determine all the seven constants. It is not excluded that in the scattering only waves with a zero angular momentum take part and that the determination of the phases is hindered by the insufficient accuracy of the experiments.

B. Scattering of Neutrons and Protons by α -particles

Staub and Tatel¹⁷ have investigated the scattering of neutrons with energies in the neighbourhood of 1 MeV by α -particles, measuring the total number of the recoil α -particles in a solid angle $\pm 12^\circ$. Analysing their data they arrived at the conclusion that the *P*-state participates in the scattering, the system possessing two closely lying levels $P_{1/2}$ and $P_{3/2}$ with a small energy difference (0.3 MeV).

On the other hand, Wheeler and Barshall¹⁸ have shown that for neutrons with an energy 2.5 MeV the phases corresponding to the waves $P_{1/2}$ and $P_{3/2}$ differ approximately by 60° .

Similar results have been obtained by us for the scattering of protons by α -particles, according to the data of Heydenburg and Ramsey¹⁹.

It turned out that in this case for the description of the scattering it is necessary to take into account three states: S_0 , $P_{1/2}$ and $P_{3/2}$, the phases of the latter differing strongly with respect to their magnitude.

These data show that the conclusion of Staub and Tatel that a narrow doublet exists is wrong and is connected with their overestimate of the accuracy of their own calculations. The existence of such a narrow doublet is even quite unexpected, if one takes into account the fact that the spin orbit interaction in the case of nuclear particles is not small (this follows from the presence of a relatively large quadrupole moment of the deuteron).

It must be noted here that Bloch's formula which has been used by Staub and Tatel is correct for a small range of the energy only, since its derivation was based on the incorrect assumption that the width of the neutron level does not depend on the energy.

After obtaining the phases K_0 , $K_{1/2}$ and $K_{3/2}$ as functions of the energy, the values of the corresponding α -phases can be obtained according to the formula (5.9).

The calculation has shown, however, that the points do not follow, in this case, a smooth curve, which is connected with the fact that the measurements

were carried out for wide energy intervals (through 0.5 MeV). At the same time, the larger the charge and the mass of the nucleus, the smaller must be the energy intervals taken, since with increase of the number of particles in the compound nucleus the number and density of the levels are also increased.

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56. ON THE ENERGY LOSS OF FAST PARTICLES BY IONISATION

The energy distribution function has been determined for fast particles which have traversed a layer of matter of a given thickness and lost energy in the latter as a result of ionisation collisions.

If a fast particle traverses a layer of matter its energy on exit will be less than the initial value due to ionisation losses. For a given thickness of the layer this energy loss is not constant, the probability of considerable fluctuations being quite appreciable.

By fast particles we mean here particles (electrons, mesotrons, etc.) which possess sufficiently large energies so that the usual ionisation theory may be applied. The layer of matter is considered not to be very thick so that the mean total energy loss is small compared with the initial energy E_0 .

The unknown distribution function will be denoted by $f(x, \Delta)$; this is the probability that a particle of a given initial energy E_0 on traversing a layer x will lose an amount of energy lying between Δ and $\Delta + d\Delta$ (the function f is normalised so that $\int f d\Delta = 1$). We shall write the kinetic equation which defines this function. Let $w(E, \varepsilon)$ be the probability (per unit length of the path) of an energy loss ε for a particle of energy E . As the ionisation losses are considered to be small compared with E_0 , instead of E we may write E_0 in $w(E, \varepsilon)$; we shall write $w(E_0, \varepsilon)$ simply as $w(\varepsilon)$. As usual the kinetic equation can be obtained by equating the change of the distribution function $(df/dx)dx$ on a length dx to the "collision integral" which expresses the difference in the number of particles which acquire, due to ionisation losses along dx , a given energy E , and the number of particles, which leave the given energy interval. We then obtain the following equation:

$$\frac{\partial f}{\partial x} = \int_0^{\infty} w(\varepsilon) [f(x, \Delta - \varepsilon) - f(x, \Delta)] d\varepsilon. \quad (1)$$

For the upper limit of integration we may write ∞ as $f(x, \Delta) = 0$ for $\Delta < 0$ and $w(\varepsilon) = 0$ for $\varepsilon > E_0$. Equation (1) does not contain explicitly the independent variables x and Δ . This permits us to find the solution by applying the Laplace transformation. Carrying out this transformation with respect to the

L. Landau, On the energy loss of fast particles by ionisation, *J. Phys. U.S.S.R.* 3, 201 (1944).

independent variable Δ we write

$$\varphi(p, x) = \int_0^{\infty} f(\Delta) e^{-p\Delta} d\Delta. \quad (2)$$

Then $f(\Delta)$ can be expressed in terms of $\varphi(p)$ as follows:

$$f(x, \Delta) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} e^{p\Delta} \varphi(p, x) dp, \quad (3)$$

where the integration is carried out over a straight line parallel to the imaginary axis and shifted to the right of the latter ($\sigma > 0$).

Multiplying both sides of the equation (1) by $e^{-p\Delta}$ and integrating with respect to $d\Delta$ we easily obtain

$$\frac{\partial \varphi(p, x)}{\partial x} = -\varphi(p, x) \int_0^{\infty} w(\varepsilon) (1 - e^{-p\varepsilon}) d\varepsilon \quad (4)$$

For $x = 0$, i.e. on the incidence surface, we must have $f(0, \Delta) = \delta(\Delta)$, which means that on this surface there is a single particle of energy $E = E_0$. We then find from (2) that for $x = 0$, $\varphi(p) = 1$. Integrating equation (4), with this initial condition we get

$$\varphi(p, x) \exp \left[-x \int_0^{\infty} w(\varepsilon) (1 - e^{-p\varepsilon}) d\varepsilon \right].$$

Inserting this in (3) we obtain the following general expression for the distribution function f in terms of the probability $w(\varepsilon)$

$$f(x, \Delta) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} e^{p\Delta - x \int_0^{\infty} w(\varepsilon) (1 - e^{-p\varepsilon}) d\varepsilon} dp. \quad (5)$$

In principle formula (5) is the solution of our problem in the general case. In order to be able to apply it, the function $w(\varepsilon)$ must be known; generally speaking, the form of this function has been determined only for energies which are large compared with the energy of the atomic electrons. It can be shown, however, that in case the energy losses are not too small a complete knowledge of the function $w(\varepsilon)$ is not necessary. The condition of applicability (see equation (20)) of the corresponding formulae will be deduced below.

In order to evaluate the integral in the exponent of the integrand we shall proceed as follows. Let ε_0 be a certain characteristic atomic energy (of the order of the mean binding energy of the atomic electrons). Further, let ε_{\max} be the maximum energy which can be transferred to an electron by the particle during ionisation. We assume that in integral (5) only those values of p are important for which

$$p\varepsilon_0 \ll 1, \quad p\varepsilon_{\max} \gg 1. \quad (6)$$

The limitations which this assumption imposes on the region of applicability of the results will be examined below. We now introduce a certain energy ε_1 such that $\varepsilon_1 \gg \varepsilon_0$ and $p\varepsilon \ll 1$; this can always be done due to (6). We shall split the integral over $d\varepsilon$ into two integrals with limits from 0 to ε_1 and ε_1 to ∞ , respectively. In the first of them we may (due to $p\varepsilon_1 \ll 1$) write $e^{-p\varepsilon} \cong 1 - p\varepsilon$. Thus

$$\int_0^{\infty} w(\varepsilon)(1 - e^{-p\varepsilon}) d\varepsilon = p \int_0^{\varepsilon_1} \varepsilon w(\varepsilon) d\varepsilon + \int_{\varepsilon_1}^{\infty} w(\varepsilon)(1 - e^{-p\varepsilon}) d\varepsilon. \quad (7)$$

For $\varepsilon_0 \ll \varepsilon \ll \varepsilon_{\max}$ the following well known formula holds (see, for instance ref. 1):

$$w(\varepsilon) = \frac{2\pi N e^4 \rho \sum Z}{m v^2 \sum A} \frac{1}{\varepsilon^2} \quad (8)$$

(m —mass of electron, e —charge of electron coinciding with that of the incident particle; v —velocity of the particle with the energy E_0 ; N is Avogadro's number, ρ —density of the substance; $\sum Z$ —sum of the atomic numbers in the molecule of the substance; $\sum A$ —sum of the atomic weights). In particular, it can be seen that for $p\varepsilon \gg 1$ the second integral in (7) rapidly converges so that these intervals of energy are not important. As $p\varepsilon_{\max} \gg 1$ it follows that for this integral expression (8) may be applied over the complete region of integration.

As is well known¹ for the integral $\int_0^{\varepsilon_1} w(\varepsilon) \varepsilon d\varepsilon$, i.e. for the energy lost by a particle (per unit path) and averaged over the interval from 0 to ε_1 , the following formula holds

$$\int_0^{\varepsilon_1} w(\varepsilon) \varepsilon d\varepsilon = \frac{2\pi N e^4 \rho \sum Z}{m v^2 \sum A} \ln \frac{\varepsilon_1}{\varepsilon'},$$

$$\ln \varepsilon' = \ln \frac{\left(1 - \frac{v^2}{c^2}\right) I^2}{2m v^2} + \frac{v^2}{c^2} \quad (9)$$

where I is a certain ionisation potential of the atom, the value of which is usually accepted to equal $I = I_0 Z$ where $I_0 = 13.5$ eV.

Substituting (8) in the second term of the right-hand side of (7) we obtain an integral of the form

$$\int_{\varepsilon_1}^{\infty} \frac{1 - e^{-p\varepsilon}}{\varepsilon^2} d\varepsilon = \frac{1}{\varepsilon_1} (1 - e^{-p\varepsilon_1}) + p \int_{\varepsilon_1}^{\infty} \frac{e^{-p\varepsilon}}{\varepsilon} d\varepsilon.$$

Remembering that $\varepsilon_1 p \ll 1$ and introducing $z = p\varepsilon$ as the integration variable, we write:

$$\frac{1}{p} \int_{\varepsilon_1}^{\infty} \frac{1 - e^{-p\varepsilon}}{\varepsilon^2} d\varepsilon = 1 + \int_{\varepsilon_1 p}^{\infty} \frac{e^{-z}}{z} dz = 1 + \int_{\varepsilon_1 p}^1 \frac{dz}{z} + \int_0^1 \frac{e^{-z} - 1}{z} dz + \int_1^{\infty} \frac{e^{-z}}{z} dz.$$

The sum of two latter integrals in the right-hand side of the equality, as is well known, is $-C$ where $C = 0.577\dots$ is Euler's constant (see, for instance ref. 2). Thus

$$\int_{\varepsilon_1}^{\infty} \frac{1 - e^{-p\varepsilon}}{\varepsilon^2} d\varepsilon = p(1 - C - \ln p\varepsilon_1).$$

Substituting this expression in equations (9) and (7) we obtain:

$$x \int_0^{\infty} w(\varepsilon) (1 - e^{-p\varepsilon}) d\varepsilon = \xi p(1 - C - \ln p\varepsilon'), \quad (10)$$

where instead of the co-ordinate x we have introduced the quantity

$$\xi = x \frac{2\pi N e^4 \varrho \sum Z}{m v^2 \sum A}. \quad (11)$$

Finally, inserting (10) in (5) and introducing a new integration variable $u = \xi p$ we obtain the following integral representation of the distribution function:

$$f(x, \Delta) = \frac{1}{\xi} \varphi(\lambda), \quad (12)$$

where

$$\varphi(\lambda) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{+i\infty + \sigma} e^{u \ln u + \lambda u} du, \quad (13)$$

$$\lambda = \frac{\Delta - \xi \left(\ln \frac{\xi}{\varepsilon'} + 1 - C \right)}{\xi}. \quad (14)$$

Thus the function of two variables $f(\Delta, x)$ turns out to equal the product of $1/\xi$ by a universal function $\varphi(\lambda)$ of a non-dimensional variable λ . This function was computed by the Calculation Bureau of the Mathematical Institute of the Academy of Sciences of the U.S.S.R., to whom the author acknowledges his gratitude. The function $\varphi(\lambda)$ has a maximum at $\lambda = -0.05$. Thus the most probable value of the energy loss is given by the expression

$$\Delta_0 = \xi \left(\ln \frac{\xi}{\varepsilon'} + 0.37 \right), \quad (15)$$

which is more precise than the usual formula for this quantity.

Let us introduce the notation

$$\eta = \frac{2\pi V e^4 \varrho \sum Z}{m c^2 \sum A} x = \frac{1.54 \times 10^3 \mu \sum Z}{\sum A}, \quad (16)$$

where μ is the mass of the layer of matter per cm^2 of its surface and η is measured in electron volts. Then $\xi = \eta/(v/c)^2$ and we have

$$\Delta_0 = \frac{\eta}{\left(\frac{v}{c}\right)^2} \left[\ln \frac{3 \times 10^3 \eta}{Z^2 \left(1 - \frac{v^2}{c^2}\right)} + 1 - \frac{v^2}{c^2} \right]. \quad (17)$$

The probability of an energy loss lying between Δ and $\Delta + d\Delta$ is

$$f(x, \Delta) d\Delta = \varphi\left(\frac{\Delta - \Delta_0}{\xi}\right) d\left(\frac{\Delta - \Delta_0}{\xi}\right), \quad (18)$$

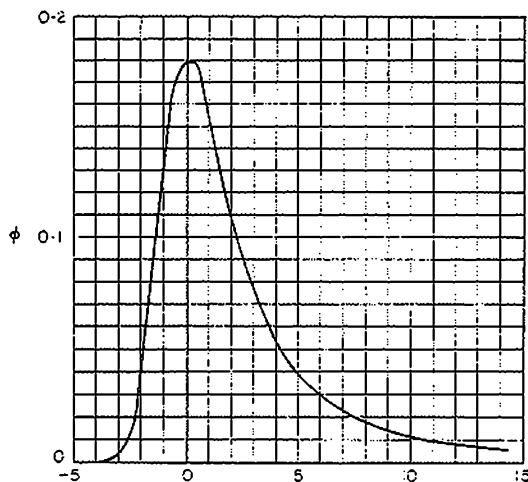


FIG. 1.

and the integral probability for an energy loss exceeding Δ is

$$\int_{\Delta}^{\infty} f(x, \Delta) d\Delta = \psi\left[\frac{\Delta - \Delta_0}{\xi}\right], \quad (19)$$

where φ and ψ are two universal functions which are shown in Figs. 1 and 2.

To the left of the maximum of Fig. 1 ($E < E_0$, i.e. the energy loss is less than the most probable) the curve decreases very rapidly; to the right of the maximum ($E > E_0$) the curve decreases at a considerably slower rate.

It can be seen from Fig. 2 that the median of the probability distribution (i.e. the vertical line which divides the area of this curve into two equal parts) lies approximately by one unit to the right of the axis of ordinates.

Let us determine what limitations on the region of applicability of the obtained results are imposed by the assumptions (6) made above. Evidently, in integral (13) the region $u \sim 1$ of the integration variable is significant. As

$u = \xi p$ assumption (6) reduces to the conditions

$$\xi \gg \varepsilon, \quad (20)$$

$$\xi \ll \varepsilon_{\max}. \quad (21)$$

The first condition means that the observed energies must be sufficiently large in comparison with the atomic energies. This is the main limitation of applicability of the results obtained here. If the fast particle is an electron or positron then $\varepsilon_{\max} = E_0$; condition (21) $\xi \ll E_0$ is at any rate fulfilled as at the very start it was assumed that the total energy loss of the particle is small compared to E_0 . However, if the mass of the fast particle is large in comparison with the

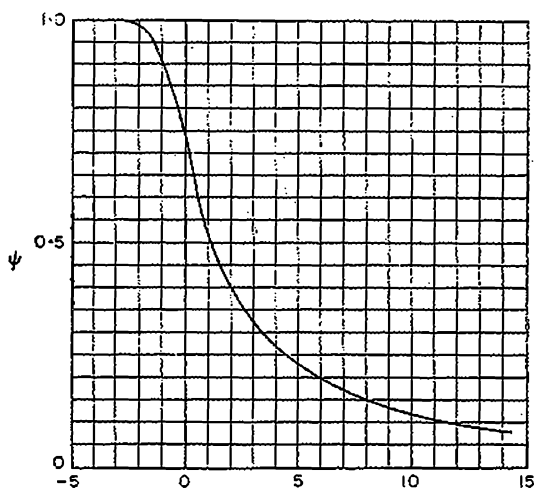


FIG. 2.

mass of the electron (mesotrons, protons), ε_{\max} may be considerably less than E_0 and then relation (21) may limit the validity of the formulae. It is easy to show that in this case the condition

$$\xi \ll \frac{2m v^2}{1 - \frac{v^2}{c^2}}, \quad \eta \ll 2m c^2 \frac{\left(\frac{v}{c}\right)^4}{1 - \frac{v^2}{c^2}} \quad (22)$$

must be fulfilled. It should be mentioned that if large fluctuations, i.e. large values of $(\Delta - \Delta_0)/\varepsilon$ interest us then instead of (22) we must write

$$\Delta - \Delta_0 \ll \frac{2m v^2}{1 - \frac{v^2}{c^2}},$$

as can be shown in an elementary way.

The opposite limiting case to (22) can also easily be examined with aid of (5); as has repeatedly been shown it leads to a gaussian distribution of the fluctuations.

We shall now derive the asymptotic formula for the function $\varphi(\lambda)$ at large values of λ . We shall first consider negative λ of large absolute values. We use the "saddle point method" to compute integral (13). The exponent $u \ln u - |\lambda|u$ in the integrand of (13) is minimum at $u = e^{|\lambda|-1}$. Let us use this value as the quantity σ , which defines the line of integration. Writing $u = e^{|\lambda|-1} + i\zeta$ we reduce the integral over du to an integral of $d\zeta$ from $-\infty$ to $+\infty$, in this integral only small values of ζ of importance. Correspondingly, expanding the expression in the exponent into a series in powers of ζ , we obtain

$$\varphi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left\{ -e^{|\lambda|-1} - \frac{\eta^2}{2e^{|\lambda|-1}} \right\} d\eta,$$

or, finally

$$\varphi(\lambda) = \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{|\lambda| - 1}{2} - e^{|\lambda|-1} \right\}, \quad (23)$$

which shows that $\varphi(\lambda)$ very rapidly decreases with increasing $|\lambda|$.

For large positive λ it is convenient to take the integral along the contour shown in Fig. 3. Then after simple transformations it takes the form

$$\varphi(\lambda) = \frac{1}{\pi} \int_0^{\infty} e^{-u \ln u - \lambda u} \sin \pi u \, du. \quad (24)$$

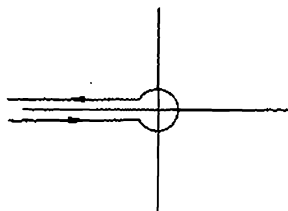


FIG. 3.

For large λ this integral tends to $1/\lambda^2$ and correspondingly $\int_{\lambda}^{\infty} \varphi(\lambda) \, d\lambda$ tends to $1/\lambda$. It will easily be seen that this means that the probability of such a fluctuation corresponds chiefly to the production of a single particle of the energy $A - A_0$. In order to obtain a more precise formula we introduce instead of λ the quantity ω which is related to λ by the equation

$$\lambda = \omega + \ln \omega + A,$$

where the value of the constant A is chosen below. Evidently,

$$\int_{\lambda}^{\infty} \varphi(\lambda) \, d\lambda = \int_0^{\infty} e^{-u \ln u - \lambda u} \frac{\sin \pi u}{\pi u} \, du = \int_0^{\infty} e^{-u \ln \omega u - \omega u - A u} \frac{\sin \pi u}{\pi u} \, du$$

or denoting ωu by U :

$$\int_{\lambda}^{\infty} \varphi(\lambda) d\lambda = \frac{1}{\omega} \int_0^{\infty} e^{-U - \frac{U}{\omega} (\ln U + A)} \frac{\sin \frac{\pi U}{\omega}}{U/\omega} dU$$

This integral can be expanded in a series of powers of $1/\omega$. Then

$$\int_{\lambda}^{\infty} \varphi(\lambda) d\varphi = \frac{1}{\omega} - \frac{1}{\omega^2} \int_0^{\infty} e^{-U} U (\ln U + A) dU + \dots$$

The second term vanishes if $A = C - 1$. Hence the following parametric relation exists between φ and $(\Delta - \Delta_0)/\xi$ with an accuracy involving terms of the order $1/\omega^2$,

$$\frac{\Delta - \Delta_0}{\xi} = \omega + \ln \omega - 0.37, \quad \varphi = \frac{1}{\omega}, \quad (25)$$

and

$$\varphi = \frac{1}{\omega(\omega + 1)}. \quad (26)$$

These formulae are valid already for $(\Delta - \Delta_0)/\xi = 10$ with an accuracy of several per cent.

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57. ON A STUDY OF THE DETONATION OF CONDENSED EXPLOSIVES

THE modern hydrodynamic theory of the detonation of condensed explosives developed by Becker¹ and Schmidt² is based on the application to the detonation products of the following equation of state:

$$p(v - \alpha) = RT \quad (1)$$

where P is the pressure, v is the volume, T is the temperature, and α is the co-volume, representing the incompressible part of the detonation products.

Application of this equation leads to numerous discrepancies, since for a number of explosives it is found that $\alpha \geq v_i$ where v_i is the initial volume of the explosive. Various hypotheses on a decrease in α with a rise in pressure, though confirmed by the experimental data of Bridgman³ and Schmidt⁴, are nevertheless insufficient to determine with accuracy the function $\alpha = \alpha(p)$, and hence do not remove the above discrepancies.

Inasmuch as the density of an explosive after it has decomposed is sufficiently high, it is in general rather futile to use any equation of state for a gas, but it is more expedient to compare the decomposition products with some liquid whose particles are in a state of oscillation, thus conditioning the nature of the course of the process of expansion of these products, and only when the detonation products expand to a definite volume v_k , when $v > v_k$, is it possible to make use of the equation of state for an ideal gas

$$p v = RT. \quad (2)$$

It is most expedient to build up such elements of the theory of detonation that are based on experimental data determined with a great accuracy, and satisfy all that has been said above. For a number of local action explosives it is found possible to determine the dependence of the detonation velocity D upon the density of the explosive δ_0 over a wide range of values of the latter (from 0.5 to 1.8). This dependence is shown on the accompanying graph and may be approximated by the simple formula

$$D = B \delta_0^a. \quad (3)$$

Л. Д. Ландау и К. П. Станюкович, Об изучении детонации конденсированных взрывчатых веществ, *Доклады Академии Наук СССР*, 46, 399 (1945).

L. Landau and K. P. Staniukovich, On a study of the detonation of condensed explosives, *C. R. Acad. Sci. URSS*, 46, 363 (1945).

Making use of the fundamental equations of the hydrodynamic theory of detonation

$$E = Q + \frac{p + p_0}{2} (v_0 - v), \quad (4)$$

$$p - p_0 = - \frac{\partial p}{\partial v} (v_0 - v), \quad (5)$$

$$D^2 = - v_0^2 \frac{\partial p}{\partial v}, \quad (6)$$

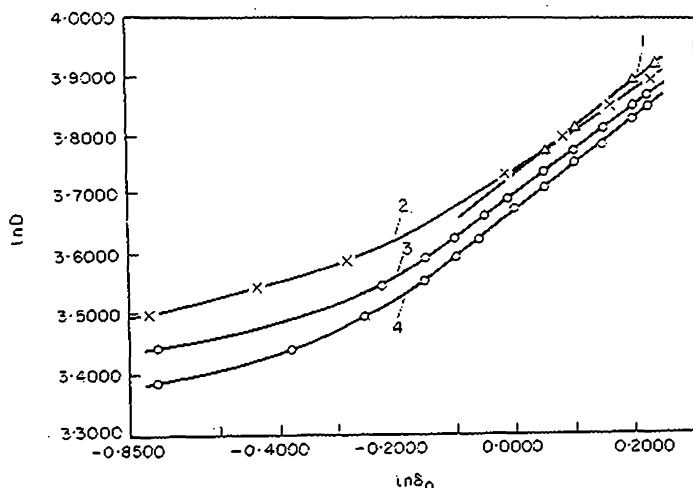


FIG. 1. 1—ten; 2—tetyl; 3—picric acid; 4—trinitrotoluene.

where E is the energy at the front of the detonation wave and Q is the heat of reaction, it may be shown that the following expression is true:

$$\frac{d \ln D}{d \ln v_0} = \frac{2v - v_0}{2(v - v_0)} + \frac{v_0 \frac{\partial p}{\partial T}}{2 \left(2c_e + (v - v_0) \frac{\partial p}{\partial T} \right)}. \quad (7)$$

Since $v_0 = 1/\delta_0$, we get from equation (3)

$$\frac{d \ln D}{d \ln v_0} = -\alpha. \quad (7a)$$

According to our hypothesis in which we assume that the free energy of the detonation products F depends upon the vibration frequency of their molecules, we may write:

$$p = p_0 + L k T \frac{\partial \ln \omega}{\partial v}, \quad (8)$$

where p_0 is that part of the pressure which is independent of the temperature, L is the number of degrees of freedom for the molecules moving as a whole, k is the Boltzmann constant, and ω is the frequency of vibrations. Since $\omega \approx c/r$, where c is the velocity of sound in the medium considered and r is the distance between the molecules, then inasmuch as c is a power function of the volume (which will become clear from what follows) and $r \approx v^{1/3}$, the equation of state may be written in the following form:

$$p = \bar{p} + \frac{AT}{v} \quad (9)$$

where \bar{p} is the part of the pressure independent of the temperature, and A is a certain coefficient depending upon the number of degrees of freedom. It follows from (9) that $\partial p / \partial T = A/v$, and hence (7) may be rewritten in the form

$$-a = \frac{2v - v_0}{2(v - v_0)} + \frac{v_0}{2 \left[2 \frac{c_0}{A} v + (v - v_0) \right]}, \quad (7b)$$

whence it follows that $v \approx v_0$. And upon assuming on the basis of equation (9) that in the region considered

$$p v^n = \text{const} \quad (10)$$

instead of the analogous equation for an ideal gas

$$p v^\gamma = \text{const}, \quad (10a)$$

this will signify that $c \approx v^{\frac{1-n}{2}}$. The value of n , as will be shown, may be determined from (7b). Indeed, on the basis of the classical theory of detonation and making use of equation (10) we may write

$$v_i = \frac{n}{n+1} v_0, \quad (11)$$

$$p_i v_0 = D u_i, \quad (11a)$$

$$u_i = \frac{D}{n+1}, \quad (11b)$$

$$c_i = \frac{n}{n+1} D, \quad (11c)$$

where p_i is the pressure at the front of the detonation wave, c_i is the velocity of sound, and u_i is the velocity of the medium behind the detonation wave front; whence it follows that

$$a = \frac{n-1}{2} - \frac{n+1}{2 \left[2n \frac{c_0}{A} - 1 \right]}, \quad (7c)$$

inasmuch as c_v and A depend upon the number of degrees of freedom, i.e. we may write:

$c_v = 12$, $A = 5n - 3$, for diatomic molecules;

$c_v = 18$, $A = 6n - 3$, a simple system for triatomic molecules;

$c_v = 12$, $A = 5n - 3$, a linear system for triatomic molecules.

Thus, it is evident that by employing the method of successive approximations we may determine the value of n from equation (7c) as well. Now it becomes possible to determine the co-ordinates of the point where equation (10) is conjugate to equation (10a).

With this aim in view let us determine the work of expansion w of the detonation products when the volume changes from v_i to v_k :

$$w = \frac{p_i v_i}{n-1} - \frac{p_k v_k}{n-1} + \frac{p_k v_k}{\gamma-1} = E. \quad (12)$$

Making use of equations (11), (11a), (11b), (11c) and disregarding the quantity $p_k v_k/(n-1)$ which is small in comparison with $p_k v_k/(\gamma-1)$, as well as the change in c_v with T , we get

$$\frac{p_k v_k}{\gamma-1} = Q - \frac{D^2}{2(n^2-1)}, \quad (12a)$$

or, denoting $Q - \frac{D^2}{2(n^2-1)}$ by ΔQ

$$c_v T_k = \Delta Q. \quad (12b)$$

The most interesting parameters computed for several explosives are:

Explosives	Q cal degree	D m/sec	δ_0	α	n	w_i m/sec	c m/sec	δ_i	p_i atm	ΔQ	p_k atm	δ_k	T
Nitrotoluene	950	7000	1.60	0.78	3.2	1680	5320	2.10	190000	320	2150	0.52	1580
Tetryl	1150	7600	1.63	0.76	3.1	1850	5750	2.15	230000	350	2000	0.46	1540
Picric acid	1000	7100	1.63	0.76	3.1	1730	5370	2.15	210000	300	2360	0.49	1530
Ten	1400	8400	1.69	0.80	3.2	2000	6400	2.28	275000	490	2900	0.52	1940

For local action explosives it is quite allowable to use an approximate value of $n = 3$, which greatly simplifies all the calculations without any great sacrifice in accuracy.

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58. THE DETERMINATION OF THE FLOW VELOCITY OF THE DETONATION PRODUCTS OF SOME GASEOUS MIXTURES

THE distribution of the detonation products of gaseous mixtures for a one-dimensional case may be directly obtained from the Riemann solution of the fundamental equations of hydrodynamics. This solution is based upon the assumption that $u = f(v)$, where v is the specific volume, and u is the velocity of the particles.

It follows then that

$$du = \pm \sqrt{-dp dv}, \quad (1)$$

where p is the pressure.

Integrating this expression, we get for a perfect gas:

$$u = u_{\text{initial}} + \frac{2}{\gamma - 1} [c_{\text{initial}} - c], \quad (2)$$

where u_{initial} is the velocity of the medium behind the detonation-wave front, c_{initial} is the initial, and c the local velocity of sound, $\gamma = c_p/c_v$:

Since

$$u_{\text{initial}} = \frac{D}{\gamma + 1}, \quad (3)$$

$$c_{\text{initial}} = \frac{\gamma D}{\gamma + 1}, \quad (3a)$$

where D is the detonation velocity, we may write

$$u = \frac{3\gamma - 1}{\gamma^2 - 1} D - \frac{2c}{\gamma - 1}. \quad (4)$$

When the gases flow into a vacuum, $c = 0$; when they flow into the atmosphere, the value of c may be found from the condition:

$$c = c_{\text{initial}} \left(\frac{p}{p_{\text{initial}}} \right)^{\frac{\gamma - 1}{2\gamma}}.$$

Л. Д. Ландау и К. П. Станюкович, Определение скорости истечения продуктов детонации некоторых газовых смесей, *Доклады Академии Наук СССР*, 47, 205 (1945).

L. Landau and K. P. Staniukovich, The determination of the flow velocity of the detonation products of some gaseous mixtures, *C. R. Acad. Sci. URSS*, 47, 199 (1945).

Substituting (3a) for c_{initial} , we obtain:

$$c = \frac{\gamma D}{\gamma + 1} \left(\frac{p}{p_{\text{initial}}} \right)^{\frac{\gamma-1}{2\gamma}}.$$

The value of p is determined from the condition that the pressure of the decomposition products (p_x) and that of the shock-wave moving in front of them (p_{shock}) are equal.

Since

$$p_{\text{shock}} = \frac{\gamma_{\text{atm}} + 1}{2} \varrho_{\text{atm}} u^2,$$

where ϱ_{atm} is the density of the atmosphere, and γ_{atm} is the ratio of the specific heats, then

$$u = \frac{D}{\gamma^2 - 1} \left[3\gamma - 1 - 2\gamma u^{\frac{\gamma-1}{\gamma}} \left(\frac{(\gamma_{\text{atm}} + 1) \varrho_{\text{atm}}}{2\varrho_{\text{initial}}} \right)^{\frac{\gamma-1}{2\gamma}} \right]. \quad (4a)$$

The value of p_{initial} may be determined by the formula:

$$p_{\text{initial}} = \frac{\varrho_0}{\gamma + 1} D^2, \quad (5)$$

where ϱ_0 is the density of the gaseous mixture.

The maximum density and temperature at the wave front are determined by the formulae:

$$\varrho_{\text{initial}} = \frac{\gamma + 1}{\gamma} \varrho_0, \quad (6)$$

$$T_{\text{initial}} = \frac{2\gamma}{\gamma + 1} T_2, \quad (6a)$$

where T_2 is the reaction temperature. In the above formulae we did not take into account the change in the specific heats with temperature, and if instead of γ we use its value for some mean temperature between T_2 and T_{initial} , as is usually done, the results will not be quite accurate, especially in a determination of the flow velocity.

The value of γ_{atm} should correspond to the temperature at the front of the shock wave; however, it may be assumed that $\gamma_{\text{atm}} = 1.4$. To determine the velocity of flow more accurately, we may employ the empirical formula $\gamma = \bar{\gamma} - \alpha T$, where the values of $\bar{\gamma}$ and α for a given gas may be computed. In this case we may write the following formula:

$$u_0 = u_{\text{initial}} + \int_T^{T_{\text{initial}}} \sqrt{\frac{c_p c_v}{RT}} dT. \quad (2a)$$

Since $c_v = R/(\bar{\gamma} - 1 - aT)$ and $c_p = R(\bar{\gamma} - aT)/(\bar{\gamma} - 1 - aT)$, integration gives

$$u_0 = \frac{D}{\gamma_{\text{initial}} + 1} + 2\sqrt{\frac{R}{a}} \left[\arcsin \sqrt{\frac{\gamma - \gamma_{\text{initial}}}{\gamma_{\text{initial}}}} - \arcsin \sqrt{\frac{\bar{\gamma} - \gamma}{\bar{\gamma}}} \right. \\ \left. + \frac{1}{\sqrt{\gamma - 1}} \left(\operatorname{artanh} \sqrt{\frac{(\bar{\gamma} - 1)\gamma_{\text{initial}}}{\bar{\gamma} - \gamma_{\text{initial}}}} - \operatorname{artanh} \sqrt{\frac{(\bar{\gamma} - 1)\gamma}{\bar{\gamma} - \gamma}} \right) \right]. \quad (4b)$$

The difference in the rates, as determined by formulae (4) and (4b), is not very large, although it still amounts to as much as 8 per cent of the actual value of the rate.

The velocity of the shock wave and the temperature at its front were determined by the formulae:

$$D_{\text{shock}} = \frac{\gamma_{\text{atm}} + 1}{2} u_x, \quad (7)$$

$$T_{\text{shock}} = \frac{\gamma_{\text{atm}} - 1}{2} u_x^2. \quad (7a)$$

The velocity of flow into the atmosphere has to be computed by the method of successive approximations or graphically, proceeding as before from the condition that the pressure at the front of the shock wave should be equal to the residual pressure of the detonation products. The results of our computations are summarised in the following table.

Gaseous mixture	D (in m/sec)	T_{initial} (in °K)	u_{initial} (in m/sec)	p_{initial} (in kg/cm ²)	u_0 (in m/sec)	u_x (in m/sec)	p_{shock} (in kg/cm ²)	D_{shock} (in m/sec)	T_{shock} (in °K)
2H ₂ + O	2820	4600	1330	19	21500	1350	19	1600	1000
2H ₂ + O ₂ + N ₂	2400	3900	1100	19	15000	1200	18	1400	900
2H ₂ + O ₂ + 3N ₂	2060	2700	930	18	11600	1000	16	1200	850
2H ₂ + O ₂ + 5N ₂	1820	2700	840	17	7900	900	14	1100	770

It is evident that the limiting velocities that arise in an unsteady flow may be quite large, well above those of a stationary process. For the case of flow into the atmosphere, the velocities have no time to increase owing to the low density of the detonating gas, and are damped by the shock wave that forms.

59. DETERMINATION OF THE FLOW VELOCITY OF THE DETONATION PRODUCTS OF CONDENSED EXPLOSIVES

In determining the flow velocity of detonation products it was established that in certain cases this velocity exceeds the detonation velocity. An especially high flow velocity was observed by G. I. Pokrovsky¹ and R. Becker² in experiments with cumulative charges during detonation in a tube. To-day this is a well-known and repeatedly verified phenomenon. Moreover, for experiments approaching the case of one-dimensional detonation, the flow velocities for commonly employed explosives (tol, tetryl) lie within the limits 8000–12000 m/sec.

The case of one-dimensional detonation lends itself comparatively easily to theoretical analysis, while the flow velocity u of the detonation products is determined from the Riemann solution of the hydrodynamics equation, which may be written in the following form:

$$u = u_{\text{initial}} + \int_{v_{\text{initial}}}^v v_{\text{initial}} \sqrt{-dp dv}, \quad (1)$$

where p is the pressure; v is the specific volume; u_{initial} is the velocity of the detonation products; v_{initial} is their specific volume at the front of the detonation wave.

As we have shown elsewhere³, the following law connecting p and v (which replaces the usual adiabatic law for an ideal gas) holds for the case of detonation:

$$p v^n = \text{const}, \quad (2)$$

where $n = 3$ for standard local action explosives.

Hence, the following formulae will hold:

$$u_{\text{initial}} = \frac{D}{n+1}, \quad (3)$$

$$c_{\text{initial}} = \frac{n D}{n+1}, \quad (3a)$$

$$v_{\text{initial}} = v_0 \frac{n}{n+1}, \quad (4)$$

Л. Д. Ландау и К. П. Станюкович, Определение скорости истечения продуктов детонации конденсированных взрывчатых веществ, *Доклады Академии Наук СССР*, 47, 273 (1945).

L. Landau and K. P. Staniukovich, Determination of the flow velocity of the detonation products of condensed explosives, *C. R. Acad. Sci. URSS*, 47, 271 (1945).

where D is the detonation velocity; c_{initial} is the velocity of sound at the front of the detonation wave, and v_0 is the initial volume of the explosive.

Let us evaluate the integral $\int_{v_{\text{initial}}} \sqrt{-dp} dv$. To do this we will divide the domain of integration in two. Moreover, at the point (p_k, v_k) we will conjugate the solution obtained for the expansion products that obey equation (2), and the solution obtained for an ideal gas. Then

$$u = u_{\text{initial}} + \frac{2c_{\text{initial}}}{n-1} - \frac{2c_k}{n-1} + \int_0^{T_k} \sqrt{\frac{c_p c_v}{RT}} dT, \quad (5)$$

where T_k is the temperature at the point of conjugation; c_v and c_p are the specific heats at constant volume and constant pressure; R is the gas constant. This temperature may be obtained from the equation

$$\Delta Q = \int_0^{T_k} c_v dt,$$

where ΔQ is the residual energy at the point of conjugation determined by the equations:

$$\frac{p_{\text{initial}} v_{\text{initial}}}{n-1} - \frac{p_k v_k}{n-1} + \Delta Q = E = \frac{p_{\text{initial}} + p_0}{2} (v_0 - v_{\text{initial}}) + Q,$$

where Q is the heat of reaction.

From this, making use of formulae (3) and (4) and discarding the quantities $p_k v_k/(n-1)$ and $2c_k/(n-1)$ which are small in comparison with ΔQ , we get

$$\Delta Q = Q - \frac{D^2}{2(n^2-1)}.$$

Now for the determination of u we will have the formula:

$$u = \frac{3n-1}{n^2-1} D + \int_0^{T_k} \sqrt{\frac{c_p c_v}{RT}} dT. \quad (5a)$$

If we disregard the dependence of c_p and c_v upon the temperature, the formulae are considerably simplified, becoming:

$$u = \frac{3n-1}{n^2-1} D + \frac{2}{\gamma-1} [\sqrt{\gamma RT_k} - \sqrt{\gamma RT}]$$

($\gamma = c_p/c_v$). Inasmuch as $T_k = \Delta Q/c_v$, the formula for the determination of u will have the following final form:

$$u = \frac{3n-1}{n^2-1} D + 2\sqrt{\frac{\gamma}{\gamma-1}} \Delta Q - \frac{2}{\gamma-1} \sqrt{\gamma RT}. \quad (6)$$

As computations show, the value of T_k is comparatively small (of the order of 1000–1500°), and it indeed becomes possible to discard the dependence of the specific heat upon the temperature. Moreover, the error in determining u will not be greater than 5 per cent.

For the case of gas flow into a vacuum, T has to be put equal to zero. For the determination of the flow velocity into the atmosphere u_x , it is necessary to take into account the resistance of the shock wave forming before the front of the moving gas.

With this aim in view let us determine the velocity D_{shock} and the pressure p_{shock} at the front of the shock wave:

$$D_{\text{shock}} = \frac{\gamma_a + 1}{2} u_x, \quad (7)$$

$$p_{\text{shock}} = \frac{\gamma_a + 1}{2} \varrho_a u_x^2, \quad (8)$$

where ϱ_a is the density of the air, and γ_a is the ratio of its specific heats. It is obvious that the condition $p_{\text{shock}} = p_x$, where p_x is the residual pressure of the detonation products, makes it possible to determine the velocity sought for. For this it is necessary to know p_k and v_k —the pressure and volume at the point of conjugation.

These quantities are determined from the equations:

$$p_k v_k^n = p_{\text{initial}} v_{\text{initial}}^n \quad \text{and} \quad p_k v_k = (\gamma - 1) \Delta Q.$$

Since the value of γ_a cannot be determined exactly owing to the high temperature at the front of the shock wave, it may be assumed, without more than slightly affecting the accuracy of the computation, that $\gamma_a = 1$, and expressions (7) and (8) may be re-written in the form:

$$D_{\text{shock}} = u_x, \quad (7a)$$

$$p_{\text{shock}} = \varrho_a u_x^2. \quad (8a)$$

Further it is obvious that $\sqrt{\gamma R T_x} = c_k (p_x/p_k)^{(\gamma-1)/2\gamma}$, and from this it follows that

$$u = \left(\frac{2}{\gamma + 1} \right)^{1/2} \left(\frac{p_x}{\varrho_a} \right)^{1/2} = \frac{3n-1}{n^2-1} D + 2 \sqrt{\frac{\gamma}{\gamma-1}} \Delta Q \left[1 - \left(\frac{p_x}{p_k} \right)^{\frac{\gamma-1}{\gamma}} \right], \quad (6a)$$

from which p_x , u_x , T_x and ϱ_x may be determined.

The results of computations are given in the adjoined table.

Explosive	u_k	u	u_x	p_x	ϱ_x	T_x
Trotyl	6600	11,300	7800	750	0.23	1200
Picric acid	6900	11,550	8000	800	0.21	1200
Ten	7900	18,500	9800	1200	0.26	1600
Tetryl	7400	11,950	9200	900	0.25	1300

The temperature at the front of the shock wave turns out to be of the order of 20,000–30,000° if the equation of state derived for an ideal gas is used, and neither its dissociation nor ionization is taken into account. It is obvious that both these processes occur and considerably lower the temperature at the front of the shock wave. Inasmuch as this is accompanied by considerable radiation, it is in general difficult to draw any conclusions regarding the nature of the shock wave, and it may turn out that an abrupt shock waves is not formed under these conditions. On the photographs made by Prof. G. I. Pokrovsky's method a shockwave before the front of the gas is indeed absent.

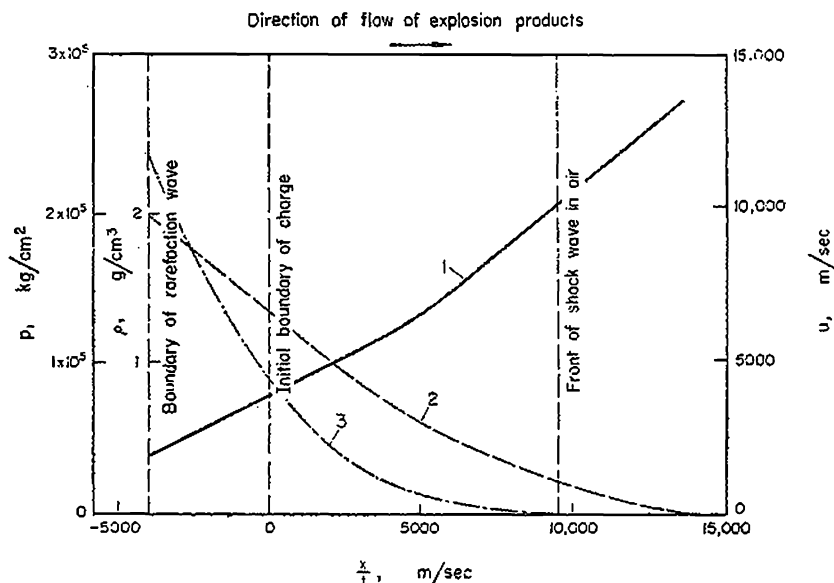


FIG. 1 1—velocity (u); 2—density (ρ); 3—pressure (p).

Let us carry the Riemann solution to the end, analysing separately both regions of the expanding detonation products and connecting the solutions at their point of conjugation. We have the following equations:

$$\frac{x}{t} = u - c, \quad (9)$$

$$u = u_{\text{initial}} + \frac{2c_{\text{initial}}}{n-1} - \frac{2c}{n-1} \quad \text{when} \quad u_{\text{initial}} \leq u \leq \frac{3n-1}{n^2-1} D, \quad (10)$$

$$u = \frac{3n-1}{n^2-1} D + \frac{2c_k}{\gamma-1} - \frac{2c}{\gamma-1}$$

when

$$\frac{3n-1}{n^2-1} D \leq u \leq \frac{3n-1}{n^2-1} D + \frac{2c_k}{\gamma-1}.$$

(10a)

Let us analyse any ideal explosive, closely resembling ten in its properties assuming that $n = 3$, $\gamma = 1.3$, $D = 8000$ m/sec, $Q = 1400$ cal/g and $\varrho_0 = 1.5$. The results of the computation are given in the accompanying figure.

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4. F. A. KOROLEV and G. I. POKROVSKY, *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS.* 42, 6 (1944).

60. ON SHOCK WAVES AT LARGE DISTANCES FROM THE PLACE OF THEIR ORIGIN

It is shown that at large distances from the body, moving with a velocity exceeding that of the sound, there exists not one (as is usually assumed) but two subsequent shock waves. The shape of these waves and the law of the decrease of their intensity with distance is determined. The propagation of a spherical shock (explosive) wave at large distances from the place of explosion is examined.

At large distances from the place of their origin shock waves are faint and have, therefore, the character of sound waves. However, for our purpose the ordinary linear approximation is insufficient and it is necessary to consider the properties of sound waves with a small amplitude in the second approximation. We shall have to do below with cylindrical and spherical waves; since, however, at large distances cylindrical or spherical waves can be considered in every small region as a plane one, we shall dwell preliminarily on some properties of plane waves.

As is well known, a one-dimensional progressive wave with an arbitrary amplitude is described by the so-called Riemann solution of the equations of motion

$$x = t[v + c(v)] + f(v),$$

where $f(v)$ is an arbitrary function of the velocity v of the gas, while c —the local velocity of sound, connected with v by the relation

$$v = \int \frac{dp}{\rho c} = \int \left(- \frac{\partial V}{\partial p} \right)^{1/2} dp$$

(ρ —density, V —specific volume of the gas). These two formulae determine, implicitly, the velocity v (and along with it the other quantities characterising the wave) as a function of x and t , i.e. the shape of the wave at any given instant. For $t = 0$ we have $x = f(v)$, i.e. the function inverse to $f(v)$ determines the shape of the wave at the initial moment.

The quantity

$$u = v + c(v) \tag{1}$$

is the velocity at which the points of the wave profile are travelling. This velocity is different for different points of the profile, and that is why it does not remain constant and varies its shape with time. Expressing u as function,

Л. Ландау, Об ударных волнах на больших расстояниях от места их возникновения, *Прикладная Математика и Механика*, 9, 286 (1945).

L. Landau, On shock waves at large distances from the place of their origin, *J. Phys. U.S.S.R.* 9, 496 (1945).

say, of the pressure p in the wave, we have for the derivative du/dp :

$$\frac{du}{dp} = \frac{dc}{dp} + \frac{1}{\rho c},$$

but $c = \sqrt{\partial p / \partial \rho} = V \sqrt{-\partial p / \partial V}$ so that calculation yields

$$\frac{du}{dp} = \frac{1}{2} c^3 \left(\frac{\partial^2 V}{\partial p^2} \right)_S.$$

The adiabatic derivative $(\partial^2 V / \partial p^2)_S$ (S —entropy) is positive for all gases, so that $dV/dp > 0$. The velocity of the displacement of a given point of the wave profile is thus the larger the larger is the pressure at this point; the compressions are, therefore, gradually advancing with respect to the rarefactions.

For a wave with a small amplitude the displacement velocity u of the points of the profile in the first approximation is obtained by putting in (1) $v = 0$, i.e. $u = c_0$ (the letters with the index zero denote the equilibrium values of the corresponding quantities), which corresponds to a displacement of the wave profile without any change of its shape. In the next approximation we have

$$u = c_0 + \frac{du}{dp_0} p',$$

or

$$u = c_0 \left(1 + \alpha \frac{p'}{p_0} \right), \quad (2)$$

$$\alpha = \frac{p_0}{2} \frac{c_0^2}{V_0^2} \left(\frac{\partial^2 V}{\partial p^2} \right)_S$$

(p' —denotes the variable part of the pressure in the wave). For an ideal gas $\alpha = (\gamma + 1)/2\gamma$ (for air $\alpha = 0.86$), where $\gamma = c_p/c_v$ is the ratio of the specific heats at constant pressures and volume.

When the profile of the wave is distorted to such an extent that the uniqueness of the solution disappears, a shock wave arises. The Riemann solution becomes, generally speaking, inapplicable after the formation of discontinuities. It is, however, natural that for waves of small amplitude, in the second approximation under consideration, this solution remains valid in the presence of such discontinuities. This can be seen in the following way. The jumps in the velocity, pressure, and specific volume in the discontinuity are connected with each other by the relation

$$v_2 - v_1 = \sqrt{(p_2 - p_1)(V_1 - V_2)}.$$

The change of the velocity v over a certain segment of the length of the x -axis in the Riemann solution is equal to the integral

$$v_2 - v_1 = \int_{p_1}^{p_2} \left(-\frac{\partial V}{\partial p} \right)^{1/2} dp.$$

A simple calculation with the help of an expansion into a series shows that the preceding expressions differ in the terms of the third order only (in carrying out the calculation it must be kept in mind that the change of entropy in the discontinuity is a quantity of the third order, while in the Riemann solution the entropy is constant). Hence it follows that which an accuracy up to terms of the second order the motion in the travelling wave in the presence of the discontinuity can be described on each side of the latter by the Riemann solution with suitable boundary condition on the discontinuity. In the following approximations this will, however, no longer be the case, which is connected with the appearance of a reflection from the surface of the discontinuity.

The position of the discontinuity in the wave is determined by a simple geometrical condition, which can easily be derived with the help of formula (2) and the condition of the continuity of the flow of the substance through the discontinuity. Namely, the discontinuity is characterised by the condition that the area of the curve, representing the profile of the wave should remain the same as that of the multivalued curve, determined by Riemann's solution.

Let us now consider a body, performing a steady motion with a velocity U exceeding that of sound. Let the x -axis be parallel to the direction of the motion of the body and let r be the distance from this axis. At large distances from the body the velocity potential $\varphi(r, z)$ of the gas is determined, in the first approximation, by the wave equation

$$\frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right).$$

The steadiness of the motion of the body is expressed by the equation

$$\frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x} = 0.$$

Combining these two equations we get

$$\left(\frac{U^2}{c_0^2} - 1 \right) \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right).$$

Replacing x by the variable

$$\tau = x / \sqrt{U^2 - c_0^2}$$

we obtain the equation

$$\frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial \tau^2} = \frac{1}{r} \left(r \frac{\partial \varphi}{\partial r} \right), \quad (3)$$

i.e. the equation of a cylindrical wave in which the role of the time is played by the variable τ .

At sufficiently large distances the cylindrical wave within each small region can be treated as a plane one. The displacement velocity of each point of the profile of the wave will then be determined by formula (2). If we wish, however, to follow with the help of this formula the displacement of a point of the wave

profile during large time intervals, it is necessary to take into account the fact that the amplitude of the cylindrical wave decreases with the distance already in the first approximation as $1/\sqrt{r}$. Writing

$$\frac{p'}{p_0} = \frac{\chi}{\sqrt{r}}, \quad (4)$$

and substituting in equation (2), we get

$$u = c_0 \left(1 + \frac{\alpha \chi}{\sqrt{r}} \right) \quad (5)$$

The first term corresponds to a displacement of the wave without change of profile (aside from the general decrease of the amplitude as $1/\sqrt{r}$), while the second leads to a distortion of the profile. The magnitude δr of the additional displacement of the point of the profile at a distance from a certain given (large) r_0 to r is obtained by multiplying by dr/c_0 and integrating from r to r_0 for a constant χ :

$$\delta r = 2 \alpha \chi (\sqrt{r} - \sqrt{r_0}).$$

If the profile of the wave is defined by the change of p' with τ , for a given r then the distortion $\delta \tau$ of the profile will be $\delta \tau = \delta r/c_0$, i.e.

$$\delta \tau = \frac{2 \alpha \chi}{c_0} (\sqrt{r} - \sqrt{r_0}). \quad (6)$$

A diverging cylindrical wave can be written, in the linear approximation, in the following way:

$$\varphi = \int_r^\infty \frac{f(\tau + \xi/c) d\xi}{\sqrt{\xi^2 - r^2}}. \quad (7)$$

The positive sign in $\tau + \xi/c$ corresponds to the fact that in the present case the wave is propagated from positive values to the negative ones (here and below the index zero, corresponding to the equilibrium values of the different quantities, is dropped for the sake of brevity). In our case the "time" τ means in reality the co-ordinate x . We shall take the origin of the co-ordinates inside the body (at a given instant), in this case the regions in front of the body correspond to positive x . Since in the case of a motion with a velocity exceeding that of sound the perturbations are not propagated in the region of space, lying in front of the body, it can, in any case, be asserted that $\varphi \rightarrow 0$ for $\tau \rightarrow \infty$. Further, at sufficiently large distances behind the body, where the perturbations caused by it are small, even on the axis itself, the potential of the divergent wave determined by formula (7) must remain finite for $r = 0$. For the convergence of the integral

$$\varphi(0, \tau) = \int_0^\infty f(\tau + \xi/c) \frac{d\xi}{\xi}$$

at the lower limit (for large negative τ) it is necessary, that $f(\tau) \rightarrow 0$ for large negative τ . Hence it is easy to conclude that for $\tau \rightarrow -\infty$ we have also $\varphi \rightarrow 0$.

On the other hand, the variable part of the pressure in the linear approximation is connected with φ by the equation $p' = -\rho c(\partial\varphi/\partial\tau)$. Integrating with respect to φ we get, consequently,

$$\int_{-\infty}^{+\infty} p' d\tau = 0 \quad (8)$$

This means that if there is a compression in the gas (the region with $p' > 0$), then there must also exist a region of rarefaction, where $p' < 0$ (in this respect a cylindrical wave—the same refers to a spherical one—differs substantially from a plane wave, which can consist of compressions or of rarefactions only).

As is well known, if the velocity of the body is larger than the velocity of sound, there arises in the gas a shock wave: in the space lying in front of this wave the gas remains at rest and directly behind the wave there is a

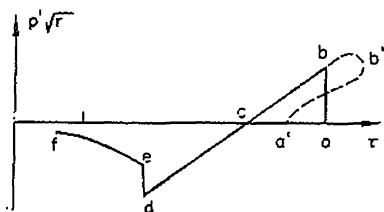


FIG. 1.

region of compression. It follows from (8) that the compression must necessarily be followed by a rarefaction and that, consequently, there must exist a point in which the rarefaction reaches its maximum; owing to the gradual distortion of the profile this point will lag with respect to those situated behind it; this will, finally, lead to a loss of the uniqueness of the solution and one more shock wave will arise.

We arrive at the result that, at least at large distances from the moving body, there exists not a single shock wave, as is usually assumed, but two shock waves following each other. In the first wave the pressure suffers a positive jump; thereafter the pressure is gradually decreased, the compression being replaced by a rarefaction, the pressure, finally, jumping again in the second shock wave.

Figure 1 represents schematically (by a continuous line) the resulting picture of the dependence of pressure p' on τ (i.e. on the co-ordinate x) for a given (large) value of r ; ab is the first shock wave, de —the second wave. In the latter the pressure increases up to a certain negative value only, p' tending to zero asymptotically when $\tau \rightarrow -\infty$.

Passing to the quantitative calculation of the profile represented in Fig. 1, let us consider the region between the two shock waves. Let the function $\tau = f(\chi)$ (where χ denotes $p'\sqrt{r/p}$) determine the profile at a certain distance r_0 .

Taking into account the effect of the distortion of the profile, we obtain the profile at a distance $r \gg r_0$ by adding to τ the displacement $\delta\tau$ from (6):

$$\tau = f(\chi) + \frac{2\alpha}{c} (\sqrt{r} - \sqrt{r_0}) \chi. \quad (9)$$

For large r χ is small, and one can with a sufficient accuracy write in (9) the value of the function $f(\chi)$ for $\chi = 0$, while $\sqrt{r_0}$ can be neglected when compared with \sqrt{r} :

$$\tau = \frac{2\alpha}{c} \sqrt{r} \chi + \text{const.} \quad (10)$$

Denoting by x_0 the value of the co-ordinate x at the point c (Fig. 1), where $\chi = 0$ (it depends, of course, on r , according to the law $x_0 = \text{const } r/c$) and passing to p' and x instead of χ and τ , we get hence

$$\frac{p'}{p} = \frac{1}{2\alpha \sqrt{\frac{U^2}{c^2} - 1}} \cdot \frac{x - x_0}{r}. \quad (11)$$

The segment ab of the profile thus proves to be rectilinear. The profile, which is obtained directly by the application of the Riemann solution throughout the whole interval, is represented in Fig. 1 by a dotted line. In reality there exists a discontinuity at a certain point a . The position of this point is determined by the above mentioned geometrical condition of the equality of the areas $a'b'c'$ and abc . Noting that in the points a' and c' $\chi = 0$, we find with the help of (9) for the area $a'b'c'$:

$$\int_{(c'b'a')} \chi d\tau = \int_{(c'b'a')} \chi f'(\chi) d\chi,$$

i.e. a value, which is independent of r . The same must, consequently, hold for the area abc . Taking into account the dependence (10) χ on τ we find without difficulty that the length l_1 of the segment from the point c (where $p' = 0$) up to the front shock wave ($p' = p_1$) is proportional to

$$l_1 \sim r^{1/4}. \quad (12)$$

Hence the dependence of the jump p'_1 of the pressure in the front shock wave on the distance is expressed by

$$p'_1 = \frac{\text{const}}{r^{3/4}}. \quad (13)$$

As regards the second discontinuity (ed in Fig. 1), it can easily be shown that the ratio of the pressure behind it (pressure at the point e) to the jump p'_2 of the pressure in the discontinuity (length of the segment ed) tends slowly

to unity for $r \rightarrow \infty$. The pressure behind this discontinuity can be regarded as equal to zero at very large distances r only; the jump p'_2 is equal here to p'_2 (the total area of the profile must be equal to zero in virtue of (8)).

Let us consider now the spherically symmetric propagation of a shock wave arising in the case of an explosion (and investigated at large distances from the place of the explosion). All the arguments here are exactly similar to those considered above.

In the case of a spherical propagation the amplitude of the wave decreases in the first approximation, as const/r , where r is now the distance from the centre. We, therefore, get for the velocity u of the displacement of the points of the profile instead of (5)

$$u = c \left(1 + \frac{\alpha \chi}{r} \right), \quad (14)$$

where χ denotes now the product

$$\chi = \frac{p'}{p} r. \quad (15)$$

We find accordingly for the displacement δr of the points of the profile on the path from a certain r_0 to r

$$\delta r = \alpha \chi \ln \frac{r}{r_0}.$$

If, however, the profile is considered as the course of the change of p' with the time τ , the distortion $\delta \tau$ is

$$\delta \tau = - \frac{\alpha \chi}{c} \ln \frac{r}{r_0}. \quad (16)$$

The distortion of the profile of a spherical wave is thus increased with the distance, according to the logarithmical law, i.e. much more slowly than in the case of a plane or a cylindrical wave (where it is proportional respectively to the first power, or to the square root of the distance). Inasmuch as the propagation of an actual wave in gas is accompanied by usual sound absorption, which is connected with the viscosity and the thermal conductivity, then, because of the slowness of the increase of distortion, the spherical sound wave can be absorbed, before the distortion of the profile will lead to the formation of discontinuities. In the particular case of the propagation of an explosive shock wave it may happen that the second shock wave by which (just as in the cylindrical case) it should be followed, may not appear because of the lack of time necessary for its formation.

Instead of equation (9) we now have the equation

$$t = f(\chi) - \frac{\alpha}{c} \ln \frac{r}{r_0} \cdot \chi. \quad (17)$$

Expanding $f(\chi)$ into series of powers of χ and limiting ourselves to the terms of the first order we get

$$t = -\frac{\alpha}{c} \chi \ln \frac{r}{a} + \text{const}, \quad (18)$$

where a is a certain constant. Hence we again obtain for p' a linear dependence of t in the form

$$\frac{p'}{p} = \frac{1}{\alpha} \frac{c(t_0 - t)}{r \ln \frac{r}{a}}. \quad (19)$$

Taking into account the law of conservation of the area, we get in the spherical case

$$l_1 \sim \sqrt{\ln \frac{r}{a}}, \quad p'_1 \sim \frac{1}{r \sqrt{\ln \frac{r}{a}}}. \quad (20)$$

61. ON THE VIBRATIONS OF THE ELECTRONIC PLASMA

The vibrations of the electronic plasma are considered, which arise as a result of an arbitrary initial non-equilibrium distribution in it. It is shown that the vibrations of the field in plasma are always damped, and the dependence of the frequency and of the damping decrement on the wave vector is determined for small and for large values of the latter.

The penetration of a periodical external electric field into the plasma is considered. The case of the frequency of the external field being almost at resonance with the proper frequency of plasma is considered separately.

THE high frequency vibrations of the electronic plasma are described by comparatively simple equations. If the frequency is high enough, the collisions of the electrons with the ions and with each other are inessential, and in the kinetic equation the collision integral can be neglected. The distribution function of ions can be considered as invariable, and only the distribution of electrons vibrates. Let $F(v, r, t)$ be the electronic distribution function, if $f_0(v)$ is the equilibrium function (the Maxwell distribution), then

$$F = f_0(v) + f(v, r, t) \quad (1)$$

f being a quantity small as compared with f_0 . The kinetic equation (without the collision integral) is

$$\frac{\partial f}{\partial t} + (v \cdot \nabla) f - \frac{e}{m} \left(\nabla \varphi \cdot \frac{\partial f_0}{\partial v} \right) = 0 \quad (2)$$

(φ —the electric field potential). The Poisson equation is

$$\nabla^2 \varphi = -4\pi e \int f d\tau \quad (d\tau = dv_x dv_y dv_z) \quad (3)$$

(the equilibrium electronic charge $e \int f_0 d\tau$ is of course compensated by the positive charge of the ions). Equations (2) and (3) form a complete set of equations.

These equations were used by A. A. Vlasov² for an investigation of the vibrations of plasma. However, most of his results turn out to be incorrect. Vlasov looked for solutions of the form $\text{const } e^{-i\omega t + i(k \cdot r)}$ and determined the dependence of the frequency ω on the wave vector k . The equation, which he obtained for this dependence contains a divergent integral; this already indicates the mathematical incorrectness of his method. Vlasov² (and also

Л. Ландау, О колебаниях электронной плазмы, *Журнал Экспериментальной и Теоретической Физики*, 16, 574 (1946).

L. Landau, On the vibrations of the electronic plasma, *J. Phys. U.S.S.R.* 10, 25 (1946).

Adirovich³⁾ tries to escape from this difficulty by taking the principal value of the integral involved, however, without any foundation. Actually there exists no definite dependence of ω on k at all, and for a given value of k arbitrary values of ω are possible. The fact that solutions of the form of $e^{-i\omega t + i(k \cdot r)}$ are insufficient can be seen already by observing that they give only a ∞^3 multiple of solutions (according to three independent parameters k_x, k_y, k_z), whereas there must actually exist a ∞^6 multitude of solutions (the equations contain six independent variables x, y, z, v_x, v_y, v_z).

1. THE VIBRATIONS WITH A GIVEN INITIAL DISTRIBUTION

In order to obtain a correct solution of equations (2) and (3), it is necessary to consider the problem concretely stated; we shall discuss here two of such problems.

Let us assume, that a definite (non-equilibrium) electronic distribution in a plasma is given in the initial moment. The problem is to determine the resulting vibrations. As equations (2) and (3) are linear and do not contain the co-ordinates explicitly, the function $f(r, v, t)$ can be expanded into a Fourier integral with respect to co-ordinates, and the equation can be written for every Fourier component separately. This means, that it is sufficient to consider solutions of the form

$$f_k(v, t) e^{i(k \cdot r)}$$

Further we shall, for the sake of convenience, omit the index k in f_k so that $f(v, t)$ will denote the Fourier component of the distribution function in question. By $g(v)$ we denote the Fourier component of the initial distribution $f(r, v, 0)$, we shall write shortly $g(v)$ for $g_k(v)$. Finally, we choose the x -axis along the direction of the considered value of the vector k .

Taking the Fourier components of equations (2) and (3), we obtain

$$\frac{\partial f}{\partial t} + i k v_x f - i k \frac{e}{m} \varphi \frac{\partial f_0}{\partial v_x} = 0, \quad (4)$$

$$k^2 \varphi(t) = 4\pi e \int f dv, \quad (5)$$

$\varphi(t)$ is the Fourier component of the potential $\varphi(r, t)$. These equations can be solved by using the operational method. Following this method, we introduce the function $f_p(v)$ defined by means of

$$f_p(v) = \int_0^\infty f(v, t) e^{-pt} dt; \quad (6)$$

then

$$f(v, t) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{+i\infty + \sigma} f_p(v) e^{pt} dp, \quad (7)$$

the integration being performed here in the plane of the complex variable p along a straight line parallel to the imaginary axis and passing to the right of it ($\sigma > 0$).

We multiply both sides of equation (4) by e^{-pt} and integrate over dt . Noting that

$$\int_0^{\infty} \frac{\partial f}{\partial t} e^{-pt} dt = f e^{-pt} \Big|_0^{\infty} + p \int_0^{\infty} f e^{-pt} dt = p f_p - g$$

(we insert $f(v, 0) \equiv g(v)$) we obtain

$$(p + i k v_x) f_p - i k \frac{e}{m} \varphi_p \frac{\partial f_0}{\partial v_x} = g.$$

In the same way (5) gives

$$k^2 \varphi_p = 4\pi e \int f_p d\tau.$$

The first of these equations yields

$$f_p(v) = \frac{1}{p + i k v_x} \left\{ g(v) + i k \frac{e}{m} \varphi_p \frac{\partial f_0(v)}{\partial v_x} \right\}, \quad (8)$$

and inserting this into the second one, we obtain for φ_p :

$$\varphi_p = \frac{4\pi e}{k^2} \frac{\int \frac{g(v)}{p + i k v_x} d\tau}{1 - \frac{4\pi i e^2}{k m} \int \frac{\partial f_0}{\partial v_x} \frac{d\tau}{(p + i k v_x)}}. \quad (9)$$

These formulae solve, in principle, the problem considered. They determine the electronic distribution and the electric field for an arbitrarily given initial distribution.

Before proceeding to the investigation of the formulae obtained, we note that in (9) the integration over $dv_y dv_z$ can be performed directly. Introducing for the following the notation $v_x \equiv u$ and

$$g(u) = \int g(v) dv_y dv_z$$

we write

$$\varphi_p = \frac{4\pi e}{k^2} \frac{\int_{-\infty}^{+\infty} \frac{g(u)}{p + i k u} du}{1 - \frac{4\pi i e^2}{k m} \int_{-\infty}^{+\infty} \frac{df_0(u)}{du} \frac{du}{(p + i k u)}}, \quad (10)$$

the equilibrium function being

$$f_0(u) = n \sqrt{\frac{m}{2\pi\kappa T}} e^{-\frac{mu^2}{2\kappa T}} \quad (11)$$

(κ —the Boltzmann constant, n —the equilibrium number of electrons per unit volume of the plasma).

An expression of the type of

$$\varphi_p = \int_0^\infty \varphi(t) e^{-pt} dt,$$

considered as a function of the complex variable p has a sense only in the right half-plane, i.e. for $\text{Re}(p) > 0$. The same refers correspondingly to the expression (10). However, we can define φ on the left half-plane as the analytical continuation of expression (10). It is easy to see, that if $g(u)$ (considered as a function of the complex variable u) is an entire function of u (i.e. it has no singularities at finite u), then the integral

$$\int_{-\infty}^{+\infty} \frac{g(u) du}{p + iku},$$

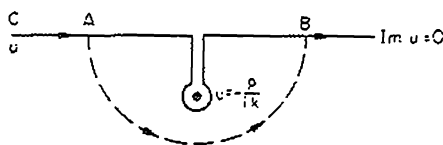


FIG. 1.

continued analytically into the left half-plane of p also defines an entire function of p . Actually, to perform the analytical continuation of the function, defined by this integral, from the right half-plane to the left one, we displace the integration path in the complex plane of u far enough into the lower half-plane so, that the point $u = -p/ik$ would lie above it. In this way we shall obtain an analytical function, defined by the integral which for $\text{Re}(p) > 0$, is taken along the real axis, and for $\text{Re}(p) < 0$ along the path, which is drawn in Fig. 1 by a full line. This function has no singularities at finite values of p , i.e. it is an entire function.

The same refers also to the integral in the denominator of expression (10), for $df_0(u)/du$ is an entire function. Thus, an analytical, in the whole plane, function φ_p is (if $g(u)$ is entire) a ratio of two entire functions. Hence the only singularities (poles) of the function φ_p are the zeros of the denominator in (10); all of these poles lie in the left half-plane.

These considerations allow us to determine the asymptotical form of the potential $\varphi(t)$ for large values of the time t . In the inversion formula

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{+i\infty + \sigma} \varphi_p e^{pt} dp \quad (12)$$

the integration is performed along a vertical line in the right half-plane. However, if φ_p is defined in the manner indicated above as a function which is analytical in the whole plane of p , we can displace the integration path into the left half-plane going around all the poles of φ_p it meets. Let p_k be that of the poles of φ_p , i.e. that of the roots of the equation

$$\frac{4\pi i e^2}{k m} \int_C \frac{df_0}{du} \frac{du}{(p + i k u)} = 1 \quad (13)$$

(integration along the path shown in Fig. 1), which has the least absolute value of its real part (i.e. which is the nearest to the imaginary axis). Let us perform the integration in (12) along the path, which is displaced far enough to the left and goes around the point $p = p_k$ in the manner shown in Fig. 2. Then in the integral (12) (with large values of t) only the residue relative to the pole p_k will be of importance. All other parts of the integral (among them

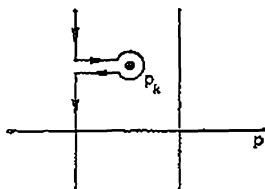


FIG. 2.

the integral along the vertical line) will be exponentially small in comparison with the residue due to the presence of the factor e^{pt} in the integrated expression, which decreases rapidly with increasing $|\operatorname{Re}(p)|$.

Thus, for large values of t the potential of the field $\varphi(t)$ is proportional to $e^{p_k t}$. With complex p_k this factor splits into a periodical part and a decreasing ($\operatorname{Re}(p) < 0$) one. We arrive, consequently, at an essential result, that the field is damped with time, the damping decrement being equal to $-\operatorname{Re}(p_k)$.

Equation (13) determines p_k , i.e. the frequency and the damping decrement of the vibrations. It coincides formally with Vlasov's equations, the difference being that here the integration is performed along the path C , whilst Vlasov integrates simply along the real axis. This difference leads, as we shall see, to qualitatively new results, namely to the presence of damping.

Consider the limiting case of long waves, $k \rightarrow 0$. The point $u = -p/ik$ (Fig. 1) is displaced to very large $|u|$ and as the function $f_0(u)$ decreases rapidly with increasing $|u|$, we can integrate in (13), in the first approximation, only along the real axis. We expand the integrand in powers of k . The first term of the expansion disappears because

$$\int_{-\infty}^{+\infty} \frac{df_0}{du} du = f_0 \Big|_{-\infty}^{+\infty} = 0.$$

The second term gives

$$\frac{4\pi e^2}{p^2 m} \int_{-\infty}^{+\infty} u \frac{df_0}{du} du = 1$$

Taking into account that

$$\int_{-\infty}^{+\infty} u \frac{df_0}{du} du = u f_0 \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f_0 du = -n_1 \quad (14)$$

we find

$$p_k = -i\omega, \quad \omega = \sqrt{\frac{4\pi n e^2}{m}} \equiv \omega_0 \quad (15)$$

(we have chosen here the sign ω which corresponds to a wave, propagating in the positive direction of the x -axis). This expression corresponds to the ordinary proper frequency of plasma; we denote it by ω_0 . In the next approximation the calculation leads to the following dependence of the frequency on the wave vector:

$$\omega = \omega_0 \left(1 + \frac{3}{2} a^2 k^2 \right) \quad (16)$$

$a = \sqrt{\kappa T / 4\pi n e^2}$ being the electronic Debye-Hückel radius. We omit here the detailed calculations because they coincide with that of Vlasov done in his first paper¹. This part of his calculations turns out to be correct due to the fact, that in calculating the frequency for small values of k , we can approximately integrate in (13) only along the real axis.

However, the vibrations are actually damped, although the damping coefficient is small for small k . To calculate this decrement we start from an assumption (which is verified by the result), that for $k \rightarrow 0$ the real part of p_k tends to zero, the imaginary part remaining finite. Hence for small k the point $u = -p_k/i k$ (Fig. 1) is situated at a finite distance from the imaginary axis and very near to the real one (under the latter). Let

$$p_k = -i\omega - \gamma,$$

γ is the damping coefficient in question ($0 < \gamma \ll \omega$). We choose a point A on the real axis (Fig. 1), situated not far from the point $u = -p_k/i k$, but so, that its distance from this point is still large as compared with $|\text{Im}(u)|$. Then we draw a semicircle AB through this point (shown by a dotted line in Fig. 1) and use it instead of the corresponding part of the integration path C .

The integral along the straight parts of the integration path is real in the limiting case of $\text{Re}(p) = 0$, in the approximation considered we can put it equal to $-4\pi n e^2 / m p^2$. As to the integral along the semicircle, it equals the residue relative to the pole, multiplied by πi (a half of the total circle!). In

this way we obtain equation (13) in the form

$$-\frac{4\pi n e^2}{m p^2} + i \frac{4\pi^2 e^2}{m k^2} \frac{df_0\left(-\frac{p}{ik}\right)}{du} = 1.$$

Putting here $p = -i\omega - \gamma$ and solving the equation by means of successive approximations, we get finally the following expression for the damping decrement:

$$\gamma = \omega_0 \sqrt{\frac{\pi}{8}} \frac{1}{(ka)^3} e^{-1/2(ka)^2}. \quad (17)$$

Thus, the damping decrement decreases exponentially with decreasing k .

Formulae (15-17) are valid for $\gamma \ll \omega$. This condition leads to the inequality

$$ka \ll 1.$$

Consider now the opposite limiting case of large k . We put again $p = -i\omega - \gamma$. It will be verified by the result, that both ω and γ increase indefinitely with $k \rightarrow \infty$ but in such a way, that for large k , $\omega \ll \gamma$ and the ratios ω/k , γ/k tend to zero and infinity respectively. Then the pole $u = -p/ik$ is situated relatively near to the imaginary, but far from the real axis ($\text{Re}(u)$ is small, $\text{Im}(u)$ is large). As the function f_0 increases exponentially for large imaginary values of u , we can integrate in (13) only along the circle around the pole, neglecting the integral along the real axis. In this way we obtain from (13)

$$\frac{4\pi e^2}{m k^2} 2\pi i \frac{df_0\left(-\frac{p}{ik}\right)}{du} = 1.$$

or, using expression (11), for $f_0(u)$

$$\sqrt{2\pi} \frac{p}{\omega_0 (ka)^3} e^{\frac{p^2}{2\omega_0^2 (ka)^2}} = 1. \quad (18)$$

By taking the moduli of the expression on the both sides of the equation, and using the suggested inequality $\gamma \gg \omega$, we get

$$\xi e^{\xi^2/2} = \frac{1}{\sqrt{2\pi}} (a k)^2 \quad (19)$$

with

$$\xi = \gamma/\omega_0 k a.$$

The phase factor of the expression in the left side of equation (18) is equal, in the same approximation, to

$$-\exp\left(\frac{i\gamma\omega}{\omega_0^2 a^2 k^2}\right).$$

As on the right of the equation stands a real positive quantity, this factor must be equal to +1. Hence we find:

$$\frac{\gamma \omega}{\omega_0^2 a^2 k^2} = \pi$$

(it can be shown, that by equating it to 3π , 5π , we would get a root of the equation (13), which is not the nearest one to the imaginary axis). Together with the definition of the quantity ξ this gives

$$\omega = \pi \sqrt{\frac{\kappa T}{m}} \frac{k}{\xi}, \quad \gamma = \sqrt{\frac{\kappa T}{m}} k \xi. \quad (20)$$

These formulae determine the frequency and the damping decrement of the vibrations, the function $\xi(k)$ being defined implicitly by equation (19). $\xi(k)$ is a slowly increasing function of k (it goes approximately as $\sqrt{\ln k a}$). The ratio γ/ω increases with k as ξ^2 , i.e. approximately as $\ln k a$.

In the preceding calculations we supposed, that the given function $g(u)$ is an entire function. If this function has singularities, then φ_p will also possess singularities apart from the poles, which are zeros of the denominator in (10). The point p_k in Fig. 2, which determines the behaviour of the potential $\varphi(t)$ for large t , must be chosen as the nearest to the imaginary axis of all the roots of equation (13) and of the singularities, which arise from the singular points of $g(u)$.

In particular, if $g(u)$ is (on the real axis) a continuous function with a discontinuous derivative, then φ_p will have purely imaginary singular points $p = -i k u_s$; u_s being the discontinuity points of $g(u)$. Thus, the behaviour of $\varphi(t)$ for large t will be determined by purely imaginary values of p_k , i.e. there will be no damping of the field. Hence it follows, that it is by no means possible to use a curve with angles (e.g. composed of straight pieces) for $g(u)$ instead of a smooth one in order to get an approximate solution of a given problem. Such a substitution will lead to a qualitatively incorrect picture with an undamped field vibration.

Finally, it is necessary to discuss the electronic distribution function itself. For the distribution function, integrated over $dv_y dv_z$ we have, according to (8):

$$f_p(u) = \frac{1}{p + i k u} \left\{ g(u) + \frac{i k e}{m} \varphi_p \frac{df_0(u)}{du} \right\},$$

$$f(u, t) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{+i\infty + \sigma} f_p(u) e^{pt} dt.$$

The behaviour of the function $f(u, t)$ for large t is determined by the purely imaginary singular point $p = -i k u$ of the function $f_p(u)$. Thus, the distribution function turns out to be proportional (for large t) to a periodical factor $e^{-i k u t}$, i.e. it performs undamped vibrations with a frequency $k u$ which depends on the velocity u .

2. THE VIBRATIONS OF A PLASMA IN A EXTERNAL ELECTRIC FIELD

Suppose, the plasma is placed into an external periodical electric field. The problem is to find the law of the penetration of the field inside the plasma. The external field can be expanded into a Fourier integral with respect to time; therefore, we can confine ourselves to the consideration of a monochromatic field of a frequency ω . We suppose that the plasma is bounded by a plane wall; the distribution is a function only of one co-ordinate, say x , along the axis, perpendicular to the wall.

The electric field can be split into a longitudinal part, directed along the x -axis, and a transversal part p which is parallel to the plane of the wall. There is no need to consider the transversal field, because the behaviour of a plasma in a transverse electromagnetic wave is described by well-known formulae. Therefore, we confine ourselves to the case of a longitudinal field.

As in section 1, we use the distribution function, integrated over the essential variables v_y, v_z . We can look for this function $f(u, x, t)$ in the form of $f(u, x) e^{-i\omega t}$ (u denotes, as above v_x).

The kinetic equation (2) becomes now

$$-i\omega f + u \frac{\partial f}{\partial x} + \frac{eE(x)}{m} \frac{df_0(u)}{du} = 0 \quad (21)$$

(we write the electric field in the form $E(x) e^{-i\omega t}$). As a second equation it is convenient to use here (instead of the Poisson equation (3)) the equation, which expresses the absence of the sources for the total current (the real current j and the displacement current):

$$\operatorname{div} \left(j - \frac{i\omega}{4\pi} E \right) = \frac{d}{dx} \left(j - \frac{i\omega}{4\pi} E \right) = 0.$$

Hence we find that $4\pi j - i\omega E$ is a constant. Outside the plasma $j = 0$; therefore, this constant equals $-i\omega E_0$ where $E_0 e^{-i\omega t}$ is the external field. Thus, we have an equation

$$-i\omega E(x) + 4\pi j(x) = -i\omega E_0. \quad (22)$$

The current density $j(x)$ can be expressed through the distribution function by means of

$$j = e \int_{-\infty}^{+\infty} u f(u, x) du. \quad (23)$$

At large distances from the wall the field E in the plasma is determined directly by the condition of the constancy of the longitudinal component of the induction $D = \epsilon E$, the electric constant ϵ of the plasma being equal to the well-known expression

$$\epsilon = 1 - \frac{4\pi n e^2}{m \omega^2}. \quad (24)$$

Outside the plasma $D = E_0$; hence the boundary condition at infinity is

$$E = \frac{E_0}{\varepsilon} \quad \text{for } x = +\infty \quad (25)$$

(the positive direction of the x -axis is into the plasma).

As to the properties at the wall, we shall suppose (as is usually done in analogous cases), that it has an ideal reflection power. This means that an electron, colliding with the wall, is reflected under the angle, equal to that of the incidence, and with the unchanged absolute value of its velocity (so that v_y, v_z remain unchanged, and $v_x = u$ changes its sign). Then the distribution function must satisfy on the wall ($x = 0$) the boundary condition

$$f(u, 0) = f(-u, 0). \quad (26)$$

We integrate formally equation (21) and find:

$$f(u, x) = -e^{i\omega x/u} \int \frac{e E(x)}{m u} \frac{df_0}{du} e^{-i\omega x/u} dx.$$

In order to determine the integration constant, we proceed in the following way. Consider ω as a complex parameter with a small positive imaginary part (which tends in the following to zero). Then the external field $E_0 e^{-i\omega t}$ increases with time, but as it is finite for every finite value of t , the distribution function must also be everywhere (for all $x = \infty$) finite.

If $u < 0$ then the factor $e^{i\omega x/u}$ increases indefinitely with x , and in order that $f(u, \infty)$ remains finite we must write for $u < 0$:

$$f(u, x) = e^{i\omega x/u} \int_x^\infty \frac{e E(\xi)}{m u} \frac{df_0(u)}{du} e^{-i\omega \xi/u} d\xi. \quad (27)$$

As to the function $f(u, x)$ for $u > 0$ it must be written so, as to fulfil the condition (26). This gives for $u > 0$:

$$f(u, x) = e^{i\omega x/u} \left[\int_0^\infty \frac{e E(\xi)}{m u} \frac{df_0(u)}{du} e^{i\omega \xi/u} d\xi - \int_0^x \frac{e E(\xi)}{m u} \frac{df_0}{du} e^{-i\omega \xi/u} d\xi \right] \quad (28)$$

(it is to be remembered, that $f_0(u)$ is an even function of u , hence df_0/du is an odd function).

Using the obtained expressions, we calculate the current density (23):

$$j = \frac{i\omega}{4\pi} \left\{ \int_0^x E(\xi) K(x - \xi) d\xi + \int_x^\infty E(\xi) K(\xi - x) d\xi - \int_0^\infty E(\xi) K(x + \xi) d\xi \right\}, \quad (29)$$

where the function $K(\xi)$ is defined by means of

$$K(\xi) = \frac{4\pi i e^2}{m \omega} \int_0^\infty \frac{df_0}{du} e^{u\xi/u} du, \quad \xi > 0 \quad (30)$$

((29) contains $K(\xi)$ only for positive values of the argument).

In the following it is convenient to split $E(x)$ into two terms, separating the value of the field for $x \rightarrow +\infty$:

$$E(x) = \frac{E_0}{\varepsilon} + E_1(x). \quad (31)$$

According to (25), $E_1(x)$ satisfies the boundary condition $E_1(\infty) = 0$. Inserting (31) into (29), we obtain easily:

$$j = j_1(x) + \frac{i\omega}{2\pi\varepsilon} E_0 \int_0^x K(\xi) d\xi \quad (32)$$

$j_1(x)$ defined by (29) with $E_1(x)$ standing instead of $E(x)$.

Inserting (31), (32) into (22) and performing some elementary transformations, we obtain the following integral equation for the function $E_1(x)$

$$\begin{aligned} E_1(x) - \int_0^x K(x-\xi) E_1(\xi) d\xi - \int_x^\infty K(\xi-x) E_1(\xi) d\xi + \int_0^\infty K(\xi+x) E_1(\xi) d\xi \\ = -\frac{2E_0}{\varepsilon} \int_0^\infty K(\xi) d\xi. \end{aligned} \quad (33)$$

In calculations we used here expression (24) for ε and the expression for the integral $\int_0^\infty K(\xi) d\xi$ which can be obtained in the following way. Consider again ω as a complex parameter with $\text{Im } \omega > 0$. Then $e^{i\omega\xi/u}$ is zero for $\xi = \infty$, and integrating over $d\xi$ under the sign of integral in (30), we get

$$\int_0^\infty K(\xi) d\xi = -\frac{4\pi e^2}{m\omega^2} \int u \frac{df_0}{du} du.$$

The integrand $u(df_0/du)$ is an even function of u hence this integral is one half of the integral (14). Finally,

$$\int_0^\infty K(\xi) d\xi = \frac{2\pi e^2 n}{m\omega^2}. \quad (34)$$

The integral equation (33) can be solved in the following way. The function $E_1(x)$ has a physical meaning only inside the plasma, i.e. for $x > 0$. We continue this function, and also the function $K(\xi)$ into the region of negative values of the argument by means of the definitions:

$$K(-\xi) = K(\xi), E_1(-x) = -E_1(x) \quad (35)$$

(the function $E_1(x)$, thus defined, has a discontinuity at $x = 0$). Then equation (33) after a simple transformation is reduced to a simpler form:

$$E_1(x) - \int_{-\infty}^{+\infty} K(x - \xi) E_1(\xi) d\xi = \begin{cases} -\frac{2E_0}{\varepsilon} \int_x^{\infty} K(\xi) d\xi & \text{for } x > 0, \\ \frac{2E_0}{\varepsilon} \int_{-\infty}^x K(\xi) d\xi & \text{for } x < 0. \end{cases} \quad (36)$$

In this form it can be solved by using the Fourier method. Multiplying both sides of the equation by e^{-ikx} and integrating over dx within the limits between $-\infty$ and $+\infty$, we obtain:

$$E_{1k}(1 - K_k) = \frac{2i E_0}{\varepsilon} \frac{K_0 - K_k}{k},$$

E_{1k} , K_k being the Fourier components:

$$E_{1k} = \int_{-\infty}^{+\infty} E_1(x) e^{-ikx} dx, \quad K_k = \int_{-\infty}^{+\infty} K(\xi) e^{-ik\xi} d\xi$$

(K_0 is the value of K_k for $k = 0$). By means of the inverse transformation

$$E_1(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{1k} e^{ikx} dk$$

we get the function $E_1(x)$ in question in the form of an integral:

$$E_1(x) = \frac{i E_0}{\pi \varepsilon} \int_{-\infty}^{+\infty} \frac{K_0 - K_k}{k(1 - K_k)} e^{ikx} dk. \quad (37)$$

The function K_k can be presented in the following form:

$$K_k = \frac{4\pi e^2}{m \omega} \int_{-\infty}^{+\infty} \frac{u \frac{df_0}{du}}{ku - \omega} du \quad (38)$$

(we used the definitions (30), (35) and the integration over $d\xi$ is performed under the sign of the integral over du with ω considered again as complex with $\text{Im } \omega > 0$). If this integral is taken simply along the real axis, it diverges at the point $u = \omega/k$. However, it is easy to see which must be actually the path of integration. In deducing (38) we assumed that $\text{Im } \omega > 0$ and the integral was taken along the real axis, i.e. along a path, passing below (if $k > 0$), or above (if $k < 0$) the singular point $u = \omega/k$. Therefore, after putting $\text{Im } \omega$ equal to zero, the integral (38) must be taken (if $k > 0$) along the path C_1

(Fig. 3), which proceeds along the real axis and goes around the singular point below it, or (if $k < 0$) along the path C_2 which goes around the singular point above it.

We introduce the notations:

$$K_k = K_1(k) \quad \text{for } k > 0; \quad K_k = K_2(k) \quad \text{for } k < 0. \quad (39)$$

The functions $K_1(k)$ and $K_2(k)$, defined formally by (38) with the integral taken along the path C_1 or C_2 , are analytical functions in all the plane of the complex variable k .

Expression (37) is inconvenient for calculations. Introducing the functions K_1 , K_2 we can represent it, after a simple transformation, in the form of

$$\begin{aligned} E_1(x) = \frac{iE_0}{\pi\varepsilon} \int_{-\infty}^{+\infty} \frac{K_0 - K_2(k)}{k[1 - K_2(k)]} e^{ikx} dk \\ + \frac{iE_0}{\pi\varepsilon} \int_{-\infty}^{+\infty} \frac{K_2(k) - K_1(k)}{k[1 - K_1(k)][1 - K_2(k)]} e^{ikx} dk. \end{aligned} \quad (40)$$

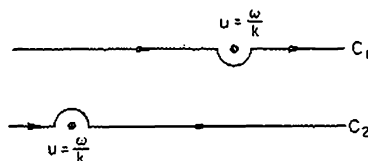


FIG. 3.

In this transformation we used the fact that according to (24), (38) and (14) we have

$$K_0 = 1 - \varepsilon. \quad (41)$$

The difference $K_2(k) - K_1(k)$ is evidently expressed by the same formula (38), the integration being performed simply along a closed contour enclosing the pole (in the negative direction). According to the theorem of the residues, we have, consequently,

$$K_2(k) - K_1(k) = -\frac{4\pi e^2}{m\omega k} 2\pi i \left(u \frac{df_0}{du} \right)_{u=\omega/k} \quad (42)$$

or

$$K_2(k) - K_1(k) = \frac{i\sqrt{2\pi\omega}}{\omega_0 a^3 k^3} e^{-\frac{\omega^2}{2\omega_0^2 a^2 k^2}}.$$

It is easy to see, that the functions K_1 , K_2 are connected with each other by means of the following relations:

$$[K_2(k)]^* = K_1(k^*), \quad K_1^*(-k^*) = K_1(k), \quad K_2^*(-k^*) = K_2(k). \quad (43)$$

At infinity both functions K_1, K_2 , vanish. An investigation which we omit here, shows, that the functions $K_1(k), K_2(k)$ have in the whole plane of the complex variable k only one singular point—namely, an essential singularity at $k = 0$. The quantity K_0 is the limit to which K_1, K_2 tend when k tends to zero along the real axis. It can also be shown, that $K_1(k)$ tends to the same limit K_0 when k tends to zero along an arbitrary path, passing outside a right-angled sector in the upper half-plane, bounded by two straight lines, which intersect at the co-ordinate origin and make an angle of 45° with the imaginary axis. The same holds for $K_2(k)$ outside an analogous sector in the lower half-plane.

In the integrals (40) those points are of importance, at which K_1, K_2 are equal to unity. It can be shown, that the equation $K_1(k) = 1$ has an infinite number of roots in the upper half-plane, which converge to a condensation point at $k = 0$. In the lower half-plane there are no roots at all if $\varepsilon > 0$ (i.e. if $K_0 < 1$), or there is one root on the imaginary axis if $\varepsilon < 0$ (i.e. if $K_0 > 1$). Analogous results for the function $K_2(k)$ follow directly from the relations (43): the equation $K_2(k) = 1$ has an infinite number of roots in the lower half-plane, and has no roots at all (if $\varepsilon > 0$), or has one root on the imaginary axis (if $\varepsilon < 0$) in the upper one.

Consequently, if $\varepsilon > 0$ the integrand of the first integral in (40) has no poles in the upper half-plane and by pushing the path of integration to infinity in this half-plane, we find, that the integral vanishes. If, on the other hand, $\varepsilon < 0$, there is a pole in the upper half-plane and the integral is reduced to the residue relative to this pole. Its dependence on x is, consequently, given by an exponentially decreasing factor $e^{-\alpha x}$, $\alpha \geq 0$.

A complete evaluation of the integrals in (40) can be performed only numerically. It is, however, possible to obtain an asymptotical expression, which determines $E_1(x)$ for large values of x ($x \gg a$). We shall see, that in this region the second integral in (40) is larger as compared with the first one and we must calculate only it. We shall do it with the aid of the well-known "method of steepest descent". Inserting (42) into (40) we obtain in the integrand an exponential factor

$$\exp \left\{ -\frac{1}{2} \left(\frac{\omega}{\omega_0 a k} \right)^2 + i k x \right\}.$$

Following the method of steepest descent we expand the exponent in powers of $\delta k = k - k_0$ where

$$k_0 = e^{\frac{3}{2} \sqrt{\frac{\omega^2}{\omega_0^2 a^2 x}}} e^{i\pi/6}$$

is the extremum point of the exponent, and then integrate along the path of "the steepest descent". In the non-exponential factor we can put $k = k_0$ and take it out of the integration sign. In the denominator we can put $1 - K_2(k_0) = 1 - K_1(k_0) \cong 1 - K_0 = \varepsilon$ (k_0 is small for large x). After a simple calculation we obtain the following final result

$$E_1(x) = \frac{2E_0}{\sqrt{3\varepsilon^2}} \left(\frac{\omega}{\omega_0} \right)^{4/3} \left(\frac{x}{a} \right)^{2/3} e^{-\frac{3}{4} \left(\frac{\omega x}{\omega_0 a} \right)^{2/3}} e^{i \left[\frac{3\sqrt{3}}{4} \left(\frac{\omega x}{\omega_0 a} \right)^{2/3} + \frac{2\pi}{3} \right]}. \quad (44)$$

Thus, the field $E_1(x)$ decreases according to an exponential law with $x^{2/3}$ in the exponent (as to the first term in (40), we have seen that it decreases according to a stronger law $e^{-\alpha x}$ and is, consequently, insignificant for large x). Expression (44) contains also a periodical factor.

The case of the frequency ω , being nearly at resonance with the proper frequency of the plasma, needs a separate consideration. The dielectric constant is here small, $|\varepsilon| \ll 1$ (and is connected with the frequency by means of a simple relation $\varepsilon = 2(\omega - \omega_0/\omega_0)$). The calculations proceed differently for $\varepsilon < 0$ and for $\varepsilon > 0$.

Suppose first that ε is small and negative. We have seen, that for $\varepsilon < 0$ the first term in (40) decreases as $e^{-\alpha x}$, i.e. faster than the second one. But with $|\varepsilon| \ll 1$ the coefficient α turns out to be small, and therefore, the second term becomes predominant only for very large x ; for smaller values of x the first term prevails.

We shall see, that the integrand of the first term has (for small $|\varepsilon|$) a pole, lying on the imaginary axis near to the co-ordinate origin (we are speaking of the only root of the equation $K_2(k) = 1$ in the upper half-plane). To calculate this root we can, therefore, expand $K_2(k)$ in powers of k . As to the path of integration C_2 in the integral (38), which defines $K_2(k)$, it is reduced simply to the whole real axis—this path passes above the singular point $u = \omega/k$ (which lies now on the negative half of the imaginary axis). A simple calculation gives in the second approximation

$$K_2(k) = 1 - \varepsilon + 3(ka)^2.$$

Hence we find for the root of the equation $K_2(k) = 1$:

$$k = \frac{i}{a} \sqrt{\frac{|\varepsilon|}{3}}.$$

Evaluating the first integral (40) as the residue relative to this pole, we find, finally, the following expression for the total field $E(x)$

$$E(x) = \frac{E_0}{\varepsilon} \left(1 - e^{-\frac{x}{a} \sqrt{\frac{|\varepsilon|}{3}}} \right). \quad (45)$$

Thus, if ε is small and negative, the field increases monotonically, according to a simple exponential law, tending to the limit E_0/ε . For $x = 0$ (45) gives $E(x) = 0$ instead of the correct value E_0 , this is connected with the fact that in the adopted approximation the quantities of the order of ε are neglected.

Consider, finally, the case of small positive values of ε . For $\varepsilon > 0$ the first term in (40) vanishes. However, the second integral contains, except the expression (44), also a term, which decreases according to a law $e^{-\alpha x}$. For very small ε this term becomes predominant for all values of x , except the largest. This term is due to the residue relative to the integrand, which lies in the upper half-plane near the real axis. It turns out, that among the infinite

number of the roots of the equation $K_1(k) = 1$ in the upper half-plane there exists one, which lies (for small ε) very near to the real axis. Expanding the $K_1(k)$ in powers of k , it is easy to obtain the following expression for the root in question:

$$k = \frac{1}{a} \left[\sqrt{\frac{\varepsilon}{3}} + i \frac{3}{2\varepsilon^2} \sqrt{\frac{\pi}{2}} e^{-3\varepsilon/2} \right].$$

Calculating the residue relative to this pole, we obtain, finally, the following expression for the field:

$$E(x) = \frac{E_0}{\varepsilon} \left[1 - \exp \left\{ \frac{i}{a} \sqrt{\frac{\varepsilon}{3}} x - \frac{3}{2a} \sqrt{\frac{\pi}{2\varepsilon}} e^{-3x/2\varepsilon} \right\} \right]. \quad (46)$$

Thus, in this case we find that the amplitude of the field increases, first, from zero (actually from E_0) up to $2E_0/\varepsilon$, and then it performs damped oscillations (with a very small damping decrement) around the value E_0/ε to which it tends on large distances.

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62. ON THE THERMODYNAMICS OF PHOTOLUMINESCENCE

The author derives the conditions imposed by thermodynamics upon the total energy yield of photoluminescence and the intensity of radiation in the anti-Stokes region.

THERMODYNAMICS imposes essential restrictions on the process of photoluminescence, giving rise to certain inequalities which must be satisfied in all cases.

According to the second law of thermodynamics the entropy must increase in an isolated system, which in the case under examination is a system consisting of luminescent body + radiation. Therefore, to write these inequalities, the entropy of the radiation as such must be included in the discussion of the problem in question.

The distribution of radiation with respect to frequencies and directions can be described by the density $\varrho(\nu, n)$ of the energy. $\varrho(\nu, n) d\nu d\omega$ is the energy of the radiation, referred to a unit volume, in the interval $d\nu$ of frequencies, possessing a direction n in the element $d\omega$ of the solid angle. Instead of the function $\varrho(\nu, n)$ it is convenient to introduce a function $f(\nu, n)$ defined as follows:

$$\varrho(\nu, n) d\nu d\omega = f(\nu, n) h\nu \frac{2\nu^2 d\nu d\omega}{c^3}, \quad (1)$$

$h\nu$ being the energy of the photons and $2\nu^2 d\nu d\omega/c^3 = 2k^2 dk d\omega$ —the “element of the volume” in the “space” of the wave vector k . (The factor 2 stands for two possible directions of polarization; for the sake of brevity we shall not consider distribution with respect to polarization altogether, as it does not have a substantial effect on the following discussion.) $f(\nu, n)$ is the number of photons per unit volume of space, referred to unit volume of the “ k -space”. The intensity dI of the radiation emitted by a unit of the surface of the considered body will be determined by

$$dI = c \varrho \cos\theta d\nu d\omega = f(\nu, n) h\nu d\tau, \quad (2)$$

where we have introduced the notation

$$d\tau = \cos\theta \frac{2\nu^2 d\nu d\omega}{c^2} \quad (3)$$

(θ is the angle between n and the normal to the surface).

The radiation entropy is determined by the well-known formulae of Bose statistics applied to a “photon gas”. The entropy S of the radiation passing

L. Landau, On the thermodynamics of photoluminescence, *J. Phys. U.S.S.R.* 10, 503 (1946).

through an area of 1 cm^2 in 1 sec is equal to

$$S = \kappa \int [(f + 1) \ln(f + 1) - f \ln f] d\tau, \quad (4)$$

the integration being extended over the total spectrum of frequencies and over all directions n ; κ is the Boltzmann constant.

If the total radiation spectrum is concentrated in a small interval $\Delta\nu$ of frequencies, and the directions—in the small interval $\Delta\phi$ of angles, then, evidently, the total radiation intensity is

$$I = \int dI \sim \bar{f} h \nu \Delta\tau, \quad (5)$$

where \bar{f} is a certain mean value of f , and

$$\Delta\tau = \frac{2\nu^2 \cos\theta \Delta\nu \Delta\phi}{c^2}.$$

For the entropy we obtain:

$$S \sim \kappa [(\bar{f} + 1) \ln(\bar{f} + 1) - \bar{f} \ln \bar{f}] \Delta\tau = \frac{\kappa}{h\nu} \frac{(\bar{f} + 1) \ln(\bar{f} + 1) - \bar{f} \ln \bar{f}}{\bar{f}} I. \quad (6)$$

If $\Delta\tau$ tends to zero for a given total intensity I , \bar{f} tends to infinity. But in the case of large values of f

$$(f + 1) \ln(f + 1) - f \ln f$$

is approximately equal to $\ln f + 1$ and $(\ln f + 1)/f$ tends to zero. Therefore, it follows from (6) that S tends to zero. The condition that $\Delta\tau$ tends to zero does not require that $\Delta\nu$ and $\Delta\phi$ should simultaneously tend to zero. It is sufficient that one of them should tend to zero. In other words, both the entropy of the strictly monochromatic ($\Delta\nu = 0$) and the entropy of the strictly directed ($\Delta\phi = 0$) radiation are equal to zero.

Let us now turn our attention to the process of photoluminescence. We shall assume, for the sake of brevity, that luminescence is accompanied by complete light scattering over all directions, i.e., that the light emitted by the body is isotropic with respect to direction and the luminescence does not depend upon the direction of the incident light. Then the process of luminescence will be specified by a certain function $w(\nu', \nu)$, which determines the intensity of radiation of light with a frequency ν upon irradiation by light with a frequency ν' .

Let us consider the restrictions imposed by thermodynamics on the properties of this function for a certain value of ν' . With this purpose in mind we may formally consider the body as being irradiated by strictly monochromatic light (frequency ν' and intensity I_a). The light that is emitted has a spectrum which is determined by the function $w(\nu', \nu)$. If the emitted light is either sufficiently monochromatic or consists of separate sufficiently narrow spectral lines, then the light entropy will equal zero, and it suffices to consider the

change in the body entropy. But this, obviously, in a unit of time (referred to a unit of the body surface) will be

$$\frac{I_a - I_l}{T},$$

where T is the temperature of the body, and I_l —the luminescence intensity. Hence in this case the second law leads to the inequality

$$I_l \leq I_a, \quad (7)$$

i.e. the relative luminescence yield cannot exceed unity.

If, however, upon irradiation by monochromatic light the luminescence does not consist of separate, sufficiently narrow lines, it is necessary to introduce a correction involving the radiation entropy. It should be pointed out that it is possible to disregard the simply reflected light owing to its obviously monochromatic nature.

Let $\Delta \nu$ be the main interval of frequencies of the luminescence spectrum. The entropy of the incident light is equal to zero, whereas the entropy of the radiated light is approximately equal to

$$\approx [(\bar{f} + 1) \ln(\bar{f} + 1) - \bar{f} \ln \bar{f}] \Delta \tau,$$

where $f(\nu)$ determines the luminescence spectrum, and we have substituted for the integral (4) the value of the function under the integral sign, corresponding to a certain mean \bar{f} multiplied by the region $\Delta \tau$ of integration. The full change of the entropy of the body together with the radiation must be positive:

$$\frac{I_a - I_l}{T} + \approx [(\bar{f} + 1) \ln(\bar{f} + 1) - \bar{f} \ln \bar{f}] \Delta \tau \geq 0.$$

The intensity I_l of the luminescent radiation can be expressed by means of f as follows:

$$I_l = \int f h \nu d\tau \sim h \nu \bar{f} \Delta \tau. \quad (8)$$

Consequently the inequality obtained may be re-written in the following form:

$$I_a \geq I_l \left[1 - \frac{\approx T}{h \nu} \frac{(\bar{f} + 1) \ln(\bar{f} + 1) - \bar{f} \ln \bar{f}}{\bar{f}} \right]. \quad (9)$$

The expression on the right-hand side depends on \bar{f} , i.e. upon the absolute light intensity. On the other hand, as long as the photoluminescence increases linearly with the intensity of the incident light, I_l/I_a is independent of the intensity. Therefore, we are justified in substituting for \bar{f} any value corresponding to the photoluminescence intensity in the linear region. Inasmuch as $[(f + 1) \ln(f + 1) - f \ln f]/f$ is a monotonically decreasing function of f , it is evident that the greatest restriction will be imposed when we take the value

of \bar{f} corresponding to the maximum photoluminescence intensity for which the linearity is still retained.

Let us introduce an "effective temperature" T_{eff} of the luminescent light, as a temperature of black radiation, in the spectrum of which the specified interval $\Delta \nu$ possesses an intensity equal to the greatest intensity of luminescence in the linear region. This temperature is connected with \bar{f} by Planck's formula:

$$\bar{f} = \frac{1}{e^{h\nu/\kappa T_{\text{eff}}} - 1}. \quad (10)$$

Upon substituting (10) into (9) a rather lengthy expression is obtained; it is sufficient to write it out for the two limiting cases—the large and small values of $h\nu/\kappa T_{\text{eff}}$. When $h\nu \gg \kappa T_{\text{eff}}$, we have $\bar{f} \cong e^{-h\nu/\kappa T_{\text{eff}}}$ and (9) yields

$$I_a \geq I_l \left(1 - \frac{T}{T_{\text{eff}}}\right). \quad (11)$$

The ratio T/T_{eff} is in general small; therefore, (11) may also be written as follows:

$$I_l \leq I_a \left(1 + \frac{T}{T_{\text{eff}}}\right) \quad (h\nu \gg \kappa T_{\text{eff}}). \quad (12)$$

In the opposite limiting case of small $h\nu/\kappa T_{\text{eff}}$, we have $\bar{f} \cong \kappa T_{\text{eff}}/h\nu$, and (9) leads to the inequality

$$I_l \leq I_a \left[1 + \frac{T}{T_{\text{eff}}} \left(\ln \frac{\kappa T_{\text{eff}}}{h\nu} + 1\right)\right] \quad (h\nu \ll \kappa T_{\text{eff}}) \quad (13)$$

differing but insignificantly from equation (12).

The inequalities thus obtained express the thermodynamic limitation of the luminescence yield. The temperature T_{eff} is usually of the order of several tens of thousands of degrees, whereas T is the room temperature. Hence, T/T_{eff} is very small, and we arrive at the conclusion that the consideration of the radiation entropy leads to but an insignificant correction in the simple inequality (7). Thus the relative luminescence yield, even when thermodynamically capable of exceeding unity, does so by an insignificant quantity.

Formulae (12) and (13) do not represent the sole restriction imposed by thermodynamics upon the process of luminescence. Consider a system consisting of the luminescent body and black radiation of the same temperature T . We can then assert that the body and the radiation are in thermal equilibrium with respect to one another. This means that in every interval $d\nu$ of frequencies, the intensity of the light emitted by the body is equal to the intensity of the black radiation spectrum in the same interval. But the light emitted by the body consists of two parts: of the light which arises under the influence of the black radiation falling on the body (this part includes photoluminescence)

and of the spontaneous emission. Therefore, under these conditions the following inequality must hold for every interval of frequencies:

$$\text{luminescence intensity} \leq \text{intensity of black radiation.} \quad (14)$$

Let us express this inequality in a mathematical form. We shall assume that for all the considered frequencies the inequality $h\nu' \gg \kappa T$ is valid. According to Wien's formula, for such frequencies the intensity of black radiation is expressed in the form

$$\text{const} \cdot \nu'^3 e^{-h\nu'/\kappa T}.$$

Multiplying by the intensity $w(\nu', \nu)$ of the emission of light with a frequency ν upon irradiation by light with a frequency ν' , and integrating over $d\nu'$, we obtain the intensity in the left-hand side of inequality (14) in the following form

$$\text{const} \int w(\nu', \nu) \nu'^3 e^{-h\nu'/\kappa T} d\nu'.$$

In the right-hand side of inequality (14) we have the expression $\text{const} \nu^3 e^{-h\nu/\kappa T}$. Thus

$$\int w(\nu', \nu) \nu'^3 e^{-h\nu'/\kappa T} d\nu' \leq \nu^3 e^{-h\nu/\kappa T}. \quad (15)$$

Here the integration in the left-hand side does not necessarily to be extended over all the values of ν' . The inequality can evidently only become stronger if we confine ourselves only to a part of them.

Let us consider a small interval of frequencies $\Delta\nu'$, in which the function $w e^{-h\nu'/\kappa T}$ does not change considerably. Then (15) may be re-written in the following form

$$\Delta\nu' \cdot w(\nu', \nu) \nu'^3 e^{-h\nu'/\kappa T} \leq \nu^3 e^{-h\nu/\kappa T}$$

or

$$w(\nu', \nu) \leq \left(\frac{\nu}{\nu'}\right)^3 \frac{e^{-h(\nu-\nu')/\kappa T}}{\Delta\nu'}. \quad (16)$$

This inequality shows that the radiation intensity in the anti-Stokes region ($\nu > \nu'$) for appreciable $\nu - \nu'$ must become insignificant as compared with the intensity of the incident light.

In the Stokes region ($\nu < \nu'$) for large $\nu' - \nu$ the inequality (16) becomes meaningless as it gives merely an upper limit for w , which is altogether too high.

I would like to point out here, that this paper arose from discussions with S. I. Vavilov, who drew my attention to the importance of the rigorous application of thermodynamics to photoluminescence.

63. ON THE THEORY OF SUPERFLUIDITY OF HELIUM II

THE velocity of the "second sound" in helium II has been measured by V. Peshkov¹ with a great precision. His results give an opportunity to perform a quantitative comparison of the theory developed by the author² with the experiment. Such a comparison gives full support to the general picture given by the theory, but at the same time it reveals a noticeable discrepancy between the calculated and observed values of the velocity (e.g. 25 m/sec calculated and 19 m/sec observed at the temperature of 1.6°K). Although this discrepancy is not very large, it is too large to be attributed to the inaccuracy of the experimental data on the thermodynamic quantities of helium II.

For calculating the velocity of the second sound the formulae were used for the thermodynamic quantities (entropy, specific heat), derived in ref. 2 under the assumption of the energy spectrum of the liquid to consist of two branches—the phonon and roton ones. The direction of the observed discrepancy indicates in what way these assumptions must be altered. Using the experimental data, one can formally compute the roton mass μ according to formulae

$$\varrho_n = N \mu, \quad F_r = -N k T. \quad (1)$$

Here N is the number of rotons per unit volume, F_r —the "roton part" of the free energy (i.e. the free energy without the vibrational part), ϱ_n —the density of the "normal part" of the liquid (the phonon part in ϱ_n is negligible as compared with the roton part). The mass μ calculated in this way appears to be approximately inversely proportional to the temperature (in temperature interval 1.3–1.7°K), instead of being constant. It is, however, to be noted, that although the very fact of the variation of μ is apparent, the quantitative law of its variation can be established only in a very approximate way (owing mainly to the scarcity of experimental data on the specific heat of helium II).

If one does not make the assumption $\varepsilon = \Delta + p^2/2\mu$ for the dependence of the energy ε of a roton on its momentum p , but considers the general dependence $\varepsilon(p)$, then the calculation according to the general formulae derived in ref. 2 shows, that in the formula $\varrho_n = N \mu_0$ the quantity $\overline{p^2}/3kT$ enters instead of μ ($\overline{p^2}$ is the mean square of the momentum). If this quantity is inversely proportional to the temperature, then $\overline{p^2} = \text{const}$, i.e. the values of the roton momenta lie mainly in the neighbourhood of a certain p_0 . At the first glance this fact appears to be very strange, but it can be explained in a natural manner by assuming, that the energy spectrum of helium II is of the type shown in

Fig. 1. For small momenta p of an elementary excitation its energy ϵ increases linearly (phonons), then reaches a maximum, begins to decrease and at a certain value $p = p_0$ the function $\epsilon(p)$ has a minimum. In the neighbourhood of the latter we can write

$$\epsilon = \Delta + \frac{(p - p_0)^2}{2\mu} \quad (2)$$

μ being a constant. With such a spectrum it is of course impossible to speak strictly of rotons and phonons as of qualitatively different types of elementary excitations. It would be more correct to speak simply of the long wave (small p) and short wave (p in the neighbourhood of p_0) excitations. It is to be stressed, that all the conclusions concerning the superfluidity and the entire macroscopical hydrodynamics of helium II, developed in ref. 2, maintain their validity also with the spectrum proposed here.

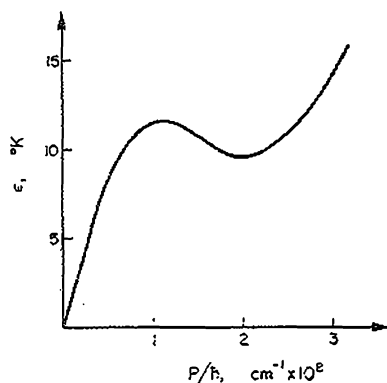


FIG. 1.

Only the formulae for the thermodynamic quantities must be changed. Instead of formulae (3, 4-7) in ref. 2 we have for the "roton" parts of the free energy, entropy, specific heat (per unit mass) and the density of the "normal liquid":

$$F_r = - \frac{2\mu^{1/2} (kT)^{3/2} p_0^2}{(2\pi)^{3/2} \rho \hbar^3} e^{-\Delta/kT}; \quad (3)$$

$$S_r = \frac{2(k\mu)^{1/2} p_0^2 \Delta}{(2\pi)^{3/2} \rho T^{1/2} \hbar^3} \left(1 + \frac{3kT}{2\Delta} \right) e^{-\Delta/kT}; \quad (4)$$

$$C_r = \frac{2\mu^{1/2} p_0^2 \Delta^2}{(2\pi)^{3/2} \rho k^{1/2} T^{3/2} \hbar^3} \times \left[1 + \frac{kT}{\Delta} + \frac{3}{4} \left(\frac{kT}{\Delta} \right)^2 \right] e^{-\Delta/kT}; \quad (5)$$

$$\frac{(\rho_n)_r}{\rho} = \frac{2\mu^{1/2} p_0^4}{3(2\pi)^{3/2} \rho (kT)^{1/2} \hbar^3} e^{-\Delta/kT}. \quad (6)$$

In such a form the theory contains three constants: Δ , p_0 and μ . It is, therefore, difficult to check it on the basis of the experimental data which are now available. For the values of Δ , p_0 and μ one gets:

$$\frac{\Delta}{k} = 9.6^\circ, \quad \frac{p_0}{\hbar} = 1.95 \times 10^8 \text{ cm}^{-1}, \quad \mu = 0.77 m_{\text{He}}. \quad (7)$$

Note that μ is of the order of the mass m_{He} of the helium atom and \hbar/p_0 is even less than the atomic dimensions. The values (7) have been used in drawing the curve in Fig. 1.

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64. ON THE MOTION OF FOREIGN PARTICLES IN HELIUM II

It has recently been suggested that the isotope of helium with mass 3, which is present in helium in small quantities (10^{-6} to 10^{-7}), does not participate in the superfluid motion of helium II¹. The aim of the present note is to show that any foreign atoms and molecules (including the isotopes ^3He and ^4He , and also electrons which have come from outside the mass of helium II), are not able to participate in superfluid motion when their concentration is low.

To prove this let us consider the energy spectrum of helium II containing a small quantity of impurity atoms. In doing this, one may ignore the interaction of the impurity atoms one with another and consider only their interaction with the atoms of the basic isotope of helium. The energy spectrum thus obtained contains, besides the phonon and roton spectra of pure helium II^{2,3}, additional energy levels due to the presence of the impurity atoms. Each such atom can not be fixed at a definite point of liquid helium II but should, like electrons in metals, move throughout the whole of the helium II (the existence of local levels in crystals is connected with the crystal lattice being quasi-classical, which is not so in helium II).

A definite momentum vector, on which the energy depends in a continuous way, is related to the motion of each impurity atom through the helium. We obtain a continuous energy spectrum which is characterised, for each sort of impurity atom, by a dependence of the energy on momentum. This dependence may be different for different sorts of atoms. To the lowest energy state there may correspond either a momentum P equal to zero, or a certain finite momentum P_0 . Expanding the energy in powers of $P - P_0$ near to the minimum energy, we obtain:

$$\varepsilon = \varepsilon_0 + \frac{(P - P_0)^2}{2\mu} \quad (1)$$

Here μ is an effective mass, equal, as a rule (to within an order of magnitude) to the mass of the impurity atom. This excludes the electron, for which the effective mass may be considerably larger than the electron mass.

If $P_0 = 0$ then we have instead of (1):

$$\varepsilon = \varepsilon_0 + \frac{P^2}{2\mu}$$

For small P an expansion of the energy in a power series in the momentum begins with the quadratic term, and not with the linear term as in the phonon case.

Л. Ландау и И. Померанчук, О движениях посторонних частиц в гелии II, *Доклады Академии наук СССР* 59, 669 (1948).

It is well known that the presence of the linear term in the phonon case is caused by some special reasons, which do not occur under the conditions now being considered.

Because of the conservation laws for energy and momentum, the excitation of phonons by impurity atoms with energy (1) is impossible once the velocity of the atom $(P - P_0)/\mu$ becomes less than the velocity of sound. As far as the excitation of rotons is concerned, this requires an energy larger than $\Delta = 9.6 \text{ K}^2$. When the velocity of the atom is less than a certain velocity, which depends on the reduced mass, excitation of rotons is impossible. In the case of ^3He this critical velocity happens to be of the order of the velocity of sound. Thus practically none of the impurity atoms, moving with a velocity less than some critical velocity, can transmit their energy to the superfluid part of Helium II. The path of such atoms is determined by their collisions with phonons and rotons. At temperatures which are not too low, the probability of a collision with the phonons is less than that with rotons, since the roton cross-section is of the order of atomic cross-sections and the phonon cross-section is small because of the large wavelength of the phonons which are present in helium II (the scattering of phonons by impurities is proportional to the fourth power of the frequency). Due to collisions with phonons and rotons, the impurity atoms will move together with the normal part of helium II, and will not be able to participate in the superfluid motion. Thus in the normal part of helium II, besides the mass connected with rotons and phonons, there will appear also an additional mass, due to the impurity atoms. The amount of this mass per unit volume is equal to

$$\frac{n P_0^2}{3 k T}$$

(n is the number of impurity atoms contained in one cm^3). If $P_0 = 0$, then the additional mass is equal to $n \mu$.

Since ^3He must move with the normal part of helium II, when the superfluid leakage of helium takes place the ^3He must remain in the original vessel. Exactly this result was obtained recently by Daunt *et al.*⁴, who were able to diminish the concentration of ^3He in a mass of He II leaking in a superfluid way by a factor of 50. Daunt *et al.* attempt to connect the non-participation of ^3He in the superfluid motion with the question of the properties of the superfluidity of pure ^3He . However, according to the theoretical statements given above, the question as to whether pure ^3He is superfluid or not superfluid bears no relation to its lack of participation in the superfluid motion of He II. Any impurity atoms contained in He II will not take part in the superfluid motion of the He II. In particular, this applies also to ^6He , although ^6He by itself probably has the property of superfluidity.

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65. ON THE ANGULAR MOMENTUM OF A SYSTEM OF TWO PHOTONS

WHEN considering the question of the annihilation of slow positrons and electrons, I. Pomeranchuk¹ discovered that in the limiting case of stationary particles, the probability of two-photon annihilation vanishes if the spins of the electron and positron are parallel.

In this connection there arises the question as to whether or not this circumstance is the result of some general, exact, selection rule, which restricts the possible values of the angular momentum of the system of two photons with the sum of the momenta equal to zero. Such a rule could be a consequence of special properties of photons connected with the transversality of the electromagnetic waves.

The wave function of the system of two photons can be represented in the form of a second rank tensor E_{ik} constructed bilinearly from the components of the electromagnetic field of both photons. If the sum of the momenta of the photons is equal to zero, then E_{ik} will be a function only of the difference $r_2 - r_1$ of the co-ordinates of the two particles. Introducing the unit vector n , directed from the first photon to the second ($r_2 - r_1 = n r$), we shall consider E_{ik} as a function of n (for given r).

The transversality of the field of each of the photons leads to the tensor E_{ik} being orthogonal to their direction of motion:

$$E_{ik} n_k = 0, \quad E_{ik} n_i = 0. \quad (1)$$

The permutation of the two photons means the permutation of the indices of the tensor E_{ik} together with a simultaneous change of the sign of n . Since the photons obey Bose statistics it is necessary that

$$E_{ik}(n) = E_{ki}(-n). \quad (2)$$

Splitting E_{ik} into parts which are symmetric and antisymmetric with respect to the indices i, k one has

$$E_{ik} = S_{ik} + A_{ik}.$$

The relation (2) (and also the orthogonality relation (1)) must, obviously, be independently satisfied by each of the tensors. Hence we find for the symmetric part the relation

$$S_{ik}(n) = S_{ik}(-n), \quad (3)$$

¹И. Д. Ландау, О моменте системы из двух фотонов, *Доклады Академии Наук СССР*, 60, 207 (1948).

and for the antisymmetric part

$$A_{ik}(n) = -A_{ki}(-n). \quad (4)$$

The behaviour of the wave function when the sign of n is changed determines the parity of the state of the system of particles (photons); thus the function S_{ik} corresponds to even states and A_{ik} to odd states. Let us now clarify what the angular momenta of the system can be in the states described by these functions.

It is well known that an antisymmetric tensor of the second rank is dual to some vector A ($A_i = e_{ikl} E_{kl}$, where e_{ikl} is the completely antisymmetric third rank unit pseudotensor). The orthogonality of the tensor A_{ik} to the vector n means that the vectors A and n are parallel. Because of this one may write $A = n \varphi(n)$ where φ is a scalar which, according to (4), must satisfy $\varphi(n) = \varphi(-n)$.

It follows from this last equation that the scalar φ can be constructed linearly from the spherical harmonics of even order (including the order zero).

The correspondence with the total angular momentum of the system J is established in the following way. Although, as is well known, the division of the photon angular momentum into orbital angular momentum and spin is without any physical meaning, we can formally introduce a "spin" S which corresponds to the rank of the tensor wave function and an "orbital angular momentum" L corresponding to the order of the spherical harmonic appearing in the wave function (such a division is a purely mathematical formality expressing the transformation properties of the wave function with respect to the rotation group).

In the given case, the wave function is reduced to a scalar φ , to which corresponds $S = 0$, so that $J = L$. Thus the tensor A_{ik} corresponds to odd states with even angular momentum.

It is well known that an arbitrary symmetric tensor of the second rank S_{ik} can be reduced to a scalar (S_{ii}) and to a symmetric tensor (S'_{ik}) with zero trace ($S'_{ii} = 0$). In view of (3) both of these are even with respect to a change of the sign of n . Thus they are expressible in terms of the spherical harmonics of even order L . The "spin" $S = 0$ corresponds to the scalar S_{ii} , therefore the total angular momentum $J = L$ is even (including the value zero).

The "spin" $S = 2$ corresponds to the tensor S'_{ik} . Adding, by the rules for the addition of angular momenta, this "spin" to an even "orbital angular momentum", we find that for given even $J \neq 0$ there are three possible states (with $L = J - 2, J, J + 2$) and for odd $J \neq 1$ there are two possible states (with $L = J - 1, J + 1$). The exceptions are $J = 0$ with one state ($L = 2$), and $J = 1$ with one state ($L = 2$).

In these calculations, however, the condition of orthogonality between the tensor S_{ik} and the vector n is not yet taken into account. Because of this, it is necessary to exclude from the number of states obtained, a number of states which correspond to a symmetric tensor of second rank parallel to the vector n . Such a tensor can be written down in the form $B_i n_k + B_k n_i$, where

$\mathbf{B}(n)$ is a vector which, according to (3), must satisfy the relation $\mathbf{B}(n) = -\mathbf{B}(n)$.

This vector must, therefore, be expressible in terms of spherical harmonics of odd order L . Noting also that to this vector there corresponds "spin" $S = 1$, we find that for every even total angular momentum $J \neq 0$ there are two possible states (with $L = J - 1, J + 1$) and for odd J one state (with $L = J$). The exception is $J = 0$ with one state ($L = 1$).

Collecting these results together we obtain Table I, which gives the number of possible even and odd states of the two-photon system (with zero total momentum) for different values of the total angular momentum J (k is a positive integer different from zero).

TABLE I

J	Even	Odd
0	1	1
1	—	—
$2k$	2	1
$2k+1$	1	—

We see that for odd J the odd states are absent, and the value $J = 1$ is completely impossible†.

If this result is applied to the annihilation of positronium we come to the conclusion (remembering that the parity of the positronium state is determined by the quantity $(-1)^l$, where l is the orbital angular momentum) that the usual two-photon annihilation is strictly forbidden in all states with odd orbital and total angular momenta, and also in all states with total angular momentum equal to one.

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† Notice that the impossibility of the value $J = 1$ in the particular case of two dipole quanta (with zero total momentum) was recently proved by Balseiro².

66. ON THE THEORY OF SUPERFLUIDITY

SINCE my paper on the theory of the superfluidity of helium II was published in 1941¹ and in later papers, this problem has been repeatedly discussed in scientific literature. I should like to express my opinion on some of the statements which have been put forward, especially in connection with the last paper by L. Tisza.^{4†}

It follows unambiguously from quantum mechanics that for every slightly excited macroscopic system a conception can be introduced of "elementary excitations," which describe the "collective" motion of the particles and which have certain energies ε and momenta p (leaving aside the question as to the actual dependence $\varepsilon(p)$, i.e. the actual form of the energy spectrum). It is this assumption, indisputable in my opinion, which is the basis of the microscopical part of my theory. On the contrary, every consideration of the motion of individual atoms in the system of strongly interacting particles is in contradiction with the first principles of quantum mechanics.[‡]

As to the actual form of the energy spectrum, the general principles allow one merely to assert¹ that for small energies the dependence $\varepsilon(p)$ must be of the "phonon" type, i.e. $\varepsilon = c p$, c being the velocity of sound. This fact is in itself sufficient to deduce strictly the superfluidity of the liquid at sufficiently low temperatures (ref. 1, section 5). It is useful to note that N. N. Bogoliubov⁵ has succeeded recently, by an ingenious application of second quantisation, in determining the general form of the energy spectrum of a Bose-Einstein gas with a weak interaction between the particles. As it should be, the "elementary excitations" appear automatically, and their energy ε as a function of the momentum p is represented by a single curve, which has a linear initial part. Although the model of such a gas does not have any direct bearing on the actual liquid helium II, it shows the manner in which the quantum-mechanical mathematical formalism leads, in fact, from a macroscopical body to an energy spectrum with the indicated properties.

Л. Д. Ландау, К теории сверхтекучести, Доклады Академии Наук СССР, 61, 253 (1948).

L. Landau, On the theory of superfluidity, *Phys. Rev.* 75, 884 (1949).

† I am glad to use this occasion to pay tribute to L. Tisza for introducing, as early as 1938, the conception of the macroscopical description of helium II by dividing its density into two parts and introducing, correspondingly, two velocity fields. This made it possible for him to predict two kinds of sound waves in helium II. (Tisza's detailed paper (*J. phys. rad.* 1, 165, 350 (1940)) was not available in U.S.S.R. until 1943 owing to war conditions, and I regret having missed seeing his previous short letter (*Comptes Rendus*, 207, 1035, 1186 (1938)).) However, his entire quantitative theory (microscopic as well as thermodynamic-hydrodynamic) is in my opinion, entirely incorrect.

‡ Such reasonings are also present in Tisza's recent paper. No quantum meaning can be given to such assertions, as, e.g. "every vortex element can be associated with a definite mass contained in the volume in which the vorticity is different from zero" (ref. 4, p. 852).

The further trend of the ϵ vs. p curve cannot be established in a general form by purely theoretical considerations. The spectrum with two branches ($\epsilon = cp$ and $\epsilon = \Delta + p^2/2\mu$), which I originally postulated, consisted of two intersecting curves; the latter fact alone makes this spectrum unsatisfactory. These considerations, together with an elaborate investigation of the new experimental data, lead³ to a spectrum consisting of a single curve; after a linear initial part, the function $\epsilon(p)$ passes through a maximum, then has a minimum and increases again.[†]

Apart from the microscopic theory and the calculation of the thermodynamic quantities based on this theory, my paper of 1941 contained also the derivation of the hydrodynamic equations for helium II. This part of the theory does not depend on the assumptions concerning the energy spectrum and the equations can be deduced starting merely from the conservation laws and the Galilean relativity principle. From these hydrodynamic equations the formula which determines the velocity of the "second sound" in terms of the thermodynamic quantities of helium II was deduced. I would like to emphasise that, at present, this formula can be directly checked by the experimental data on the entropy and the specific heat of helium II and the values of p_n directly measured by E. Andronikashvili⁶. Such a comparison shows excellent agreement between the theory and the experiment well inside the limits of the experimental error. Therefore, one must consider as a mere misunderstanding Tisza's assertion that this formula is in conflict with experiment.

The hydrodynamic equations given by Tisza are, in my opinion, quite unsatisfactory. It is easy to see that in their exact form they even violate the conservation laws![‡] If one tries to obtain my results starting from these equations, it can be done only as far as equations of the first approximation are concerned, in which the terms of the second order in the velocities are neglected. In this case Tisza's equations can be readily reduced to my equations by means of a suitable definition of the arbitrary quantity p_n , which enters Tisza's theory; this is exactly what he does in his recent paper. Unfortunately, however, he obtains the correct result by using an incorrect assumption of the proportionality between the entropy and the normal part of the density ρ_n of helium II. Tisza's effort to give a thermodynamical foundation for this assumption is quite unconvincing and the formulas given^{1,3} actually show that such a proportionality is absent.^{††}

Tisza excludes phonons from the "normal part" of the liquid, the argument being that the phonons are "associated with the liquid as a whole," contrary to the "elementary excitations" which "correspond to helium atoms in translational Bloch-type states." Such exclusion of the phonons evidently presumes

[†] Tisza's remark that this assumption "tends to modify the theory in the wrong direction" (ref. 4, p. 852) can hardly be justified.

[‡] For instance, the time derivative of the total momentum $\int (\rho_n V_n + \rho_s V_s) dV$ is not equal to zero.

^{††} Accidentally, the temperature dependences of the roton parts of ρ_n and of the entropy are very similar (the only difference being the factor $1 + 3T/2\Delta$). It is this circumstance that enabled Tisza to attain a good agreement with the experimental data on the second sound velocity in the region of not too low temperatures, where the rotons prevail over the phonons.

either that (1) the phonons penetrate freely through narrow slits, without scattering by the walls (I do not mention their mutual collisions, which can be calculated hydrodynamically and turn out to be by no means improbable), or (2) the moving phonon gas has no momentum, whereas the opposite ("the sound wind"!) is well known. Both alternatives are so obviously incorrect that I can hardly imagine which is the one adopted by Tisza. It should also be noted that the part of the density ρ_n which is due to phonons can be strictly calculated.^{1†} The experimental data which are available at present are yet insufficient to disprove Tisza's assertion, because of the comparatively small role of the phonons in the temperature region explored. But I have no doubt whatever that at temperatures of 1.0–1.1°K the second sound velocity will have a minimum and will increase with the further decrease in temperature. This follows from the values of the thermodynamic quantities of helium II calculated by me.

Tisza's paper contains also some considerations concerning the viscosity of helium II. These considerations are, however, confined to some unconvincing remarks on the necessity of distinguishing the "liquid-type viscosity" from the "gas-type or transport viscosity," and as a result, Tisza concludes that the viscosity must decrease with decreasing temperature. Actually, this problem is theoretically rather complicated, and its solution requires an elaborate investigation of different elementary collision processes of phonons and rotons with each other. Such an investigation shows that the viscosity coefficient of helium II can be represented as a sum of two parts—the "roton part" and the "phonon part." The first one turns out to be independent of the temperature, whereas the second increases experimentally with decreasing temperature (a temperature region which is not too near to the λ -point is implied, thus allowing one to consider the phonons and rotons as a "perfect gas"). These results are entirely in accord with the recent viscosity measurements by E. Andronikashvili², which correspondingly are in conflict with Tisza's considerations.

Finally, I should like to dwell upon the question of behavior of foreign atoms dissolved in helium II (e.g. atoms of the isotope ³He). In a recent paper by I. Pomeranchuk and the author³ it was shown by considering the energy spectrum of a quantum liquid, together with a foreign atom that the presence of such atoms gives rise to the appearance of a new kind of "elementary excitation" connected with these atoms. These atoms enter the "normal part" of the liquid together with the phonons and rotons and thus cannot penetrate narrow slits (a fact actually observed by Daunt *et al.*⁴). It is to be emphasised that this fact has nothing to do with the question as to the

† Tisza remarks that the argument given in ref. 1 "is not convincing as it tends to obtain information on a kinetic coefficient (viscosity) from equilibrium considerations" (ref. 4, p. 852). However, this is mere misunderstanding. It is generally known that the uniform rotation admits a thermodynamic consideration, and the argumentation given¹ uses such considerations only for the calculation of that part of the helium mass which rotates together with the rotating vessel, whereas no conclusions on the magnitude of the helium viscosity can be obtained in such a way. Of course, my paper does not make an attempt of this kind.

superfluidity of the substance of the admixture in itself (in particular of the pure ^3He), contrary to the opinion expressed in the literature (J. Franck¹⁰ and Tisza⁴).

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67. THE EFFECTIVE MASS OF THE POLARON

It is assumed that the carriers in crystals with ionic lattices are not electrons from the conductivity band, but polarons. The conservative motion of the polaron as a whole is considered. The dependence of the energy of the system on the velocity of the translational motion of the polaron is given, and the equation of motion of the polaron in an external field is derived.

1. INTRODUCTION

The existing theory of the electronic conductivity of dielectrics and semiconductors assumes that conduction electrons move in one of the "allowed" energy bands of the crystal, and that the wave function ψ of the electron is a wave which has constant amplitude throughout the crystal. The existence, in a crystal, of forbidden bands is supported by experimental findings; these are the existence of dielectrics, investigations of the temperature dependence of the conductivity, the study of the internal photoeffect, the photoeffect from a metal to a dielectric, and so on. Other features of the theory mentioned above have not found direct experimental support, and have sometimes even been contradictory to experimental data.

But, above all (at least for ionic crystals), one should point out the self-contradiction of the theory. Usually, when considering the quantum states of a conduction electron, it is assumed that the ions are fixed at the lattice points so that the electrons move in a periodic field. In reality, however, the ions are in motion and the state of the electron follows this motion adiabatically. S. I. Pekar considered such a motion¹ and showed that, by its electric field, a conduction electron will dielectrically polarise the ionic crystal. It happens that at the very beginning of the process of polarisation the polarised crystal already represents, for the electron, a potential well with a discrete spectrum. The electron, having spent part of its energy on the polarisation of the crystal, falls into one of the discrete levels, and is thus localised. The polarisation of the crystal then increases, the energy level of the electron and the energy of the whole system being lowered. The equilibrium occurs when the energy of the whole system reaches a minimum; this was considered in detail in previous papers by one of the authors^{2, 3}. Thus the state of an electron in the conduction band does not correspond to an extremum of the energy of this system, i.e. it is unstable. The slow conduction electrons should continuously fall into polaron states. The decay time of the band state of a slow electron should be of the order of 10^{-13} sec. Thus it is necessary to reconsider the foundation of the existing theory of conductivity.

Л. Д. Ландау и С. И. Пекар, Эффективная масса полярона, *Журнал Экспериментальной и Теоретической Физики*, 18, 419 (1948).

S. I. Pekar^{1, 4} gave a new point of view on the electronic conductivity of ionic crystals, according to which it is the polaron, and not a free electron from the conduction band, which is the current carrier. The polaron should move in an external electric field like a negative charge, the localised state as a whole being forced to move along the field (the inertial polarisation of the crystal should follow the motion of the polaron). The calculated mobility of the polaron⁴ is in agreement, to within an order of magnitude, with the carrier mobility determined experimentally as the product of the electrical conductivity and the Hall constant.

In the present article the effective mass of the polaron is calculated.

2. EFFECTIVE MASS OF THE POLARON

It is possible to separate, in the dielectric polarisation of the ionic crystal, the inertial and transient parts². The transient part of the polarisation is characterised by the square of the refractive index for light (in the region where the dispersion curve has a plateau); it is suggested that this transient part completely follows the motion of the conductivity electron. As a result of the transient polarisation of the lattice points of the crystal by the field of the electron, an additional force which has the periodicity of the lattice acts on this electron. This force we include in the periodic potential of an electron in the crystal. For a solution of the problems before us it is possible to use the well known method of the effective mass of the electron⁵, i.e. we ignore the periodic potential and at the same time change the mass of the electron to an effective mass μ .

The assumption that the electronic state $\psi(r, t)$ adiabatically follows the motion of the ions and also that the radius of the state ψ is much greater than the lattice constant, leads to the following expression for the energy H of the system:

$$H = \frac{\hbar^2}{2\mu} \int |\nabla \psi|^2 d\tau - \int (\mathbf{p} \cdot \mathbf{D}[\psi]) d\tau + U_p \quad (1)$$

Here $\mathbf{p}(r, t)$ is the inertial part of the specific polarisation of the dielectric², and is

$$\mathbf{D}[\psi] = e \int |\psi(r_1)|^2 \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3} d\tau_1 \quad (2)$$

the quantum mechanical expectation value of the electrostatic induction of the electron. In formula (1) the first term represents the kinetic energy of the electron, the second term is the interaction energy of the electron with the polarised crystal and U_p is the total energy of the crystal when the electron is suddenly removed (the potential and kinetic energy of the ions).

According to the variational principle of quantum mechanics (which is interchangeable with the Schrödinger equation) for any given polarisation

$p(r, t)$, the function $\psi(r, t)$ is determined as that function which minimises the functional (1) subject to the additional condition

$$\int |\psi|^2 d\tau = 1; \quad (3)$$

hence

$$\frac{\hbar^2}{2\mu} \delta \int |\nabla \varphi|^2 d\tau - \int (p \cdot \delta D[\psi]) d\tau = 0. \quad (4)$$

We assume, in the following, that ψ is already determined in this way; then H is the Hamiltonian function which describes the motion of the ions in the crystal.

Let us assume that the crystal is a system of ions undergoing small harmonic oscillations and express H in terms of the corresponding normal co-ordinates, and velocities of the ions. The motion of the ions is governed by the usual canonical Hamiltonian equations (one should allow for (4) when differentiating H). These canonical equations are the equations of forced oscillations of the ions under the influences of the force $D(r, t)$.

Let us consider the particular case of a motion in which the polaron moves as a whole translationally with velocity v , i.e. when

$$\psi = \psi(r - vt), \quad D = D(r - vt), \quad p = p(r - vt). \quad (5)$$

Expanding D and p in Fourier series

$$D = \sum_k D_k e^{i(k \cdot r - vt)} = \sum_k D_k e^{i[(k \cdot r) - \omega_k t]}, \quad \omega_k = (k \cdot v), \quad (6)$$

$$p = \sum_k e^{i[(k \cdot r) - \omega_k t]}, \quad (7)$$

we notice that the forced oscillations of the ions induced by the force D may be obtained as a simple superposition of the oscillations induced by separate harmonics of the form

$$D_k e^{i[(k \cdot r) - \omega_k t]}. \quad (8)$$

These harmonics, of course, are not the usual electromagnetic waves (their velocity $v \neq c$, they obey another dispersion law, and they are not transverse). Since, however, the size of the polaron is greater than the lattice constant, the expansion (6) is dominated by those harmonics having a wavelength which is also greater than the lattice constant. In this case the problem is reduced to the known limiting case of long waves, and when considering the forced oscillations of the ions it is possible to replace the harmonics (8) by electromagnetic waves which have an induction vector with the same amplitude and frequency. This replacement allows us to express the polarisation of the dielectric in terms of the dielectric constant $\epsilon(\omega)$ determining the dispersion of the electromagnetic waves in the crystal:

$$p_k = \frac{c(\omega_k)}{4\pi} D_k, \quad c(\omega) = \frac{1}{n^2} - \frac{1}{\epsilon(\omega)} \quad (9)$$

(In the derivation of the expression for $c(\omega)$ one should bear in mind that p is only the inertial part of the specific polarisation²). It is assumed that the

crystal is isotropically polarised. The relations (9) give the solution of the equations for forced oscillations. The equations (2), (4) and (9) determine the functions ψ , \mathbf{D} , \mathbf{p} . It is obvious that it is possible to satisfy all three equations simultaneously by functions of the form (5).

For a determination of the effective mass of the polaron it is necessary to calculate the energy (1) exactly up to terms of the order of v^2 . In accordance with this and taking the evenness of the function $\varepsilon(\omega)$ into consideration (the imaginary part of ε , i.e. the absorption, is neglected), we write $c(\omega)$ in the form of a series

$$c(\omega_k) = c_0 + c_2 \omega_k^2 + \dots, \quad (10)$$

$$c_0 = \frac{1}{n^2} - \frac{1}{\varepsilon(0)}, \quad c_2 = \frac{c_0}{\omega_1^2}, \quad (11)$$

ω_1 is the limiting frequency for the optical oscillations of the ions (which should not be confused with the "dispersion frequency"). Since

$$\omega_k^2 \mathbf{D}_k e^{i[(\mathbf{k} \cdot \mathbf{r}) - \omega_k t]} = - \frac{\partial^2}{\partial t^2} \mathbf{D}_k e^{i[(\mathbf{k} \cdot \mathbf{r}) - \omega_k t]},$$

it is convenient to write the equations for the forced oscillations (9) in the summed form:

$$\mathbf{p}(\mathbf{r}, t) = \frac{1}{4\pi} \left[c_0 \mathbf{D}(\mathbf{r}, t) - c_2 \frac{\partial^2 \mathbf{D}}{\partial t^2} \right]. \quad (12)$$

It is now easy to obtain the expression for the energy of the crystal U_p . Taking into consideration that the work per unit time and unit volume performed on the crystal is given by the equation

$$(\mathbf{D} \cdot \dot{\mathbf{p}}) = \left(\frac{\mathbf{D}}{4\pi} \cdot [c_0 \dot{\mathbf{D}} - c_2 \ddot{\mathbf{D}}] \right), \quad (13)$$

and putting the energy per unit volume in the absence of the field (when the polaron is far away) equal to zero, we obtain

$$U_p = \frac{c_0}{8\pi} \int \mathbf{D}^2 d\tau + \frac{3c_2}{8\pi} \int \dot{\mathbf{D}}^2 d\tau. \quad (14)$$

Inserting (12) and (14) into the expression for the energy of the system (1) we obtain

$$H = \frac{\hbar^2}{2\mu} \int |\nabla \psi|^2 d\tau - \frac{c_0}{8\pi} \int \mathbf{D}^2[\psi] d\tau + \frac{c_2}{8\pi} \int \dot{\mathbf{D}}^2[\psi] d\tau. \quad (15)$$

The case of the stationary polaron ($\mathbf{v} = 0$, $\mathbf{D} = 0$) was considered in detail by one of the authors in some previous papers^{2,3}. The ground state of the polaron $\psi = \psi_0(\mathbf{r})$ and the corresponding energy

$$H_0 = -0.0547 \frac{\mu e^4}{\hbar^2} c_0^2 \quad (16)$$

were obtained. Taking this solution as the zero approximation, and going to the case of a uniformly moving polaron, it is possible to show that the wave function ψ , and the energy, will have only small corrections provided that the velocity

$$v \ll r_m \omega_1, \quad (17)$$

where r_m is the radius of the polaron².

The energy, up to terms of the order of v^2 will be written in the form

$$H = H_0 + \frac{1}{2} M v^2, \quad M = \frac{c_2}{3} \left[\int_0^\infty \left[\frac{\partial D[\psi_0]}{\partial r} \right]^2 r^2 dr + 2 \int_0^\infty D^2[\psi_0] dr \right]. \quad (18)$$

If one inserts into this $D[\psi_0]$ from ref. 3, the effective mass of the polaron is equal to

$$M = 5.8 \times 10^{-3} \left(\frac{\mu e^2}{\hbar^2} \right)^3 e^2 c_0^3 c_2 = 9.08 \times 10^3 \left(\frac{\mu}{m} \right)^3 c_0^3 c^2 \text{ g}, \quad (19)$$

where n is the mass of the free electron and μ is the effective mass of the electron in the original conduction band. Usually M happens to be considerably larger than the electron mass (for an NaCl crystal it is 432 times larger).

Considering the motion of the polaron in an external electric field E , and equating the power introduced by this field to the increase in the energy (18) per second, we obtain for the polaron, the usual equation of motion

$$M \dot{v} = e E. \quad (20)$$

It was assumed above, that the uniform motion of the polaron is conservative, i.e. the harmonics (8) maintain the stationary forced oscillations of the ions, but do not generate ionic oscillations having eigenfrequencies. In reality, however, because of a certain anharmonicity, the forced oscillations of the ions generate oscillations with eigenfrequencies (heat is dissipated). By this means there is some retardation of the polaron. The energy, transmitted by the polaron to the lattice can be formally calculated as the absorption by the crystal of the electromagnetic waves, equivalent to the harmonics (8), which were mentioned above. Such a calculation has shown⁴ that the retarding force is proportional to the velocity and the mobility of the polaron in the external field is

$$u = 0.262 \times 10^{-13} \frac{\varepsilon(0)^{3/2}}{b \left(\frac{\mu}{m} \right)^3 c_0^3} \text{ abs. units.} \quad (21)$$

The coefficient b is determined experimentally by measuring the absorption of the electromagnetic field in the crystal; namely $b \omega$ is the imaginary part of the refractive index of the crystal for waves with frequency $\omega \ll \omega_1$. The retarding force R is expressed in terms of the mobility and the velocity of the polaron by the formula

$$R = - \frac{e v}{u}. \quad (22)$$

Neglecting the weak retardation, and quantising the motion of the polaron as a whole (without the field), it is possible to convince oneself that this motion is described by plane waves (as for the motion of the centre of mass of any complex particle with internal degrees of freedom). Because of this, the dependence of the energy level density of the translational motion of the polaron on its kinetic energy has the same form as in the case of a free particle with mass M and spin $1/2$.

The thermodynamic equilibrium concentration of polarons is given by the usual formula,

$$n = \frac{2}{h^3} (2\pi M k T)^{3/2} e^{(\zeta - H_0)/kT},$$

where ζ is the chemical potential of the electrons in the crystal. However, the coefficient in front of the exponential factor is usually many times larger than in the case of electrons. It is owing to the interaction of the electrons with the thermal oscillations of the ions that a Maxwellian distribution for the velocities of the polarons is established.

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68. ON THE THEORY OF ENERGY TRANSFER DURING COLLISIONS III

A method has been developed which allows us to perform exact calculations of the cross section for the splitting of a deuteron into a neutron and a proton, during its passage through a Coulomb field. The energy distribution of the outgoing particles has been determined, as well as their angular distribution.

In an earlier paper by L. Landau¹, a general theory has been developed which allows the calculation (in the quasi-classical approximation) of the probabilities for the different processes which accompany scattering. Later this theory was applied by E. Lifshitz² to the scattering of deuterons by heavy nuclei accompanied by one of the following reactions: (1) the decay of the deuteron with the ejection of a neutron and a proton, (2) the decay with the capture of the neutron by the nucleus and the ejection of the proton, (3) the decay with the capture of the proton and the ejection of the neutron, (4) the capture of the deuteron by the nucleus. All this refers to sufficiently slow deuterons; their energy should be sufficiently below the Coulomb barrier of the nucleus. Fulfilment of this condition allowed the use of the general quasi-classical theory of Landau. This theory in its general form, however, allows us to determine the cross section only to exponential accuracy, i.e. without coefficient, which is slowly varying with the energy of the colliding particles, in front of an exponential factor with a large negative exponent (an absolute value of the exponent, large compared to unity is the condition for the applicability of the quasi-classical method).

Corresponding to this, in ref. 2 only the variation of the cross-sections with the energies of the deuteron was determined, but not their absolute values. In that case were considered collisions in which the deuteron had zero orbital momentum relative to the nucleus ("head-on" collisions), which give the largest contribution to the cross section. Naturally, then the angular distribution of the outgoing particles could not be determined.

In the present paper a new method has been developed which allows us to perform exact calculations of the cross sections for the above mentioned processes. It has been applied to the first mentioned reaction — the decay of a deuteron with the release of both a proton and a neutron. For the other reactions there should enter into the cross-sections "sticking coefficients" of the neutron or the proton to the nucleus; the indeterminacy of this factor to a large extent deprives an exact calculation of the cross-section for these reactions of any meaning.

Л. Ландау и Е. Лифшиц, К теории передачи энергии при столкновениях III, *Журнал Экспериментальной и Теоретической Физики*, 18, 750 (1948).

1. DERIVATION OF THE GENERAL FORMULA

The Schrödinger equation which describes the system of proton + neutron, in the spirit of Bethe-Peierls theory, can be written in the form

$$\frac{\hbar^2}{2m} (\nabla_n^2 + \nabla_p^2) \Psi + (E - \varepsilon) \Psi = -\frac{\hbar^2}{m} 4\pi \left(\frac{\kappa \alpha}{2\pi} \right)^{1/2} \psi_d \delta(r_n - r_p) \quad (1)$$

where m is the proton mass, ε is the binding energy of the deuteron, κ is the constant

$$\kappa = \frac{(m \varepsilon)^{1/2}}{\hbar}, \quad (2)$$

E is the kinetic energy of the deuteron; α is a numerical factor, the nature of which is explained below, ∇_n^2 and ∇_p^2 are Laplace operators with respect to the co-ordinates r_n and r_p of the neutron and the proton; ψ_d is the wavefunction for the motion of the deuteron as a whole, i.e. a plane wave:

$$\psi_d \sim e^{i(p_d \cdot r_d)/\hbar}, \quad (3)$$

where $p_d = (4Em)^{1/2}$ is the deuteron momentum, and $r_d = (r_p + r_n)/2$ is the radius vector of its centre of inertia.

Actually, on the one hand, when $r_n \neq r_p$ equation (1) corresponds to the free motion of a proton and a neutron with the common energy $E - \varepsilon$. On the other hand it is easy to see that when $r_n - r_p \rightarrow 0$ this equation gives the correct deuteron wave function:

$$\Psi = \psi_d \left(\frac{\alpha \kappa}{2\pi} \right)^{1/2} \frac{e^{-\kappa r}}{r}; \quad (4)$$

the factor multiplying ψ_d is the normalised wave function of relative motion of the particles in the deuteron ($r = |r_p - r_n|$), which corresponds to its ground state in the Bethe-Peierls theory. The numerical factor α is introduced into the normalisation coefficient of this function (see ref. 3, section 12) in order to improve the accuracy of the Bethe-Peierls theory (and corresponds in a certain sense to taking account of the finiteness of the range of the forces between the neutron and the proton in the deuteron); from comparison with experimental data on the scattering of neutrons on protons one obtains $\alpha \cong 3/2$.⁴

In order to convince ourselves that the function (4) satisfies equation (1) let us transform the latter to co-ordinates r and r_d instead of r_n and r_p :

$$\frac{\hbar^2}{4m} \nabla_d^2 \Psi + \frac{\hbar^2}{m} \nabla^2 \Psi + (E - \varepsilon) \Psi = -4\pi \frac{\hbar^2}{m} \left(\frac{\alpha \kappa}{2\pi} \right)^{1/2} \psi_d \delta(r)$$

Substituting here for ψ from (4) and (3) one obtains the expression

$$(\nabla^2 - \kappa^2) \left(\frac{e^{-\kappa r}}{r} \right) = -4\pi \delta(r),$$

which is satisfied identically.

The Schrödinger equation for motion in the field of a heavy nucleus is obviously obtained from (1) by subtracting from the energy E the potential energy $Z e^2/r_p$ of the proton in the Coulomb field. In this way we have the basic equation:

$$\nabla_n^2 \Psi + \nabla_p^2 \Psi + \frac{2m}{\hbar^2} \left(E - \frac{Z e^2}{r_p} - \varepsilon \right) \Psi = -8\pi \left(\frac{\alpha \kappa}{2\pi} \right)^{1/2} \psi_d \delta(r_n - r_p). \quad (5)$$

Here ψ_d is the wave functions of the motion of the deuteron as a whole in the field of the nucleus. This function has been chosen in the form which is applicable to the problem of Rutherford scattering; namely, this is the function which is composed at infinity of an incoming plane wave and a scattered diverging spherical wave. We shall assume that this function is normalised such that the incoming plane wave is normalised to unit flux density.

In order to solve equation (5) let us expand the unknown function Ψ in terms of wave functions ψ_p of the proton motion in a Coulomb field. These functions satisfy the Schrödinger equation

$$\nabla_p^2 \psi_p + \frac{2m}{\hbar^2} \left(E_p - \frac{Z e^2}{r_p} \right) \psi_p = 0 \quad (6)$$

(E_p is the energy of the proton), ψ_p being composed, at infinity, of a plane wave (with the wave vector $k_p = p_p/\hbar$, p_p is the momentum of the outgoing proton) and a converging spherical wave.† We shall assume the functions ψ_p to be normalised in such a way that the plane wave in its asymptotic expression is normalised to a δ -function in momentum space.

The coefficients in the expansion of the function Ψ in functions ψ_p will be functions of the neutron co-ordinates, so that we may write

$$\Psi = \int a_n(r_n) \psi_p(r_p) d\tau_p \quad (7)$$

($d\tau_p$ is a volume element in proton momentum space; in the functions ψ_p and a_n one should add as an index the momentum of the outgoing proton p_p , but we shall omit this in order to simplify the notation).

Substituting (7) in equation (5) using equation (6) and applying in the usual way the mutual orthogonality of different functions ψ_p , we obtain the following equation for the functions $a_n(r_n)$:

$$\nabla_n^2 a_n + \left(\frac{2m}{\hbar^2} \right) E_n a_n = -8\pi \left(\frac{\alpha \kappa}{2\pi} \right)^{1/2} \psi_d(r_n) \psi_p^*(r_n), \quad (8)$$

where $E_n = E - \varepsilon - E_p$ is the energy of the neutron. This equation is of the type for retarded potentials and its solution at large distances $R = Rn$ (n is

† It is known that when considering the problem of scattering which is accompanied by the emission of a new particle from the centre to a given direction, then the wave function of the latter need not contain a diverging spherical wave in its asymptotic expression (see, for instance, the theory of the photoeffect, or the bremsstrahlung theory of X-rays accompanied by ionization of atoms in ref. 5, Chapter VI, section 4 and Chapter VII, section 2).

a unit vector in the direction of \mathbf{R}) from the origin of co-ordinates can be immediately written by analogy with the known formula of radiation theory:

$$a_n = 2 \left(\frac{\alpha \kappa}{2\pi} \right)^{1/2} \frac{e^{i k_n R}}{R} \int \psi_d(\mathbf{r}) \psi_p^*(\mathbf{r}) e^{-i(\mathbf{k}_n \cdot \mathbf{r})} dV. \quad (9)$$

Here $k_n = n k_n$, $k_n = (2mE_n)^{1/2}/\hbar$.

Obviously $a_n(\mathbf{r}_n)$ is nothing but the wave function of the neutron which is created together with the proton with momentum p_p (to which corresponds the wave function ψ_p) when the deuteron decays. Multiplying the squared modulus of a_n by the velocity of the neutron $v_n = (2E_n/m)^{1/2}$ and by the spherical surface element $R^2 d\Omega_n$, we obtain the neutron flux in the element of solid angle $d\Omega_n$ (around the direction \mathbf{n}). Because of our choice of normalisation of the wave functions this quantity represents the cross-section for the process under consideration per unit volume of proton momentum space. In other words,

$$d\sigma = v_n R^2 |a_n|^2 p_p^2 dp_p d\Omega_p d\Omega_n$$

is the cross-section for the decay of a deuteron into a neutron and a proton emitted into elements of solid angle $d\Omega_n$ and $d\Omega_p$, respectively, the momentum of the proton having a value in the interval dp_p . Substituting here the expression (9) for a_n , we obtain the final formula:

$$d\sigma = 4\alpha \left(\frac{\hbar}{2\pi} \right) (m\varepsilon)^{1/2} k_n k_p \left| \int \psi_d(\mathbf{r}) \psi_p^*(\mathbf{r}) e^{-i(\mathbf{k}_n \cdot \mathbf{r})} dV \right|^2 dE_n d\Omega_n d\Omega_p. \quad (10)$$

Here we have introduced the wave vectors of the neutron and the proton and related the cross-section to the neutron energy interval, which is connected to the energy of the proton $E_p = p_p^2/2m$ by the conservation law $E_p + E_n = E - \varepsilon$.

2. EVALUATION OF THE INTEGRAL

As has already been pointed out, we should use for the deuteron wave function the Coulomb wave function of the continuous spectrum (in a repulsive field), the asymptotic expression for which consists of an incoming plane wave (with wave vector \mathbf{k}_d , $k_d = (4mE^{1/2})/\hbar$) and a diverging spherical wave. Normalising it in the manner described above, we have (see, for instance, ref. 5, Chapter II, section 9):

$$\psi_d = v_d^{-1/2} e^{-\pi n_d/2} \Gamma(1 + i n_d) e^{i(\mathbf{k}_d \cdot \mathbf{r})} F(-i n_d, 1, i(k_d r - (\mathbf{k}_d \cdot \mathbf{r}))), \quad (11)$$

where $v_d = (E/m)^{1/2}$ is the deuteron velocity, and n_d is the quantity

$$n_d = \frac{2mZe^2}{\hbar^2 k_d} = \beta \left(\frac{\varepsilon}{E} \right)^{1/2}.$$

We have for convenience introduced the notation:

$$\beta = \left(\frac{Ze^2}{\hbar} \right) \left(\frac{m}{\varepsilon} \right)^{1/2} \quad (12)$$

$\Gamma(z)$ is the Γ -function and $F(\alpha, \gamma, z)$ is the confluent hypergeometric function.

The proton wave function at infinity should consist of a plane wave with the wave vector k_p , $k_p = (2mE_p)^{1/2}/\hbar$, and a converging spherical wave. Normalising it in the manner described in section 1, we have (see, ref. 5, Chapter II, section 9):

$$\psi_p = (2\pi\hbar)^{-3/2} e^{-\pi n_p/2} \Gamma(1 - i n_p) e^{i(k_p \cdot r)} F(i n_p, 1, -i(k_p r + (k_p \cdot r))), \quad (13)$$

where

$$n_p = \frac{mZe^2}{\hbar^2 k_p} = \beta \left(\frac{\varepsilon}{2E_p} \right)^{1/2}$$

Substituting (11) and (13) into (10) and noticing that

$$|\Gamma(1 + i n_d)|^2 = \Gamma(1 + i n_d) \Gamma(1 - i n_d) = \frac{\pi n}{\sinh n_d \pi},$$

$$|\Gamma(1 + i n_p)|^2 = \frac{\pi n_p}{\sinh n_p \pi},$$

we obtain:

$$d\sigma = \frac{\alpha \beta^2 m^{5/2} \varepsilon^{3/2}}{\pi^2 \hbar^5 \sinh \pi n_p \sinh \pi n_d} \frac{k_n}{k_d^2} |I|^2 e^{-2\pi(n_d + n_p)} \cdot dE_n d\phi_n d\phi_p, \quad (14)$$

where by I we denote the integral

$$I = \int e^{i(\alpha \cdot r)} F(-i n_d, 1, \varrho_d) F(-i n_p, 1, \varrho_p) dV, \quad (15)$$

$$\varrho_d = i(k_d r - (k_d \cdot r)), \quad \varrho_p = i(k_p r + (k_p \cdot r)), \quad q = k_d - k_p - k_n.$$

This integral is met with in Sommerfeld's theory of the continuous X-ray spectrum and is equal to (see ref. 5, Chapter VII, section 6):

$$iI = \left[\frac{d}{d\lambda} (BF(-i n_d, -i n_p, 1, \xi)) \right]_{\lambda=0}, \quad (16)$$

where

$$B = 4\pi(q^2 - 2(q \cdot k_d) - 2\lambda k_d)^{i n_d} (q^2 + 2(q \cdot k_p) - 2\lambda k_p)^{i n_p} q^{-2(i n_d + i n_p + 1)},$$

$$\xi = 2 \frac{q^2(k_d k_p + (k_d \cdot k_p)) - 2((q \cdot k_d) + \lambda k_d)((q \cdot k_p) - \lambda k_p)}{(q^2 - 2(q \cdot k_d) - 2\lambda k_d)(q^2 + 2(q \cdot k_p) - 2\lambda k_p)}, \quad (17)$$

and $F(\alpha, \beta, \gamma, z)$ is the hypergeometric function.

Formulae (14-17) are exact; in them the assumption that the problem is quasi-classical has not been made. Further we shall assume that the energy of the deuteron (and therefore of the proton) is small enough that the quasi-classical approximation is applicable; this means that $n_d, n_p \gg 1$. Therefore, in particular, in formula (14) we can write

$$\sinh n_d \pi \cong \frac{e^{\pi n_d}}{2}, \quad \sinh n_p \pi \cong \frac{e^{\pi n_p}}{2}.$$

Further, it should be noticed that the cross-section as a function of energy of the emitted neutron has its maximum value when $E_n = 0$ and exponentially decreases with increasing E_n (as was shown in ref. 2 and will be confirmed by future results). As a function of the direction of the outgoing proton the cross-section is maximal when the proton travels in the opposite direction to the incoming deuteron (this corresponds to a "head-on" collision); we shall see below that the cross-section exponentially decreases with deviations from this direction.

These circumstances allow us, in evaluating the integral I , to put in all non-exponential expressions (as we shall see the exponential factor enters into the hypergeometric function $F(-in_d, -in_p, 1, \xi)$):

$$k_n = 0, \quad q = k_d - k_p, \quad (k_d \cdot k_p) = -k_d k_p. \quad (18)$$

Correspondingly, for this case we have for the proton energy $E_p = E - \varepsilon$, and k_p should be connected with this value of E_p . The expressions (16), (17) for I are greatly simplified in this case. It is easy to see that the derivative $(d\xi/d\lambda)_{\lambda=\theta}$ under the conditions (18) becomes zero so that the term in I containing the derivative of the hypergeometric function drops out. In the argument ξ of the hypergeometric function it is self evident that we cannot simply substitute (18), since this argument enters the exponent. However, we can expand ξ in powers of the neutron energy E_n and the angle θ_p , which is formed by the vectors k_p and $-k_d$ ($\theta_p = 0$ corresponds to the proton travelling in the opposite direction to the incoming deuteron). One of the parameters (in_p) of the hypergeometric function should be subjected to a similar expansion.

An evaluation carried out in this way leads to the following expression for the modulus squared of I :

$$|I|^2 = \frac{\beta^2 m \varepsilon}{\hbar^2} \frac{64\pi^2}{(k_d + k_p)^6 (k_d - k_p)^2} |F(-in, -in\eta, 1, \xi)|^2, \quad (19)$$

where the parameters of the hypergeometric function are equal to

$$n = \frac{2mZe^2}{\hbar^2 k_d}, \quad \eta = \frac{n_p}{n_d} = \left(\frac{k_d}{2k_p}\right) \left[1 + \left(\frac{k_n^2}{2k_p^2}\right)\right], \quad (20)$$

and its argument is determined by the formulae:

$$\begin{aligned} \xi &= -4k_d k_p (k_d - k_p)^{-2} (1 + \xi_1), \\ \xi_1 &= -\frac{\theta_p^2}{4} + \frac{(3k_p^2 - k_d^2)}{2k_p^2 (k_d - k_p)^2} k_n^2 - \frac{k_n^2 \sin^2 \theta_n}{(k_d - k_p)^2} + \frac{k_n \theta_p \sin \theta_n \cos \varphi}{k_d - k_p}. \end{aligned} \quad (21)$$

Here and below k_d and k_p have the values:

$$\hbar k_d = 2\sqrt{mE}, \quad \hbar k_p = \sqrt{2m(E - \varepsilon)}, \quad (22)$$

θ_n is the angle formed by the vectors k_n and $-k_d$, i.e. the angle between the direction of the outgoing neutron and the opposite direction to the incoming

deuteron; φ is the difference of the azimuths of the vectors k_n and k_p relative to the vector $-k_d$ as polar axis.

In view of the fact that the parameter n is assumed to be large we can use the asymptotic formula for the hypergeometric function, which was obtained by Sommerfeld (see ref. 5, Appendix 16D).

For the modulus squared of that function we have:

$$|F(-in, -in\eta, 1, \xi)|^2 = \frac{\exp\{-2\pi\eta n + n f(u) + n f^*(u^*)\}}{2\pi |u|^2 |f'(u)| n}, \quad (23)$$

where

$$f(u) = i \ln[u^\eta (1-u)^{-\eta} (1-u\xi)^{-1}], \quad (24)$$

here, for u should be taken the root of the quadratic equation $f'(u) = 0$ which lies in the upper half-plane. The phases of each of the quantities u , $1-u$, and $1-u\xi$ in (24) should be understood as angles described by an arc rotating in the positive direction (i.e. anti-clockwise) from the positive real axis until it intersects the corresponding point in the complex plane.[†]

When evaluating the factor in front of the exponential in (23) one can again, of course, let $k_n = 0$, $\theta_p = 0$; the exponent should be expanded in powers of these small quantities. As a result of this calculations the following result is obtained:

$$\begin{aligned} |F|^2 &= (8\pi\beta)^{-1} (E + \varepsilon)^2 \varepsilon^{-1} [(2E)^{1/2} - (E - \varepsilon)^{1/2}]^{-2} \exp\{\dots\}, \\ \{\dots\} &= 2\pi n_p + 2\pi n_d + \beta \left[-\left(\frac{8\varepsilon}{E - \varepsilon}\right)^{1/2} \arccos\left(\frac{E - \varepsilon}{E + \varepsilon}\right)^{1/2} \right. \\ &\quad + 4\left(\frac{\varepsilon}{E}\right)^{1/2} \arccos\left(\frac{E}{E + \varepsilon}\right)^{1/2} - \frac{(2\varepsilon)^{1/2} E_n}{(E - \varepsilon)^{3/2}} \arccos\left(\frac{E - \varepsilon}{E + \varepsilon}\right)^{1/2} \\ &\quad \left. + \frac{4\varepsilon \xi_1}{[(2E)^{1/2} + (E - \varepsilon)^{1/2}]^2} \right]. \end{aligned} \quad (25)$$

3. CROSS-SECTION

Substituting (25) into (19) and later into (14), we obtain the desired cross-section:

$$d\sigma = \beta^3 \frac{\hbar^2}{m\varepsilon} \frac{\alpha \varepsilon^2 (\varepsilon E_n)^{1/2} dE_n d\alpha_n d\alpha_p}{\sqrt{2\pi E(E + \varepsilon)^2 [(2E)^{1/2} + (E - \varepsilon)^{1/2}]^2}} e^{-\beta\varphi} \quad (26)$$

(β from (12)), where

$$\Phi = \Phi_0 + E_n \Phi_1 + E_n \sin^2 \theta_n \cdot \Phi_2 + \theta_p^2 \Phi_3 + \theta_p E_n^{1/2} \sin \theta_n \cos \varphi \cdot \Phi_4, \quad (27)$$

[†] Formula (23) is applicable only when $\eta < 1$. When $\eta = k_d/2k_p$ this condition is equivalent to demanding that $E > 2\varepsilon$. If, however, $E < 2\varepsilon$ then in (19) parameters n and η should be taken as $n = 2mZe^2/\hbar^2 k_p$, $\eta = n_d/n_p \cong 2k_p/k_d$ (then again $\eta < 1$). The evaluation with these η and n leads, as should be expected, to precisely the same result (25).

and

$$\Phi_0(E) = \left(\frac{8\varepsilon}{E - \varepsilon} \right)^{1/2} \arccos \left(\frac{E - \varepsilon}{E + \varepsilon} \right)^{1/2} - 4 \left(\frac{\varepsilon}{E} \right)^{1/2} \arccos \left(\frac{E}{E + \varepsilon} \right)^{1/2}, \quad (28)$$

$$\Phi_1(E) = \frac{(2\varepsilon)^{1/2}}{(E - \varepsilon)^{3/2}} \arccos \left(\frac{E - \varepsilon}{E + \varepsilon} \right)^{1/2} - \frac{2\varepsilon(E - 3\varepsilon)}{(E + \varepsilon)^2(E - \varepsilon)}, \quad (29)$$

$$\Phi_2(E) = 4\varepsilon(E + \varepsilon)^{-2}, \quad (30)$$

$$\Phi_3(E) = \varepsilon[(2E)^{1/2} + (E - \varepsilon)^{1/2}]^{-2}, \quad (31)$$

$$\Phi_4(E) = 4\varepsilon(E + \varepsilon)^{-1} [(2E)^{1/2} + (E - \varepsilon)^{1/2}]^{-1}. \quad (32)$$

The total cross-section $\sigma_0(E)$, as a function of the deuteron energy, is obtained by integrating (26) over the neutron energy (which can be performed because of the rapid convergence of the integral in the limits 0 to ∞) and over all directions of the neutron and the proton (integration over θ_p can also be performed in the range 0 to ∞). As a result of this calculation it is found that†:

$$\sigma_0 = 4 \left(\frac{\hbar^2}{m\varepsilon} \right) \beta^{1/2} \alpha \left(\frac{\pi}{2} \right)^{3/2} \varepsilon^{3/2} E^{-1} (E + \varepsilon)^{-2} \Phi_1^{-3/2} e^{-\beta \Phi_0(E)}. \quad (33)$$

The exponent $\beta \Phi_0$ is, of course, that found in ref. 1.

After integrating the cross-section (26) only over the direction of the proton (over $d\theta_p$), the angle θ_n drops out of the expression obtained. In other words, the angular distribution of the neutrons (not correlated with the directions of the protons) is isotropic. For the energy distribution of the neutrons in this case the following expression is obtained:

$$d\sigma(E_n) = \sigma_0 2\pi^{-1/2} (\beta \Phi_1)^{3/2} E_n^{1/2} e^{-\beta E_n \Phi_1} dE_n. \quad (34)$$

The exponent in this formula coincides with that found in ref. 1 (this, however, was not obvious in advance).

The angular distribution of protons is obtained by integrating (26) over $dE_n d\theta_n$. The calculation leads to the formula‡:

$$d\sigma(\theta_p) = \sigma_0 \frac{\beta \Phi_1 \Phi_3}{\pi(\Phi_1 + \Phi_2)} \exp \left\{ -\beta \theta_p^2 \frac{\Phi_1 \Phi_3}{\Phi_1 + \Phi_2} \right\} d\theta_p. \quad (35)$$

In this way the distribution over angles θ_p appears to be Gaussian with the maximum in the direction opposite to that of the deuteron.

The limits of applicability of formula (33) for the cross-section are set by the condition that the exponent $\beta \Phi_0$ should not be small compared with

† The integration over $d\theta_p$ is carried out by introducing new variables of integration $\theta_x = \theta_p \cos \varphi$, $\theta_y = \theta_p \sin \varphi$ (where $d\theta_p \cong \theta_p d\theta_p d\varphi$ is replaced by $d\theta_x d\theta_y$, and the integration is carried out over the range $-\infty$ to $+\infty$). After this integration the result does not depend on θ_n , and integration over E_n is done directly.

‡ The integration over $dE_n d\theta_n$ is performed by going from "spherical" to "Cartesian" co-ordinates in neutron momentum space.

unity: $\beta \Phi_0(E) \geq 1$ †. This condition is violated when deuteron energies are sufficiently large, but there exists a region (for large enough Z), in which $E \gg \varepsilon$, and the formula (33) is still applicable. In this region we have

$$\Phi_0 = \frac{8\varepsilon^2}{3E^2}, \quad \Phi_1 = \frac{32\varepsilon^2}{3E^3},$$

and formula (33) takes on the form:

$$\sigma_0 = \left(\frac{\hbar^2}{m\varepsilon} \right) \beta^{1/2} 2^{-7} (3\pi)^{3/2} \left(\frac{E}{\varepsilon} \right)^{3/2} e^{-3\beta\varepsilon^2/E^2}. \quad (36)$$

For numerical calculations let us write the formula (28) in the form

$$\sigma_0 = Z^{1/2} A(E) \times 10^{-Zf(E)} \times 10^{-26} \text{ cm}^2 \quad (37)$$

The functions $A(E)$ and $f(E)$ are tabulated in Table 1. For the binding energy of the deuteron the value $\varepsilon = 2.19 \text{ MeV}$ is taken⁶.

TABLE 1

$E \text{ (MeV)}$	$A(E)/4$	$10f(E)$	$E \text{ (MeV)}$	$A(E)/4$	$10f(E)$
2.6	0.028	3.68	5.4	1.1	0.375
2.8	0.062	2.59	5.6	1.2	0.343
3.0	0.11	1.97	5.8	1.4	0.316
3.2	0.16	1.57	6.0	1.5	0.292
3.4	0.22	1.29	6.5	1.8	0.242
3.6	0.29	1.08	7.0	2.1	0.206
3.8	0.37	0.926	7.5	2.5	0.176
4.0	0.45	0.804	8.0	2.8	0.152
4.2	0.53	0.703	8.5	3.2	0.133
4.4	0.62	0.624	9.0	3.6	0.118
4.6	0.72	0.556	9.5	4.0	0.105
4.8	0.82	0.500	10.0	4.4	0.093
5.0	0.92	0.451	10.5	4.8	0.084
5.2	1.0	0.412	11.0	5.3	0.075

In Table 2 the numerical values of

$$\delta_n = \frac{Z}{\beta \Phi_1(E)}$$

are given.

According to the formula (34), written in the form

$$d\sigma(E_n) \sim e^{-Z E_n / \delta_n},$$

† The asymptotic formula (23) used by us is obtained by applying the method of steepest descent to the complex integral which determines the hypergeometric function. Investigation of the approximation applied in this (applied to our specific case) reveals the sufficiency of the condition $\beta \Phi_0 \geq 1$ (and not necessarily $\beta \Phi_0 \gg 1$).

TABLE 2

E (MeV)	δ_n (MeV)	E (MeV)	δ_n (MeV)
2.6	0.55	5.4	19
3.0	1.6	6.0	28
3.4	3.2	7.0	45
3.8	5.2	8.0	69
4.2	7.9	9.0	99
4.6	11	10.0	140
5.0	15	11.0	180

the quantity δ_n/Z determines the width of the energy distribution of the outgoing neutrons.

At the present time experimental data on the cross-sections for the (d, pn) reactions are not available. It is interesting to compare the cross-sections given by formula (37) with the experimentally measured sections for the (d, p) and (d, n) reactions. At the present time data about the absolute value of these cross-sections only exist for Bi ($Z = 83$). Tatel and Cork⁷ measured cross-sections for the reactions $^{209}\text{Bi} (d, p) ^{210}\text{RaE}$ and $^{209}\text{Bi} (d, n) ^{210}\text{Po}$. The activities of the products of the reactions (RaE and Po) served as an indicator, because, naturally, the reaction $^{209}\text{Bi} (d, pn) ^{209}\text{Bi}$, escapes observation. We are giving some values of the cross-sections for (d, p) and (d, n) reactions (the accuracy of these values is not very large) found by these authors, together with cross-sections for (d, pn) reaction calculated from formula (37).

E (MeV)	$\sigma_{d,p} \times 10^{28}$ (cm ²)	$\sigma_{d,n} \times 10^{28}$ (cm ²)	$\sigma_{d,np} \times 10^{28}$ (cm ²)
8.2	18	3.4	680
6.3	1.5	0.12	40

We see that the calculated values of $\sigma_{d,np}$ appreciably exceed the values of $\sigma_{d,p}$ and $\sigma_{d,n}$ so it can be assumed, in this case, that the (d, np) reaction is really the basic one.

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69. THE THEORY OF THE VISCOSITY OF HELIUM II: I. COLLISIONS OF ELEMENTARY EXCITATIONS IN HELIUM II

The theory of superfluidity of liquid helium II is used in a consideration of the viscosity in this substance. Calculations of the effective cross section of dispersion of elementary excitations (phonons and rotons) are given.

P.L. KAPITZA demonstrated that He II shows no viscosity in flowing through capillaries or narrow channels. This phenomenon is explained by supposing that in flowing through a narrow channel the superfluid part of He II flows without friction, whilst the normal part is held back by friction and flows through the same channel appreciably slower^{1, 2}.

At the same time, measurements of the viscosity of He II by the decay of rotational oscillations of a disk immersed in the liquid give a non-zero viscosity³. This result is not remarkable in itself and is easily explained qualitatively by the theory, according to which the disc oscillating in He II is damped by friction of the normal part. Nevertheless up to now there has not been agreement between the measurements of viscosity by different authors.

Thus Keesom and McWood^{†3} obtained a dependence of the viscosity of He II on temperature featuring a noticeable fall of viscosity on the low temperature side of the lambda point.

These authors supposed that the disc was damped by the whole liquid. In reality, as has already been pointed out previously⁴, the oscillating disc is damped only by the normal part of the He II. Thus the values obtained from the experiments of Keesom and McWood must be divided by ρ_n/ρ . This modification shows that from 2–1.5°K the viscosity is normally constant and of magnitude $\cong 1.4 \times 10^{-5}$ poise. At temperatures below 1.5°K, the scatter of the points is so great that it naturally causes doubt as to the accuracy of the experiments at these temperatures; consequently from the data of Keesom and McWood it is impossible to deduce anything with certainty about the temperature dependence of the viscosity at low temperatures, even qualitatively.

Л. Д. Ландау и И. М. Халатников, Теория вязкости гелия II, I. Столкновения элементарных возбуждений в гелии II, *Журнал Экспериментальной и Теоретической Физики*, 19, 637 (1949).

[†] The coefficients of viscosity obtained in ref. 3 are somewhat high since the authors used an erroneous formula due to McWood in calculating the correction for the penetration depth. In the final reckoning this erroneous formula gave a correction with the wrong sign.

E. Andronikashvili⁵ investigated the temperature dependence of η_n/η by the method of rotational oscillations. In these exceptionally cunning experiments a pile of aluminium disks fixed on a common axis was immersed successfully in He II, He I, and a vacuum. The ratio η_n/η was connected by a simple formula with the periods of oscillation of the system. From the logarithmic decrements observed in the same experiments on He II it was also possible to calculate the coefficient of viscosity at the various temperatures. It was established that from 1.9–1.5 °K the coefficient of viscosity was constant within the limits of accuracy of the experiments (1×10^{-5} poise). Thus these experiments gave a picture qualitatively agreeing with that obtained by a recalculation of Keesom and McWood's data in the same region of temperature.

Recently Andronikashvili⁶ made direct measurements of the viscosity coefficient of He II over a wide temperature interval, from the lambda point to 1.35°K. For temperatures above 1.5°K the results coincide on the whole with those obtained earlier by him. At lower temperatures, however, a marked rise of the viscosity coefficient with the lowering of temperature is noticeable.

The present work is devoted to a theoretical examination of the question of the viscosity of He II. The law of temperature dependence, derived unambiguously from the theory, appears to be in good agreement with these recent experiments of Andronikashvili.

We shall begin the examination of the problem of the viscosity of He II with a discussion of the question of interactions of excitations with each other.

1. INTRODUCTION

For obvious reasons, the exact form of the energy spectrum of He II cannot be estimated. However, starting from sufficiently general considerations it is possible to determine the form of the energy curve at the most significant places. According to refs. 2 and 7 the energy spectrum of He II has the following form. Initially, for small values of momentum p , the energy ε is a linear function of p :

$$\varepsilon = c p \quad (1.1)$$

where c is the speed of sound. The corresponding elementary excitations consist of sound quanta, i.e. phonons. However, from numerous experiments on He II it follows that beside phonons there must exist in He II elementary excitations of a different type. For these excitations (normally called rotons) to be possible the energy curve must have a minimum at some value $p = P_0$. Close to this minimum the roton energy E can be expressed as a series in powers of $(p - P_0)$. Thus with an accuracy up to the terms of the second order the energy of the rotons is given by:

$$E = \Delta + \frac{(p - P_0)^2}{2\mu} \quad (1.2)$$

where Δ and P_0 are suitable parameters and μ the effective roton mass. Both parts of the spectrum can be combined in the form of a single energy curve.

In such a single spectrum it is impossible to divide the elementary excitations into two types—phonons and rotons. It is possible only to speak of long wave excitations (p small) and short wave excitations (p in the neighbourhood of P_0). For considerations of convenience we retain the terms “phonons” and “rotons”, calling the long wave excitations “phonons” and the short wave excitations “rotons”.

Where the number of phonons and rotons is not large, i.e. not very close to the lambda point, they may be considered together as a mixture of two ideal gases, a gas of phonons and a gas of rotons. The phonon gas obeys Bose statistics. For the rotons, Boltzmann statistics may be used, since the condition $\Delta \gg kT$ gives a roton distribution function which is insensitive to the type of statistics.

Initially, the energy ε depends linearly on the momentum p . However, for phonons of greater energy the effect of dispersion becomes appreciable, i.e. there is a departure from the linear law. It turns out to be quite essential to take dispersion into account in the calculation of the effective cross-section for the scattering of phonons by phonons.

It is clear that since the exact form of the energy spectrum is unknown the dispersion problem cannot be solved exactly. However, from the available data on the energy curve small values of momentum and in the neighbourhood of the minimum, we can attempt to calculate the coefficients in the expansion of the energy in rising powers of p . To do this we construct a four-term formula with undetermined coefficients

$$\varepsilon^2 = A_1 p^2 + A_2 p^4 + A_3 p^6 + A_4 p^8. \quad (1.3)$$

The expressions (1.1) and (1.2) give four conditions:

$$\left(\frac{\partial \varepsilon}{\partial p}\right)_{p=0} = c, \quad \left(\frac{\partial \varepsilon}{\partial p}\right)_{p=P_0} = 0, \quad \left(\frac{\partial^2 \varepsilon}{\partial p^2}\right)_{p=P_0} = \frac{1}{\mu}, \quad \varepsilon(P_0) = \Delta, \quad (1.4)$$

which allows us to determine the four coefficients A_1, A_2, A_3, A_4 in equation (1.3). The values of these coefficients turn out to be

$$A_1 = c^2, \quad A_2 = \frac{1}{4P_0^2} \left(24 \frac{\Delta^2}{P_0^2} + \frac{\Delta}{\mu} - 12c^2 \right) \quad (1.5)$$

$$A_3 = \frac{1}{P_0^4} \left(3c^2 - 8 \frac{\Delta^2}{P_0^2} - \frac{\Delta}{2\mu} \right), \quad A_4 = -\frac{1}{4P_0^6} \left(4c^2 - 12 \frac{\Delta^2}{P_0^2} - \frac{\Delta}{\mu} \right).$$

Extracting the square root of (1.3) and retaining only the first two terms we obtain

$$\varepsilon = c(p - \gamma p^3), \quad (1.6)$$

where $\gamma = -A_2/2c^2$. The value of γ , calculated with the help of known values of Δ , μ , and P_0 is found to be $\gamma \approx 2.8 \times 10^{27} \text{ g}^{-2} \text{ cm}^{-2} \text{ sec}^2$.

For calculating the effective scattering cross-sections of phonons we make use of the Fourier expansion of the phonon field. In doing this we use the analogy between the phonon field and the roton field. Expressing the density of the helium $\varrho(\mathbf{r})$ and the velocity $\mathbf{v}(\mathbf{r})$ as Fourier series

$$\begin{aligned}\varrho(\mathbf{r}) &= \varrho_0 + \Omega^{-1/2} \left[\sum_{\mathbf{p}} \varrho_{\mathbf{p}} e^{i(\mathbf{p} \cdot \mathbf{r})/\hbar} + \varrho_{\mathbf{p}}^* e^{-i(\mathbf{p} \cdot \mathbf{r})/\hbar} \right], \\ \mathbf{v}(\mathbf{r}) &= \Omega^{-1/2} \left[\sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}} e^{i(\mathbf{p} \cdot \mathbf{r})/\hbar} + \mathbf{v}_{\mathbf{p}}^* e^{-i(\mathbf{p} \cdot \mathbf{r})/\hbar} \right].\end{aligned}\quad (1.7)$$

Here ϱ_0 is the equilibrium density in the absence of phonons, \mathbf{p} is the phonon momentum connected with the frequency ω of the relation

$$\omega = \frac{c p}{\hbar} \quad (1.8)$$

For running waves, the summation in (1.7) is to be carried out for both positive and negative values of momentum. From the commutation relations² between $\varrho(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$

$$\varrho(\mathbf{r}_1) \cdot \mathbf{v}(\mathbf{r}_2) - \mathbf{v}(\mathbf{r}_2) \cdot \varrho(\mathbf{r}_1) = \frac{\hbar}{i} \nabla \delta(\mathbf{r}_1 - \mathbf{r}_2) \quad (1.9)$$

we can obtain the commutation relation for the Fourier components. For this we transform $\varrho(\mathbf{r}_1)$ and $\mathbf{v}(\mathbf{r}_2)$ with the help of (1.7), and substitute the above expressions into the left-hand side of (1.9). The δ -function appearing on the right-hand side of (1.9) we express in the form

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{2} \Omega^{-1} \left[\sum_{\mathbf{p}} e^{i(\mathbf{p} \cdot \mathbf{r}_1 - \mathbf{r}_2)/\hbar} + e^{-i(\mathbf{p} \cdot \mathbf{r}_1 - \mathbf{r}_2)/\hbar} \right], \quad (1.10)$$

and we compare coefficients of the same exponents on the left and the right-hand sides. This gives the following relations, valid for $\text{curl } \mathbf{v} = 0$,

$$\varrho_{\mathbf{p}} \varrho_{\mathbf{s}}^* - \varrho_{\mathbf{s}}^* \varrho_{\mathbf{p}} = \frac{\varrho_0 p}{2c} \delta_{\mathbf{p}, \mathbf{s}}, \quad \mathbf{v}_{\mathbf{p}} = \frac{c p}{\varrho_0 p} \varrho_{\mathbf{p}}, \quad (1.11)$$

where $\delta_{\mathbf{p}, \mathbf{s}} = \begin{cases} 0 & \mathbf{p} \neq \mathbf{s}, \\ 1 & \mathbf{p} = \mathbf{s}. \end{cases}$

The total Hamiltonian of the volume concerned is

$$\int \left\{ \frac{1}{2} (\mathbf{v} \cdot \varrho \mathbf{v}) + \varrho \varepsilon(\varrho) \right\} d\Omega$$

($\varepsilon(\varrho)$ is the internal energy of unit mass of the liquid). Up to cubic terms in $\varrho_{\mathbf{p}}$ this is expressed in Fourier components as

$$H_0 = \frac{c^2}{\varrho_0} \sum (\varrho_{\mathbf{p}} \varrho_{\mathbf{p}}^* + \varrho_{\mathbf{p}}^* \varrho_{\mathbf{p}}) = \sum \left(n_{\mathbf{p}} + \frac{1}{2} \right) \hbar \omega. \quad (1.12)$$

Here n_p is the number of phonons of momentum p . From (1.11) and (1.12) we obtain the following non-zero matrix elements of the Fourier components

$$(\varrho_p)_{n_p, n_{p+1}} = \sqrt{\frac{\varrho_0 p}{2c}} \cdot (n_p + 1) \cdot e^{-i\omega t}, \quad (1.13)$$

$$(\varrho_p^*)_{n_p, n_{p-1}} = \sqrt{\frac{\varrho_0 p}{2c}} \cdot n_p \cdot e^{-i\omega t}. \quad (1.14)$$

2. SCATTERING OF PHONONS BY PHONONS

The effective cross-section for scattering of phonons by phonons will be calculated by the methods of perturbation theory. The Hamiltonian per unit of He II is according to ref. 2 given by

$$H = \frac{1}{2} (\mathbf{v} \cdot \varrho \mathbf{v}) + \varrho \varepsilon(\varrho). \quad (2.1)$$

The process of scattering considered is a four-phonon effect. The non-vanishing matrix elements of the energy of perturbation for transitions of two phonons into two other phonons are obtained from the cubic terms in ϱ' in the energy in the *second* approximation of the perturbation theory and from the fourth-order terms in ϱ' of the *first* approximation of the perturbation theory; here again ϱ' is the departure of the density from its value for an immobile liquid, i.e. in the absence of phonons.

Limiting ourselves to terms of the fourth order in ϱ' we write the Hamiltonian (2.1) in the form

$$H = H_0 + V_3 + V_4 \quad (2.2)$$

where H_0 is the total density of sound energy containing terms of second order in ϱ' .

$$H_0 = \frac{\varrho_0 v^2}{2} + \frac{\varrho'^2 c^2}{2\varrho_0}, \quad (2.3)$$

V_3 contains the terms of third order

$$V_3 = \frac{(\mathbf{v} \cdot \varrho' \mathbf{v})}{2} + \frac{1}{3!} \frac{\partial}{\partial \varrho} \left(\frac{c^3}{\varrho} \right) \varrho'^3, \quad (2.4)$$

and finally V_4 contains the terms of fourth order

$$V_4 = \frac{1}{4!} \frac{\partial^2}{\partial \varrho^2} \left(\frac{c^2}{\varrho} \right) \varrho'^4. \quad (2.5)$$

For scattering of phonons with momenta p and p_1 leading to the appearance of phonons with momenta p' and p'_1 , six intermediate states I–VI are possible in which the phonons have the momenta

$$\begin{aligned} \text{I } p + p_1; \quad \text{II } p - p', p_1, p'; \quad \text{III } p - p'_1, p_1, p'_1; \quad \text{IV } p_1 - p'_1, p, p'_1; \\ \text{V } p_1 - p', p, p'; \quad \text{VI } p, p_1, p', p'_1, - (p^1 + p'_1) \end{aligned}$$

In the second approximation of the perturbation theory the matrix elements for transition from the initial state (A) to the final state (F) are given by

$$H'_{AF} = \sum_{i=I}^{VI} \frac{(V_3)_{Ai} (V_3)_{iF}}{E_A - E_i} + (V_4)_{AF}. \quad (2.6)$$

The denominators in (2.6), which are the differences in energy between the initial and the intermediate states, are expressed through the phonon energies:

$$\begin{aligned} E_A - E_I &= \varepsilon(p) + \varepsilon(p_1) - \varepsilon(|p + p_1|), \\ E_A - E_{II} &= \varepsilon(p) - \varepsilon(p') - \varepsilon(|p - p'|), \\ E_A - E_{III} &= \varepsilon(p) - \varepsilon(p'_1) - \varepsilon(|p - p'_1|), \\ E_A - E_{IV} &= \varepsilon(p_1) - \varepsilon(p'_1) - \varepsilon(|p_1 - p'_1|), \\ E_A - E_V &= \varepsilon(p_1) - \varepsilon(p'_1) - \varepsilon(|p_1 - p'_1|), \\ E_A - E_{VI} &= -\varepsilon(p') - \varepsilon(p'_1) - \varepsilon(|p' + p'_1|) \end{aligned} \quad (2.7)$$

From (2.7) it is easily seen that if we neglect dispersion the denominators of the first five terms of (2.6) may become zero, leading to a considerable divergence of the matrix elements. Indeed, for a linear dependence of the phonon energy on momentum the denominators become zero whenever the angle between momentum vectors and the colliding phonons become zero, i.e. when $|p + p_1| = p + p_1$. Thus for the phonon energy we shall make use of (1.6), which contains a cubic term in the momentum besides the linear term. The matrix elements of the perturbation energies V_3 and V_4 entering into (2.6) are calculated with the help of (1.13) and (1.14) for the non-vanishing matrix elements of the density.[†]

The expression obtained in this way fundamentally involves two dimensionless parameters

$$u = \frac{\varrho_0}{c^2} \frac{\partial c^2}{\partial \varrho} \quad \text{and} \quad z = \frac{\varrho_0^3}{c^2} \frac{\partial^2}{\partial \varrho^2} \left(\frac{c^2}{\varrho} \right). \quad (2.8)$$

The effective differential cross-section for the process considered is determined by the relation

$$d\sigma(p, p_1, p', p'_1) = \frac{2\pi}{\hbar c} |H'_{AF}|^2 \delta(\varepsilon + \varepsilon_1 - \varepsilon' - \varepsilon'_1) (2\pi \hbar)^{-3} dp', \quad (2.9)$$

$$dp' = dp'_x dp'_y dp'_z.$$

The calculation of the total effective cross section for scattering of phonons with momentum p by phonons with momentum p_1 leads to most complicated expressions. We shall consider a concrete case where one of the momenta of

[†] In calculating the matrix elements, it is necessary to take account of all possible rearrangements of the phonons. For this it is necessary to multiply terms containing $(\varrho')^2$ by two, terms containing $(\varrho')^3$ by six, and finally those containing $(\varrho')^4$ by twenty-four.

the colliding phonons is small, in fact where $p \ll p_1$. As far as the momenta of the scattered phonons (p' and p'_1) are concerned, we shall suppose them to be comparable in magnitude to cp_1 .

In this case, of the three terms in the transitional matrix elements in H'_{IF} which contain resonance denominators involving corresponding intermediate states I, IV, and V, only the term

$$\frac{(V_3)_{AI} (V_3)_{IF}}{\varepsilon(p) + \varepsilon(p_1) - \varepsilon(|p + p_1|)}$$

will contain a small quantity in the denominator, in distinction to the two others proportional respectively to $(p_1 - p')^2/p_1 p'$ and $(p_1 - p'_1)^2/p_1 p'_1$; this term turns out to be of the order of p_1/p and consequently it alone is important. This consideration simplifies the expression (2.9) for $d\sigma$, which after not very complicated transformations assumes the form

$$d\sigma = \frac{(u+2)^4 p_1^2 p'_1 p'}{(16\pi \hbar^3 \varrho_0)^2 c p} \frac{\delta[\varepsilon(p) + \varepsilon(p_1) - \varepsilon(p') - \varepsilon(p'_1)]}{(1 - (n \cdot n_1) + 3\gamma p_1^2)^2} p'^2 dp' d\sigma'. \quad (2.10)$$

The consideration that (2.10) has a sharp maximum when the angles between momenta of colliding phonons are small permits us to carry out the necessary integration fairly easily. We take as polar axis of a system of spherical polar co-ordinates the direction given by the sum of the momenta, $p + p_1$. In this system let the vectors p , p_1 , p' , and p'_1 have polar angles θ , θ_1 , θ' , and θ'_1 . We now transform the δ -function containing the difference of energies into a form convenient for integration; with the help of (1.6) and the law of conservation of momentum this takes the following form for small angles:

$$c \left[\left(\frac{p}{2p_1} \right) (p + p_1) \theta^2 - \left(\frac{p'}{2p'_1} \right) (p' + p'_1) \theta'^2 + 3\gamma (p_1 - p') (p p'_1 - p_1 p') \right].$$

Let us integrate (2.10) over the phase volume dp' and then average over all angles between the momenta p and p_1 . We obtain for the total effective cross section for the scattering of phonons of momentum p by phonons of momentum p_1 the following expression†

$$\sigma(p, p_1) = \frac{\pi(u+2)^4 p_1^4}{(96\pi \hbar^3 \varrho_0 c)^2 \gamma}, \quad p \ll p_1. \quad (2.11)$$

The value of the dimensionless expression u in (2.11) can be calculated from the data of Keesom⁸ for the dependence of the density of He II on pressure. From these data we obtain a value 3.6.‡

The curious effect should be noted that the effective cross-section of scattering (2.11) is independent of the momentum p for phonons of small energy.

† To take account of the symmetry with respect to the pair of scattered phonons of momentum p and p_1 (2.11) contains an extra multiplier $\frac{1}{2}$.

‡ We may mention at the same time that the value of the parameter z (see (2.8)) obtained from the same data is approximately 20.

Substituting into (2.11) the numerical values of the parameters, we find

$$\sigma(p, p_1) = 6 \times 10^{-19} (xT)^4. \quad (2.12)$$

Here x is the momentum expressed in units of kT/c , so that $p_1 = x(kT/c)$.

As already noted, the effective cross-section for scattering of phonons by phonons attains its maximum value for small angles between the momentum of the colliding and that of the scattered phonons. From this it follows that such a process must lead fundamentally to a rapid exchange of energy between phonons, and does not lead to a significant change in the directions of the momenta of the colliding phonons.

3. SCATTERING OF PHONONS BY ROTONS

In the present section we calculate the effective cross-section for scattering of phonons by rotons. The Hamiltonian of the phonon-roton system can be written in the form

$$H = H_\phi + H_r + V, \quad (3.1)$$

where H_ϕ is the phonon energy, H_r the roton energy, and V the interaction function between phonons and rotons. Since the character of the interaction between phonon and roton is unknown, we shall treat the roton as some sort of particle finding itself in the phonon field. For such considerations the internal structure of the roton is not important.

The presence of the phonon is equivalent to small oscillations of density of the medium and its motion with velocity v . In a stationary system of reference the energy of a roton with momentum \mathbf{P} is expressed in the following way:

$$H_r = H_{r0} - (\mathbf{P} \cdot \mathbf{v}) \quad (3.2)$$

(H_{r0} is the energy of the roton when in a system of reference moving with the medium.) From (3.2) it follows that the interaction energy of phonons with rotons contains a term $(\mathbf{P} \cdot \mathbf{v})$ depending on velocity. Corresponding to this term in the perturbation energy the Hermitian quantum operator is

$$\frac{1}{2} [(\mathbf{P} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{P})] \quad (3.3)$$

where $\mathbf{P} = -i\hbar \nabla$ is the momentum operator. We shall now obtain the dependence of the interaction energy V on the density of the liquid. For this, we expand the roton energy in powers of ϱ , the fluctuation of the density from its equilibrium value, and retain only terms up to and including those of the second order in ϱ . (Here and in what follows we shall denote the difference of the density of the liquid from the equilibrium value by ϱ without a dash.) The terms of the first order in ϱ will give the transitions necessary for our problem in the *second* approximation of perturbation theory, while the terms of second order in ϱ will give the same transitions but in the *first* approximation

of perturbation theory. Let us use (1.2) for the roton energy, and carry out the expansion indicated:

$$H_r = H_{r,0} + \frac{\partial \Delta}{\partial \varrho} \varrho + \frac{1}{2} \left[\frac{\partial^2 \Delta}{\partial \varrho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \varrho} \right)^2 \right] \varrho^2. \quad (3.4)$$

Here $H_{r,0}$ is the energy of the roton in the absence of phonons. From the form of the energy spectrum it follows that the majority of rotons have momenta close to P_0 , and hence the terms in the expansion of H_r containing the difference $(P - P_0)$ are to be neglected, and have not been written down.

We have likewise excluded the term in (3.4) containing the differential $\partial \Delta / \partial \varrho$. The justification for this lies in the anomalously small coefficient of thermal expansion of He II. The magnitude of this coefficient may be calculated by total differentiation of the thermodynamic potential with respect to pressure and temperature. The most important part of the expression obtained in this way turns out to be proportional to the effective value of $\partial \Delta / \partial \varrho$.

In this way, with the help of (3.3) and (3.4), we have finally for the interaction energy of phonon and roton

$$V = -\frac{1}{2} [(\mathbf{P} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{P})] + \frac{1}{2} \left[\frac{\partial^2 \Delta}{\partial \varrho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \varrho} \right)^2 \right] \varrho^2. \quad (3.4a)$$

The derivatives $\partial^2 \Delta / \partial \varrho^2$ and $\partial P_0 / \partial \varrho$ entering into this expression can be calculated from the experimental values of the variation of the velocity of second sound with pressure obtained by V. Peshkov and K. Zinovieva⁹. The values of the derivatives

$$\partial^2 \Delta / \partial \varrho^2 \cong 2 \Delta / \varrho^2 \quad \text{and} \quad \partial P_0 / \partial \varrho \cong P_0 / \varrho$$

obtained in this way turn out to be such appreciable quantities that the effect in the scattering of phonons by rotons arising from these terms exceeds the one depending on the term containing the velocity \mathbf{v} . The magnitude of the derivative $\partial^2 \Delta / \partial \varrho^2$ can likewise be calculated from the data available on the temperature dependence on the velocity of sound in He II. In this way, values are obtained which agree with those quoted above.

Because of the smallness of the velocity of oscillations of \mathbf{v} and the departures ϱ of the density from the equilibrium value ϱ_0 , the interaction energy can be considered as a small perturbation in the Hamiltonian expression (3.1) of the phonon-roton system. Thus the calculation of the probability of scattering of phonons by rotons will be carried out according to the scheme of perturbation theory.

Since the scattering effect concerned is a two-phonon one, the terms in the perturbation energy which are linear in ϱ or \mathbf{v} will give the necessary transitions in the second approximation of perturbation theory. However, the fact that the equation of hydrodynamics are non-linear somewhat changes the usual picture in this respect. In fact, we shall solve the equations of hydrodynamics by the method of successive approximations.

In the first approximation, the density fluctuation ϱ (or the velocity \mathbf{v}) will be a superposition of plane waves. The terms in the second approximation

of the density ϱ (or v) will contain even powers of various Fourier components. The matrix elements of terms in the second approximation of the density ϱ will contain even powers of the phonon amplitudes.

Among the even powers will be included some which correspond exactly to the scattering of phonons in the process considered, i.e. they will contain amplitudes both of the incident and of the scattered phonons. Consequently, already in the first approximation of perturbation theory terms linear in ϱ (or v) in the perturbation energy will give transitions essential to our problem. In view of what has been said, we shall solve the equations of hydrodynamics taking account of terms of the second order in ϱ .

Because of the smallness of the fluctuations in density and pressure, we shall write the variables in the form $P_0 + p$, $\varrho_0 + \varrho$, where p_0 and ϱ_0 are the constant equilibrium values of the pressure and density. Excluding small quantities of the third order from Euler's equation, we obtain

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{c^2}{\varrho_0} \nabla \varrho + \frac{\varrho}{\varrho_0} \nabla \varrho \left(\frac{c^2}{\varrho_0} - \frac{\partial c^2}{\partial \varrho} \right). \quad (3.5)$$

To the same approximation, the equation of continuity reduces to

$$\frac{\partial \varrho}{\partial t} + \varrho_0 \operatorname{div} v = -\operatorname{div} \varrho v. \quad (3.6)$$

We now apply the divergence operator the equation (3.5), differentiate (3.6) with respect to p , and subtract the latter from the former. As a result, we obtain the wave equation taking account of terms of the second order, which we collect together on the right-hand side.

$$\frac{\partial^2 \varrho}{\partial t^2} - c^2 \nabla^2 \varrho = -\operatorname{div} \left\{ \frac{\partial \varrho v}{\partial t} - \varrho_0 (v \cdot \nabla) v + \left(\frac{c^2}{\varrho_0} - \frac{\partial c^2}{\partial \varrho} \right) \varrho \nabla \varrho \right\}. \quad (3.7)$$

To a first approximation, we have for the incident and scattered waves,

$$v = v_1 e^{i[(p \cdot r) - c p t]/\hbar}, \quad \varrho = \varrho_1 e^{i[(p \cdot r) - c p t]/\hbar} \quad (3.8a)$$

$$v' = v'_1 e^{i[(p' \cdot r) - c p' t]/\hbar}, \quad \varrho' = \varrho'_1 e^{i[(p' \cdot r) - c p' t]/\hbar} \quad (3.8b)$$

where the quantities without the dash refer to the incident wave, and those with the dash to the scattered one. In the second approximation we have looked only for those terms in which the incident and scattered waves are muddled up. The remaining terms do not give conservation of energy and disappear in the matrix elements of the perturbation theory.

We denote the unit vector in the direction of the incident waves by n and that in the direction of the scattered waves by n' . Using the method of successive approximations, we substitute (3.8a) and (3.8b) into the right-hand side of (3.7). Then in the second approximation we obtain for the density

$$\begin{aligned} \varrho_2 = & \frac{\varrho_1 \varrho'_1}{c^2 p p' [1 - (n \cdot n')]} \left\{ [p - p' (n \cdot n')] [p' - p (n \cdot n')] \frac{c^2}{\varrho_0} \right. \\ & \left. - \frac{1}{2} (p - p')^2 \frac{\partial c^2}{\partial \varrho} \right\} e^{\frac{i}{\hbar} [(p - p') \cdot r - c(p - p') t]}. \end{aligned} \quad (3.9)$$

In the same approximation we find the velocity from (3.5),

$$v_2 = \frac{\varrho \varrho' (p - p')(p' - p)}{\varrho_0 c p p' [1 - (n \cdot n')]} \left\{ (n \cdot n') \frac{c^2}{\varrho_0} + \frac{1}{2} \frac{\partial c^2}{\partial \varrho} \right\} e^{i[(p - p') \cdot r] - c(p - p')t/\hbar}. \quad (3.10)$$

The problem now consists in determining the probability of transition of a roton from the state with momentum P to the state with momentum P' in which the phonon p is absorbed and the phonon p' is emitted. Two intermediate states, I and II, are possible.

I. The roton in the state (A) with momentum P absorbs the phonon p and goes over into the intermediate state $P + p$, after which it emits the phonon p' and goes over into the final state (F) having momentum $P' = P + p - p'$.

II. A roton in the state (A) with momentum P emits a phonon p' and goes over to the intermediate state $P - p'$ after which it absorbs a phonon p and goes over to the final state (F) with momentum $P' = P + p - p'$. The scattering of phonons by rotons is in a certain sense analogous to the scattering of light particles by heavy ones. This analogy is connected with the fact that a phonon has a momentum which is much smaller than the momentum of the roton. From the distribution function for rotons it follows that the roton momentum is approximately equal to P_0 . Taking account of the law of conservation of momentum, the law of conservation of energy for the process considered becomes according to (1.1) and (1.2)

$$c p + \frac{(P - P_0)^2}{2\mu} = c p' + \frac{(|P + p - p'| - P_0)^2}{2\mu} \quad (3.11)$$

from which after simple transformations taking into account the smallness of the phonon momenta (p and $p' \ll P_0$), we obtain

$$c(p - p') = \frac{(P \cdot p - p')^2}{2\mu P_0^2}. \quad (3.12)$$

Let us now use the fact that the phonon energy $\varepsilon = c p$ is much less than the quantity μc^2 which is approximately equal to 20°K . This allows one to conclude that the law of conservation of energy in the process simply amounts to equality of the magnitude of the momenta of the incident and scattered phonons, i.e. $p = p'$. This result confirms the analogy pointed out above with the scattering of light particles by heavy ones, where likewise the scattered particles change only the direction of their momenta without change of their magnitudes.

According to (3.4a) the perturbation energy V contains a quantity v_2 , which is determined by (3.10). However, the quantity v_2 is significantly dependent on the factor $p - p'$. Thus taking account the law of conservation of energy, we may omit in the perturbation energy terms of the second approximation in v_2 . According to (3.4a) the transitional matrix element H'_{AF} is, in

the second approximation of perturbation theory, given by

$$\begin{aligned}
 H'_{AF} = & \frac{[(\mathbf{P} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{P})]_{AI} [(\mathbf{P} \cdot \mathbf{v}') + (\mathbf{v}' \cdot \mathbf{P})]_{IF}}{4(E_A - E_I + pc)} \\
 & + \frac{[(\mathbf{P} \cdot \mathbf{v}') + (\mathbf{v}' \cdot \mathbf{P})]_{AII} [(\mathbf{P} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{P})]_{IIF}}{4(E_A - E_{II} - p^1 c)} \\
 & + \frac{1}{2} \left[-\frac{\partial^2 \Delta}{\partial \varrho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \varrho} \right)^2 \right] \varrho_{AF}^2.
 \end{aligned} \quad (3.13)$$

The changes of energy for transitions into the intermediate state are given by

$$\begin{aligned}
 E_A - E_I &= -\frac{(|\mathbf{P} + \mathbf{p}| - P_0)^2}{2\mu} = -\frac{(\mathbf{P} \cdot \mathbf{p})^2}{2\mu P_0^2}, \\
 E_A - E_{II} &= -\frac{(|\mathbf{P} - \mathbf{p}'| - P_0)^2}{2\mu} = -\frac{(\mathbf{P} \cdot \mathbf{p}')^2}{2\mu P_0^2}.
 \end{aligned} \quad (3.14)$$

On the same basis as in (3.12), it can be asserted that

$$E_A - E_I \ll pc \quad \text{and} \quad E_A - E_{II} \ll p'c.$$

However, we cannot simply neglect the energy differences $E_A - E_I$ and $E_A - E_{II}$ in the denominators of the first two terms in (3.13), since the expressions of zero order in the phonon momenta obtained in this way cancel each other. Therefore we shall first reduce the first two terms in (3.13) to a common denominator, and only then exclude in the latter the changes of energy of the roton. In this way we obtain

$$\begin{aligned}
 H'_{AF} = & \frac{v v'}{p^2 p'^2 c^2} \left\{ \left(\mathbf{P} + \frac{1}{2} \mathbf{p} \cdot \mathbf{p} \right) \left(\mathbf{P}' + \frac{1}{2} \mathbf{p}' \cdot \mathbf{p}' \right) \left[p' + \frac{(\mathbf{P} \cdot \mathbf{p}')^2}{2\mu c P_0^2} \right] \right. \\
 & - \left(\mathbf{P} - \frac{1}{2} \mathbf{p}' \cdot \mathbf{p}' \right) \left(\mathbf{P}' - \frac{1}{2} \mathbf{p} \cdot \mathbf{p} \right) \left[p - \frac{(\mathbf{P} \cdot \mathbf{p})^2}{2\mu c P_0^2} \right] \Big\} \\
 & + \left[\frac{\partial^2 \Delta}{\partial \varrho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \varrho^2} \right)^2 \right] \varrho \varrho'
 \end{aligned} \quad (3.15)$$

where in the development of the expression ϱ_{AF}^2 account must be taken of possible transpositions of ϱ and ϱ' , leading to a doubling of the corresponding terms.

Using the conservation laws, we transform the expression in the curly brackets of (3.15) into the form

$$P_0 p^3 \left\{ (\mathbf{n} + \mathbf{n}' \cdot \mathbf{m})(\mathbf{n} \cdot \mathbf{n}') + \frac{P_0}{\mu c} (\mathbf{n} \cdot \mathbf{m})^2 (\mathbf{n}' \cdot \mathbf{m})^2 \right\}. \quad (3.16)$$

Here \mathbf{m} is the unit vector in the direction of the momentum vector of the roton \mathbf{P}^\dagger .

[†] From the conservation of momentum it follows that the direction of the momentum \mathbf{P} of the roton is conserved.

Finally, the matrix element of the transition given by (3.13) takes the form (with the help of (3.16)),

$$H'_{AF} = \frac{P_0 p}{2\varrho_0} \left\{ (n + n' \cdot m)(n \cdot n') + \frac{P_0}{\mu c} (n \cdot m)^2 (n' \cdot m)^2 + A \right\}, \quad (3.17)$$

$$\text{where } A = \frac{\varrho_0^2}{P_0 c} \left[\frac{\partial^2 A}{\partial \varrho^2} + \frac{1}{\mu} \left(\frac{\partial P_0}{\partial \varrho} \right)^2 \right].$$

The effective differential cross section for scattering of phonons by rotons is given by

$$d\sigma = \frac{2\pi}{\hbar c} |H'_{AF}|^2 \delta(E_A + p c - E_F - p' c) (2\pi \hbar)^{-3} p'^2 dp' d\sigma'. \quad (3.18)$$

We now substitute (3.17) into (3.18) and integrate over the possible momenta of the scattered phonon, and find

$$d\sigma = \left(\frac{P_0 p^2}{4\pi \hbar^2 \varrho_0 c} \right)^2 \left\{ (n + n' \cdot m)(n \cdot n') + \frac{P_0}{\mu c} (n \cdot m)^2 (n' \cdot m)^2 + A \right\}^2 d\sigma'. \quad (3.19)$$

In solving the problem of the viscosity of He II, we shall need the expression for the probability of scattering of phonons by rotons, in which the direction of the phonon momentum p is changed by a given angle ψ . Therefore we shall average the expression (3.19) over the angles of the incident and scattered rotons. After some simple calculations we find

$$d\sigma(p, \psi) = \left(\frac{P_0 p^2}{4\pi \hbar^2 \varrho_0 c} \right)^2 \left\{ \frac{2}{3} (1 + \cos \psi) \cos^2 \psi + \frac{1}{105} \left(1 + 8 \cos^2 \psi + \frac{8}{3} \cos^4 \psi \right) + \frac{2A}{15} \left(\frac{P_0}{\mu c} \right) (1 + 2 \cos^2 \psi) + A^2 \right\} d\sigma'. \quad (3.20)$$

The angle ψ entering into (3.20) is the angle between the directions of the incident and scattered phonons. Integrating (3.20) over all angles of scatter we find the total effective cross-section of scattering of phonons with momentum p by rotons

$$\sigma_{pr} = \frac{1}{4\pi} \left(\frac{P_0 p^2}{\hbar^2 \varrho_0 c} \right)^2 \left[\frac{2}{9} + \frac{1}{25} \left(\frac{P_0}{\mu c} \right)^2 + \frac{2A}{9} \frac{P_0}{\mu c} + A^2 \right]. \quad (3.21)$$

The value of the parameter A , calculated with the help of the values quoted above for the derivatives of the parameters A and P_0 , turns out to be 1.25. It should, of course, be noted that the accuracy of these derivatives and the parameter A is not particularly high.

Inserting the numerical values of all the parameters entering into (3.21), we obtain for σ_{pr} :

$$\sigma_{pr} = 7 \times 10^{-19} (xT)^4, \quad \text{where } p = x(kT/c). \quad (3.22)$$

4. THE SCATTERING OF ROTONS BY ROTONS

Theory gives no indication of the character of the interaction of rotons with rotons; thus naturally the problem of scatter of rotons by rotons cannot be solved exactly. However, since our aim is the calculation only of the temperature dependence of the viscosity of He II, such an exact solution does not appear to be indispensable. For finding the temperature variation it is sufficient to know the probability of scattering of rotons by rotons as a function of temperature, apart from a constant multiplier; such a probability is not sensitive to the choice of the roton-roton interaction function.

In calculating the probability of scattering of rotons by rotons we will use the methods of perturbation theory, considering the interaction energy of rotons by rotons as a δ -function of the distance between them. It is well known that in applying perturbation theory the choice of interaction energy as a δ -function does not cause any divergence in the expressions obtained for the probability.

Let the interaction energy of two rotons be

$$V = V_0 \delta(\mathbf{r} - \mathbf{r}_1), \quad (4.1)$$

where \mathbf{r} and \mathbf{r}_1 are the radius vectors of the rotons and V_0 a constant, for which a value may be determined from the magnitude of the viscosity of He II. Denoting the energy and momentum of the rotons by E and \mathbf{P} , without a dash for the incident rotons and with a dash for the scattered rotons, the transition probability of rotons from state (A) with momenta \mathbf{P} and \mathbf{P}_1 , to state (F) with momenta \mathbf{P}' and \mathbf{P}'_1 , is given by the following formula of perturbation theory:

$$dw = \frac{2\pi}{\hbar} |V_{AF}|^2 \delta(E + E_1 - E' - E'_1) \cdot d\mathbf{P}' d\mathbf{P}'_1 / (2\pi\hbar)^6. \quad (4.2)$$

For the wave functions for the rotons we choose plane waves symmetrised with respect to the pairs of incident and scattered rotons. Thus for the incident rotons the wave function is

$$\psi(\mathbf{P}, \mathbf{P}_1) = \frac{1}{\sqrt{2}} \left\{ e^{i[(\mathbf{P} \cdot \mathbf{r}) + (\mathbf{P}_1 \cdot \mathbf{r})]/\hbar} + e^{i[(\mathbf{P} \cdot \mathbf{r}_1) + (\mathbf{P}_1 \cdot \mathbf{r})]/\hbar} \right\},$$

and similarly for the scattered rotons,

$$\psi(\mathbf{P}', \mathbf{P}'_1) = \frac{1}{\sqrt{2}} \left\{ e^{i[(\mathbf{P}' \cdot \mathbf{r}) + (\mathbf{P}'_1 \cdot \mathbf{r}_1)]/\hbar} + e^{i[(\mathbf{P}' \cdot \mathbf{r}_1) + (\mathbf{P}'_1 \cdot \mathbf{r})]/\hbar} \right\}$$

With the help of wave functions of this type, we calculate the transition matrix element V_{AF} :

$$\begin{aligned} V_{AF} &= V_0 \Omega^{-1/2} \int \psi^*(\mathbf{P}, \mathbf{P}_1) \delta(\mathbf{r} - \mathbf{r}_1) \psi(\mathbf{P}', \mathbf{P}'_1) d\Omega d\Omega_1 \\ &= 2V_0 \Omega^{-1/2} \int e^{i(\mathbf{P}' + \mathbf{P}'_1 - \mathbf{P} - \mathbf{P}_1 \cdot \mathbf{r})/\hbar} d\Omega. \end{aligned} \quad (4.3)$$

The square of the modulus of the matrix element can easily be integrated with respect to the momenta of one of the scattered rotons:

$$\int |V_{\Delta P}|^2 \frac{dP'_1}{(2\pi\hbar)^3} = \frac{4|V_0|^2}{\Omega} \int dP'_1 \delta(P' + P'_1 - P - P_1) \int d\Omega e^{i(P' + P'_1 - P - P_1 \cdot r)/\hbar} \\ = |V_0|^2 \quad (4.4)$$

This relation allows the transition probability dw to be transformed into

$$dw = \frac{8\pi}{\hbar} |V_0|^2 \delta(E + E_1 - E' - E'_1) dP'/(2\pi\hbar)^3. \quad (4.5)$$

The probability given by (4.5) has the dimensions of $\text{cm}^3 \text{sec}^{-1}$. To obtain the effective differential cross-section of scattering, the above probability must be divided by the relative velocity of the interacting rotons, equal to

$$v = \left| \frac{\partial E}{\partial P} - \frac{\partial E_1}{\partial P_1} \right|.$$

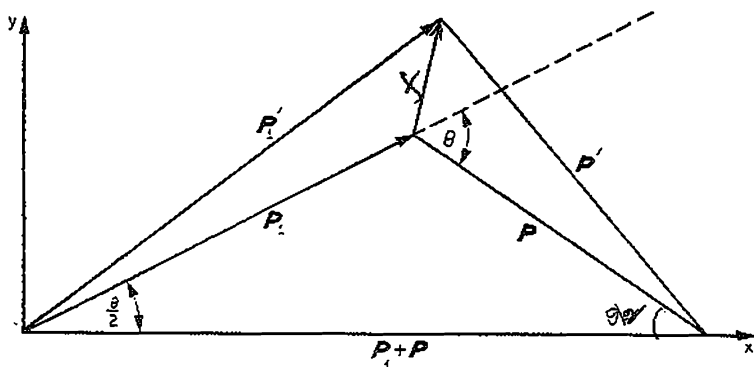


FIG. 1.

From the form of the energy spectrum it follows that the majority of rotons will have momenta close in absolute value to P_0 . Consequently the changes of momenta of the rotons in scattering will in magnitude be appreciably less than P_0 .

Let the momenta of the rotons P and P' before collision make an angle θ . Then it is easily seen from Fig. 1 that the momenta of the rotons after collision may, on introducing a variable f , be represented in the following form:

$$P' = P_0 + f_x \cos \frac{\theta}{2} + f_y \sin \frac{\theta}{2}, \quad P'_1 = P_0 + f_x \cos \frac{\theta}{2} - f_y \sin \frac{\theta}{2} \quad (4.6)$$

where $|f| \ll P_0$. In the new variables, the element of phase volume reduces to

$$dP' = 2\pi P_0 \sin \frac{\theta}{2} df_x df_y \quad (4.7)$$

and the law of conservation of energy to

$$f_x^2 \cos^2 \frac{\theta}{2} + f_y^2 \sin^2 \frac{\theta}{2} = \frac{1}{2} (P - P_0)^2 + \frac{1}{2} (P_1 - P_0)^2. \quad (4.8)$$

For calculating the total effective cross-section σ of scatter of rotons by rotons, we must integrate (4.5) over the phase volume of the scattered particles. In (4.5) it is only the δ -function containing the law of conservation of energy which depends on the co-ordinates in phase space.

If for convenience of integration we introduce the auxiliary variable g with the help of the relation $g^2 = f_x^2 \cos^2 (\theta/2) + f_y^2 \sin^2 (\theta/2)$, then taking account of (4.7) and (4.8), the integration of the δ -function over the phase volume of the scattered particles becomes very simple. In fact

$$\begin{aligned} \int \delta(E + E_1 - E' - E'_1) dP' &= \int \mu \delta \left(f_x^2 \cos^2 \frac{\theta}{2} + f_y^2 \sin^2 \frac{\theta}{2} - \frac{1}{2} (P - P_0)^2 \right. \\ &\quad \left. - \frac{1}{2} (P_1 - P_0)^2 \right) \cdot 2\pi P_0 \sin \frac{\theta}{2} df_x df_y \\ &= \int \delta(g^2 - \frac{1}{2} (P - P_0)^2 - \frac{1}{2} (P_1 - P_0)^2) \frac{2\pi P_0 \mu}{\cos \frac{\theta}{2}} \cdot 2\pi g dg = \frac{2\pi^2 P_0 \mu}{\cos \frac{\theta}{2}}. \end{aligned} \quad (4.9)$$

Now making use of (4.5) and (4.9), we write the expression for the total effective cross section σ of scattering of rotons by rotons as

$$\sigma = \frac{2P_0 \mu |V_0|^2}{\left| \frac{\partial E}{\partial P} - \frac{\partial E_1}{\partial P_1} \right| \hbar^4 \cos \frac{\theta}{2}}. \quad (4.10)$$

The reciprocal of the mean time t_r between two collisions of a roton is obtained from (4.10) multiplied by the total flux of rotons, and followed by the averaging over all angles made by the momenta of the colliding rotons. In this way we find†.

$$\frac{1}{t_r} = \sigma \overline{\left| \frac{\partial E}{\partial P} - \frac{\partial E_1}{\partial P_1} \right|} N_r = \frac{4P_0 \mu |V_0|^2 N_r}{\hbar^4}. \quad (4.11)$$

Here N_r is the number of rotons per unit volume given by

$$N_r = \frac{2P_0^2 (\mu kT)^{1/2} e^{-\epsilon/kT}}{(2\pi)^{3/2} \hbar^3}. \quad (4.12)$$

The magnitude of the constant $|V_0|^2$ entering into (4.10) and (4.11) can be calculated from the experimental values of the coefficient of viscosity of He II

† The bar denotes the averaging over angles.

(see Part II Section 11). It turns out to be of order of magnitude 0.25×10^{-26} (erg. cm³)².

In conclusion, we draw attention to the circumstance that the expression (4.10) for the effective cross-section σ diverges when the angles between the tracks of the momenta of the colliding rotons are close to π . For rotons having a mean velocity

$$\bar{v} = \left(\frac{2kT}{\pi\mu} \right)^{1/3}, \quad (4.13)$$

equation (4.10) gives, after substitution of numerical values of the parameters,

$$\sigma \simeq 5 \times 10^{-15} T^{-1/2}. \quad (4.14)$$

It should be noted that besides the processes of scattering of elementary excitations considered above, processes of emission and absorption of rotons and phonons are possible, as a result of which the total number of rotons and phonons may change. The appropriate assessments which can be carried out for these processes of emission and absorption of rotons shows, as might be expected, that these processes are less probable than the process of scattering of rotons by rotons. This is explained by the circumstance that the creation of a new roton demands a considerable threshold energy of the order of Δ . As for the process of emission and absorption of phonons, we shall devote section 8 of the second part of our investigation to this.

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70. THE THEORY OF THE VISCOSITY OF HELIUM II

II. CALCULATION OF THE VISCOSITY COEFFICIENT

The kinetic equations for rotons and phonons are solved by using effective differential cross-sections for the scattering of elementary excitations (phonons and rotons) by each other, as obtained in the first part of this work. It is shown that the viscosity coefficient of helium II is composed of two parts: one caused by scattering of rotons ("roton viscosity") and not dependent on temperature; the other, caused by scattering of phonons ("phonon viscosity") and sharply increasing with fall in temperature. The experimental values for the viscosity coefficient in helium II appear to be in good agreement with theory.

5. THE KINETIC EQUATION

The expressions obtained in the first part of this work for the probability of scattering of various types of excitations by each other permit investigation of the temperature dependence of the viscosity coefficient of helium II. The kinetic equation for elementary excitations which we have to solve may be written in the form:

$$\frac{\partial n}{\partial t} + (v \cdot \nabla n) = J(n) \quad (5.1)$$

where $n = n(r, v, t)$ is the distribution function, v the velocity, and $J(n)$ the collision integral, the exact form of which is deduced below for each type of interaction.

We will assume that in a fluid there is a macroscopic flow of velocity u , varying with the co-ordinate in such a way that the velocity gradient is so small that equilibrium is almost established in each volume element corresponding to a given value of the velocity: i.e. in each volume element the distribution functions for rotons and phonons are almost equal to their equilibrium values. Thus the distribution function n for rotons (or phonons) may be represented as the sum of the equilibrium function n_0 and a small variation δn :

$$n = n_0 + \delta n. \quad (5.2)$$

Since we are seeking a stationary solution of the kinetic equation appropriate to a given macroscopic flow (constant in time), the left-hand term

Л. Д. Ландау и И. М. Халатников, Теория вязкости гелия II, II. Вычисление коэффициента вязкости, *Журнал Экспериментальной и Теоретической Физики*, 19, 709 (1949).

$\partial n / \partial t$ in the kinetic equation should be taken as equal to zero. Furthermore, we need only substitute the equilibrium function n_0 (instead of n) in the approximation term $(\mathbf{v} \cdot \nabla n)$ in question, since this term already contains a very small quantity—the velocity gradient of macroscopic flow.

In a fluid moving with velocity \mathbf{u} , the equilibrium roton distribution function appears as:

$$n_0 = \exp \left\{ -\frac{\Delta}{kT} - \frac{(P - P_0)^2}{2\mu kT} + \frac{(\mathbf{P} \cdot \mathbf{u})}{kT} \right\}. \quad (5.3)$$

Let us choose the direction of the z -axis along the velocity vector \mathbf{u} and for the sake of simplicity let us assume that the velocity gradient is directed along an axis x perpendicular to the z -axis. It is evident that the value of the viscosity coefficient required is independent of the co-ordinate system. In calculating the first viscosity coefficient we need only consider the case where that velocity \mathbf{u} satisfies the condition $\text{div } \mathbf{u} = 0$. Let us now assume that the density and temperature are constant throughout the liquid. As a result, on the left-hand side of the kinetic equation we have:

$$(\mathbf{v} \cdot \nabla n) = v_x \frac{\partial n_0}{\partial x} = n_0 \frac{P_x v_x}{kT} \frac{\partial u}{\partial x}.$$

If we use a spherical system of co-ordinates with the polar axis lying in the z -direction, the expression obtained may be rewritten in the form

$$n_0 \frac{P_0 v}{kT} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi, \quad (5.4)$$

in which it is legitimate to write P_0 instead of P since rotons possess momenta of magnitude close to P_0 . In accordance with (1.2) the velocity v is given by

$$v = \frac{\partial E}{\partial P} = \frac{P - P_0}{\mu}. \quad (5.5)$$

The final kinetic equation for rotons becomes:

$$n_0 \frac{P_0(P - P_0)}{\mu kT} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi = J(n). \quad (5.6)$$

The kinetic equation for phonons may be written in an analogous form. In this case, however, allowance must be made for the fact that phonons obey Bose statistics, and therefore their equilibrium function for a flowing gas is

$$n_0 = \left\{ \exp \left[\frac{cp - (\mathbf{u} \cdot \mathbf{p})}{kT} \right] - 1 \right\}^{-1} \quad (5.7)$$

Carrying out similar calculations to those given above for rotons, but with the distribution function of (5.7), we obtain the kinetic equation for phonons

$$n_0(n_0 + 1) \frac{cp}{kT} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi = J(n). \quad (5.8)$$

The collision integrals on the right side of the equations (5.6) and (5.8) are dependent on the nature of the interactions of rotons with each other and with phonons in the case of equation (5.6), and on the nature of the interactions of phonons with each other and with rotons in the case of equation (5.8).

The value of the viscosity coefficient evidently does not depend on the part of the liquid considered. It is convenient to select a point at which the velocity of macroscopic motion is zero. At this point the expressions for the scattering function (5.3) and (5.7) are not dependent on velocity, and coincide with the corresponding equilibrium functions in a motionless liquid.

Due to the presence of two types of excitations in helium II we have written two kinetic equations. That portion of viscosity caused by momentum transfer by rotons will be conditionally referred to as roton viscosity; and that part due to transfer of momentum by phonons as phonon viscosity. Actually one viscosity is observed, this being equal to the sum of the roton and phonon contributions.

6. ROTON VISCOSITY

Provided the processes of their emission and absorption (see section 4) are disregarded, the change in the number of rotons in a given phase volume will take place in two ways, namely: (a) elastic scattering of rotons by rotons (section 4); (b) scattering of rotons by phonons (section 3).

However, simple calculations show that down to temperatures of the order of 0.6–0.7° K the contribution to the viscosity of the scattering of rotons by phonons will be negligible compared to the elastic scattering of rotons by rotons. This is not only because the number of phonons at higher temperatures is found to be less than the number of rotons, but mainly because the roton momenta are changed very little during the scattering of rotons by phonons; accordingly the momentum flux, which determines the magnitude of the viscosity, is found to be negligible. Since at temperatures below 1°K the roton viscosity becomes negligibly small in comparison with the phonon viscosity, we conclude that it is sufficient to take into account only roton-roton scattering in the kinetic equation.

The roton kinetic equation (5.6) obtained above cannot be solved accurately. However, if we do not attempt an exact calculation of the roton portion of the viscosity, but limit ourselves to a determination of its temperature dependence, the kinetic equation (5.6) may be simplified. For then it is permissible to replace the collision integral on the right-hand side of this equation by:

$$-\delta n/t, \quad (6.1)$$

where t differs from the mean time between roton collisions (t_r) by a factor which is independent of temperature and is of the order of unity. Because the time interval t_r found in sect. 4 contains the unknown constant V_0 , we will from now on simply write t_r everywhere instead of t , including the factor of order unity indicated above within the quantity V_0 .

Thus the kinetic equation (5.6) becomes

$$\frac{P_0(P - P_0)}{\mu kT} n_0 \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi = - \frac{\delta n}{t_r}. \quad (6.2)$$

From this we find the deviation of the distribution function δn from its equilibrium value

$$\delta n = - \frac{t_r P(P - P_0)}{\mu kT} n_0 \cos \theta \sin \theta \cos \varphi \frac{\partial u}{\partial x}. \quad (6.3)$$

We will now calculate the momentum flux. In our case only the z -component of the momentum will differ from zero, being directed towards x and equal to

$$\begin{aligned} \sigma_{xz} &= - \int P_z v_x \delta n \frac{dP}{(2\pi\hbar)^3} \\ &= \frac{t_r}{kT \mu^2 (2\pi\hbar)^3} \frac{\partial u}{\partial x} \int P_0^2 (P - P_0)^2 \cos^2 \theta \sin^2 \theta \cos^2 \varphi n_0 dP. \end{aligned} \quad (6.4)$$

Comparing (6.4) with the expression for the corresponding tensor component and pressure containing the viscosity coefficient† η_r

$$\sigma_{xz} = \eta_r \frac{\partial u}{\partial x}$$

we find the magnitude of the viscosity coefficient required:

$$\eta_r = \frac{t_r}{kT \mu^2 (2\pi\hbar)^3} \int P_0^2 (P - P_0)^2 \cos^2 \theta \sin^2 \theta \cos^2 \varphi n_0 dP. \quad (6.5)$$

We carry out the necessary integration in (6.5) over the elements of phase volume dP using the distribution function n_0 given by (5.3). The final result is

$$\eta_r = \left(\frac{t_r P_0^2}{15\mu} \right) N_r = \frac{\tau_r P_0^2}{15\mu}. \quad (6.6)$$

The quantity τ_r , equal to the product $t_r N_r$ is independent of temperature. And actually, according to (4.11) we have

$$\tau_r = t_r N_r = \frac{\hbar^4}{P_0 \mu} |V_0|^{-2}. \quad (6.7)$$

Thus the roton portion of the viscosity coefficient is given in terms of constant quantities and is therefore a constant independent of temperature.

† The suffix r is used to distinguish the roton part of the viscosity coefficient.

7. ESTABLISHMENT OF EQUILIBRIUM OF A PHONON GAS

A change in the number of phonons in the phase volume element may occur in the following ways: (a) through scattering of phonons by phonons (section 2); (b) by absorption and emission of phonons due to inelastic impacts of rotons or phonons with each other.

In considering the role of each of these processes in the viscosity, we first calculate the relaxation times (or the corresponding free paths) characterising the establishment of equilibrium in a phonon gas.

In section 2 it was shown that the effective cross section of phonon-phonon scattering reaches its maximum importance for small angles between the momenta of the colliding phonons. It follows from the laws of conservation of momentum and energy that such a process of scattering does not lead to a material change in the direction of the momenta of colliding phonons, and therefore scattering of phonons by phonons leads mainly to rapid exchange of energies between phonons. However, an exact calculation of the relaxation time characteristic of the establishment of energy equilibrium in a phonon gas cannot be accomplished for the simple reason that the problem itself cannot be precisely formulated. Nevertheless, since the time required for the establishment of energy equilibrium in a phonon gas is an extremely important characteristic of our system, we will attempt to approach its determination from two limiting cases.

In the first case we assume that by some means there has been produced in the phonon gas a change in the number of phonons possessing small energies (smaller than the average phonon energy) so that the distribution function for the region of small energies does not equal the equilibrium function; then we calculate the relaxation interval characterising the establishment of equilibrium in such a gas.

In the second limiting case we assume that in the phonon gas there has been a change in the number of phonons having large energies (greater than the average energy of the phonons) so that the distribution function in the region of large energies does not equal the equilibrium function. In this case we calculate the relaxation time characterising the establishment of equilibrium in such a gas. Comparing it with the time characterising the viscous processes of transfer, we show that the process of establishing energetic equilibrium in a phonon gas is more rapid than the viscous processes.

We start with the first case; that is, we examine the scattering of small energy phonons by phonons. In this case, it may be assumed that the momentum p of the phonon under question is considerably smaller than the momentum p_1 of the phonon with which this phonon collides. As was shown in section 2, such an assumption considerably simplifies the expression for the averaging over angles of the phonon-phonon scattering. In the case in question, the angularly averaged effective cross-section of phonon-phonon scattering is by (2.11)

$$\sigma(p, p_1) = \frac{\pi(u+2)p_1^4}{(96\pi\hbar^2\rho_0c)^2\gamma}.$$

Let us look at the kinetic equation for phonons

$$\frac{\partial n}{\partial t} + (\mathbf{v} \cdot \nabla n) = J(n),$$

where the collision integral $J(n)$ for the process of phonon-phonon scattering equals

$$J(n) = - (2\pi \hbar)^{-3} \int c \sigma(p, p_1) \{n n_1 (n' + 1)(n'_1 + 1) - n' n'_1 (n + 1)(n_1 + 1)\} d p_1. \quad (7.1)$$

We are interested in the relaxation of small energy phonons, whose distribution function equals n , with the given equilibrium distribution of the remaining phonons. Therefore the distribution functions n_1 , n' and n'_1 in (7.1) will be regarded as in equilibrium, and the deviation of the distribution function n from the equilibrium value n_0 will be regarded as small and equal to δn . Now using certain properties of equilibrium distribution functions, we transform the expression enclosed in paranthesis in (7.1).

$$\begin{aligned} & \{(n_0 + \delta n) n_1 (n' + 1)(n'_1 + 1)\} - n' n'_1 (n_1 + 1)(n_0 + 1 + \delta n) \\ &= \delta n \{n_1 (n' + 1)(n'_1 + 1) - n' n'_1 (n_1 + 1)\} = \delta n n' n'_1 (n_1 + 1)/n. \end{aligned} \quad (7.2)$$

Substituting the expression obtained in the collision integral, we have

$$J(n) = - (2\pi \hbar)^{-3} \delta n \int n^{-1} n' n'_1 (n_1 + 1) c \sigma(p, p_1) d p_1. \quad (7.3)$$

Thus the relaxation time for phonons with small energy is determined by the relationship

$$\frac{1}{t_{sm}} = (2\pi \hbar)^{-3} \int n^{-1} n' n'_1 (n_1 + 1) c \sigma(p, p_1) d p_1. \quad (7.4)$$

In order to simplify the integration of (7.4), let us replace the functions n_1 , n' , and n'_1 by the Wien function and make allowance for the fact that the momenta p and p_1 satisfy the inequality $p \ll p_1$. In this way the combination of distribution functions entering into expression (7.4) appears as: $n_1 (n_1 + 1) p c / k T$. Using this and substituting in (7.4) expression (2.11) for $\sigma(p, p_1)$, we have†

$$\frac{1}{t_{sm}} = (2\pi \hbar)^{-3} \frac{\pi(u+2)^4}{(96\pi \hbar^2 \varrho_0)^2 c \gamma} \int n_1 (n_1 + 1) p_1^4 d p_1 \frac{p c}{k T} \quad (7.5)$$

or after an elementary integration

$$\frac{1}{t_{sm}} = \frac{(u+2)^4 6!}{(48 \hbar^2 \varrho_0)^2 c \gamma (2\pi \hbar)^3} \left(\frac{k T}{c} \right)^6 p. \quad (7.6)$$

Automatic allowance is made for the fact that phonons having momentum p_1 possess high energy, since in the integration over $d p_1$ a substantial role is played by phonons with energy of the order of $6 k T$.

† Integration over $d p_1$ with a zero lower limit results in a certain error, which however is not material, since our results are of approximate nature.

Let us proceed to the case of relaxation of high energy phonons. In this case the assumption that the phonon momenta obey the relation:

$$p_1 \ll p \quad (7.7)$$

permits a simplification of the expression for the angularly averaged probability of phonon scattering by a phonon. Substituting p_1 for p in expression (2.11), we obtain

$$\sigma(p, p_1) = \frac{\pi(u+2)^4 p^4}{(96\pi \hbar^2 \varrho_0 c)^2 \gamma}. \quad (7.8)$$

In quite the same way as in the case of low energy phonons, we find the relationship for the relaxation time t_l for high energy phonons

$$\frac{1}{t_l} = (2\pi \hbar)^{-3} \int n^{-1}(n_1 + 1) n' n'_1 c \sigma(p, p_1) d p_1, \quad (7.9)$$

which coincides with expression (7.4). This agreement is natural, since in deriving equation (7.4) the assumption about the magnitude of the energy of relaxing phonons was nowhere employed. Let us now make use of the assumption about the smallness of the momentum; taking the phonon distribution functions n , n' , and n_1 as small compared to unity, we may simplify the combination of distribution functions entering into expression (7.9).

We have†

$$n^{-1} n' n'_1 (n_1 + 1) \simeq n_1 (n_1 + 1). \quad (7.10)$$

Substituting (7.8) and (7.10) in equation (7.9), we have

$$\frac{1}{t_l} = (2\pi \hbar)^{-2} \frac{\pi(u+2)^4}{(96\pi \hbar^2 \varrho_0 c)^2 c \gamma} p^4 \int n_1 (n_1 + 1) d p_1. \quad (7.11)$$

Finally, after an elementary integration, we find

$$\frac{1}{t_l} = \frac{(u+2)^4 (kT/c)^3 (\pi^2/3) p^4}{(48 \hbar^2 \varrho_0)^2 c \gamma (2\pi \hbar)^3}. \quad (7.12)$$

The fact that the momentum p_1 is small is automatically satisfied in expression (7.11), because a substantial role is played in the integral contained there by energies of the order of $2kT$. Using the numerical values of the parameters entering into formulae (7.6) and (7.12), we calculate the coefficients in these expressions. The result obtained is the relaxation time of low energy phonons:

$$\frac{1}{t_{sm}} \simeq 2 \times 10^7 T^7 x, \quad (7.13)$$

† The assumption of the smallness of functions n , n' , and n'_1 is equivalent to the condition $p, p', p'_1 \gg kT/c$.

and for phonons with high energy:

$$\frac{1}{t_i} \simeq 10^5 T^7 x^4. \quad (7.14)$$

Here the quantity x is related to the phonon energy by $\varepsilon = x k T$.

In determining the relaxation time of phonons possessing an arbitrary energy, an interpolation formula can be selected which would go over into equations (7.13) and (7.14) obtained in the two limiting cases. The following formula satisfies this condition:

$$\frac{1}{t_g} = 10^5 T^7 x(x + 6)^3. \quad (7.15)$$

Scattering of phonons by phonons does not cause a substantial change in the direction of the momenta of the scattered phonons, but on the other hand, as already noted, this process provides for the establishment of equilibrium of energy in a phonon gas. The reciprocal of the calculated relaxation time ($1/t_g$) characterises the speed with which their equilibrium is established in a phonon gas.

8. VISCOSITY DUE TO THE SCATTERING OF PHONONS BY ROTONS AND OF PHONONS BY PHONONS

The effective cross-section of various scattering processes enter into the collision integral additively, and therefore we rewrite the kinetic equation for phonons (5.8) as:

$$n_0(n_0 + 1) \frac{c p}{k T} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi = J_I(n) + J_{II}(n) + J_{III}(n) \quad (8.1)$$

where $J_I(n)$ is the collision integral for the process of scattering of phonons by rotons, $J_{II}(n)$ is the collision integral for the process of scattering of phonons by phonons, and finally, $J_{III}(n)$ is the collision integral associated with the process of small angle scattering of phonons by one other, as discussed in section 7.

The kinetic equation (8.1) as written here determines the distribution function for phonons of arbitrary energy moving in a given direction. Let us make an assumption, the correctness of which will be convincingly shown below. Let us assume that the process of establishing energy equilibrium in a phonon gas takes place noticeably faster than the scattering of phonons by rotons and of phonons by phonons. Then the phonons moving in a given direction will be distributed in energy in just the same way as they would in complete equilibrium in the absence of a gradient of the macroscopic velocity. According to section 7, establishment of energy equilibrium in a phonon gas is accomplished by the process of scattering of phonons by phonons at small angles. In such a process, the directions of motion of colliding phonons and scattered phonons coincide. At each scattering process, the energy of the

colliding phonons is conserved. Inasmuch as the phonons themselves do not change their direction of motion, it may therefore be asserted that the total energy of the phonons moving in a given direction remains unchanged.

Let us integrate the left and right-hand sides of equation (8.1) over the total range of phonon energies. As a result of such an operation, the integral over all energies derived from $J_{\text{III}}(n)$ turns out to be identically zero, because the total energy of the phonons, moving in a given direction during small-angle phonon scattering, is conserved. Thus after the integration indicated, equation (8.1) becomes:

$$\frac{c}{kT} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi \int_0^{\infty} n_0(n_0 + 1) \varepsilon p^3 dp = \int_0^{\infty} \{J_{\text{I}}(n) + J_{\text{II}}(n)\} \varepsilon p^2 dp \quad (8.2)$$

(where $\varepsilon = c p$).

A change in the total number of phonons can take place as a result of emission and absorption of phonons. The simplest of such processes is the triple-phonon one in which one phonon turns into two, or the reverse; this is forbidden because of the impossibility of simultaneous fulfillment of two conservation laws, those of energy and momentum. The prohibition indicated does not apply to the five-phonon process, i.e. for the process of absorption or emission of a phonon when two phonons collide with each other. The calculation of the probability of such a process is theoretically possible but involves extremely cumbersome calculations. It is sufficient to mention that the number of intermediate states for the transfer of two phonons into three amounts to over 50. Fortunately, it is possible to avoid these calculations, since the relaxation time corresponding to the five-phonon process can be calculated directly from experimental values for the absorption coefficient of sound in helium II.

These values show that at temperatures above 0.9°K the five-phonon process takes place more slowly than the processes of scattering characterising viscosity. At lower temperatures, the picture is the opposite—the process of emission (or absorption) of the third phonon during collision of two phonons becomes more rapid than the scattering processes characterising viscosity. This circumstance is likewise confirmed by estimates of the probability of the five-phonon process.

We have estimated the probability of emission or absorption of phonons during collisions of rotons among themselves†. These processes turn out to be slower than the processes of scattering characterising viscosity for all temperatures below the lambda-point. Let us first examine the region above 0.9°K. From the above it follows that for the times characterising viscous effects at these temperatures, the total number of phonons in motion in a given direction may be materially altered.

Nevertheless, the change in the total number of phonons in motion in a given direction does not take place during scattering of phonons by phonons at small angles. For this reason, in integrating the left and right sides of

† This question will be more fully considered in another paper.

equation (8.1) for all phonons moving in a given direction the corresponding integral from $J_{\text{II}}(n)$ becomes identically zero. As a result we have

$$\frac{c}{kT} \frac{\partial u}{\partial x} \cos \theta \sin \theta \cos \varphi \int_0^\infty n_0(n_0 + 1) p^3 dp = \int_0^\infty \{J_{\text{I}}(n) + J_{\text{II}}(n)\} p^2 dp. \quad (8.3)$$

From what has been said above about the relative slowness of the processes of emission and absorption of phonons, it follows from equation (8.3) that the change in the total number of phonons moving in a given direction is entirely determined by the scattering processes characterising viscosity.

In accordance with our assumption for the times characterising the viscous process, the energetic equilibrium for phonons moving in a given direction is not destroyed. Thus phonons moving in a given direction have some temperature T' , in general not equal to the average phonon gas temperature, and are characterised by an equilibrium distribution function appropriate to this temperature.

Nevertheless, because the number of phonons may change during the intervals characterising viscosity, the distribution function for phonons moving in a given direction will not be simply the Planck function, but will contain some function α' similar to a chemical potential, and dependent on the direction of the phonons' motion. Accordingly, at the temperatures under discussion (above 0.9°K) phonons moving in a given direction are described by the distribution function

$$n = [e^{\alpha' + (pc/kT')} - 1]^{-1}, \quad (8.4)$$

where α' and T' are functions of the direction of the momentum of the phonon in question in the system of co-ordinates selected.

The function α' may be of either sign, and for this reason the expression $\alpha' + (pc/kT')$ will become zero at certain phonon energies. However, as the magnitude α' , being proportional to the macroscopic velocity gradient, is extremely minute, this possibility could be realised only for phonons with extremely minute energy $\epsilon \ll kT$. But the number of such phonons is so minute that they play no role at all in processes of scattering characteristic of viscosity; for this reason the circumstance mentioned has no influence on further calculations.

Let the average temperatures of the phonon gas be T ; then the deviation of the distribution function n from the equilibrium value n_0 is, in accordance with (8.4) and (5.7),

$$\delta n = n - n_0 = -n_0(n_0 + 1) \left\{ \alpha' - \frac{pc}{kT} \frac{\delta T}{T} \right\}, \quad (8.5)$$

where δT is the deviation in temperature of phonons moving in a given direction, from the average temperature T ($\delta T = T' - T$).

The deviation δn of the distribution function from its equilibrium value in the temperature region under consideration depends on the two variables α' and δT ; to determine them we use equations (8.2) and (8.3).

We will now consider the temperature region below 0.9°K . In this case the processes of emission and absorption of phonons take place faster than the processes of scattering characteristic of viscosity. Because of the noticeable difference in the laws of temperature variation for the processes mentioned, the once comparatively slow processes become comparatively rapid on transition through a small interval of temperatures, so that by 0.8°K the processes of emission and absorption of phonons may already be considered rapid. Thus in the region of temperatures below 0.8°K the number of phonons moving in a given direction will be described merely by Planck's function, dependent only on the temperature T' of the phonons moving in that direction:

$$n = [e^{pc/kT'} - 1]^{-1}. \quad (8.6)$$

In this case we obtain the following expression for the deviation of the distribution function of phonons from the equilibrium value

$$\delta n = n - n_0 = n_0(n_0 + 1) \frac{pc}{kT} \frac{\delta T}{T}. \quad (8.7)$$

We will use equation (8.2) to determine δn in this region of temperatures. Equation (8.3) has no significance here because the motion of the total number of phonons moving in a given direction is not determined by viscous processes but by the more rapid processes of emission and absorption of the phonons.

Let us calculate the phonon portion of the viscosity coefficient for the temperature region above 0.9°K . To do this equations (8.2) and (8.3) must be solved. Let us begin by a transformation of the collision integral $J_I(n)$ appearing on the right-hand side of (8.2) and (8.3). The collision integral $J_I(n)$ for scattering process of a phonon by rotons equals

$$J_I(n) = - (2\pi \hbar)^{-3} \int \int c \, d\sigma(p, \psi) \{ N n (N' + 1) (n' + 1) - N' n' (N + 1) (n + 1) \} d\mathbf{P}. \quad (8.8)$$

Here N is the distribution function of rotons, n is the distribution function of phonons, \mathbf{P} is the roton momentum and \mathbf{p} is the phonon momentum. The functions for colliding phonons and rotons are written without a prime and the corresponding functions for scattered particles are indicated with a prime. The effective differential cross section $d\sigma$ of the scattering of phonons by rotons at an angle ψ equals, by (3.20)

$$d\sigma(p, \psi) = \left(\frac{P_0 p^2}{4\pi \hbar^2 \rho_0 c} \right)^2 \left\{ \frac{2}{3} (1 + \cos \psi) \cos^2 \psi + \frac{1}{105} \left(\frac{P_0}{\mu c} \right)^2 \left(1 + 8 \cos^2 \psi + \frac{8}{3} \cos^4 \psi \right) + \frac{2A}{15} \left(\frac{P_0}{\mu c} \right) (1 + 2 \cos^2 \psi) + A^2 \right\} dO$$

(dO is an element of solid angle).

Where there is a macroscopic gradient of the velocity u , the distribution of rotons and phonons is characterised by non-equilibrium functions. However, it may easily be shown that due to the rapidity of the process of scattering of rotons by rotons, the deviation of their distribution functions from the equilibrium values will be considerably less than the corresponding deviations for phonons. For this reason rotons will be described by their equilibrium distribution functions in the collision integral $J_I(n)$. For phonons, however, we will have to substitute non-equilibrium functions as in (5.2). After such substitution and integration over the elements of phase volume dP of the colliding rotons, the collisions integral $J_I(n)$ appears as follows:

$$J_I(n) = -N_r \int c d\sigma(p, \psi) \{\delta n - \delta n'\}, \quad (8.9)$$

where N_r represents the number of rotons per unit volume, and the deviation δn is determined by the expressions in the relation (8.5).

The left-hand side of equation (8.2) and (8.3) contains a function of the angles of the form $P_{21}(\cos\theta) \cos\varphi$, where $P_{21}(\cos\theta)$ is an associated Legendre function. Therefore equations (8.2) and (8.3) should be solved in such a way that the expressions for the deviation δn entering into the right-hand sides of expressions (8.2) and (8.3) should be proportional to similar functions of the angles. In future, however, in order to simplify transformations of the collision integrals, we will use in the intermediate calculations the Legendre functions of zero order in $P_2(\cos\theta)$, instead of the function $P_{21}(\cos\theta) \cos\varphi$.

Such a substitution is possible because the spherical symmetry of the scattering makes the collision integral insensitive to it. Naturally in the final results we will substitute instead of the function $P_2(\cos\theta)$ the function $P_{21}(\cos\theta) \cos\varphi$ obtained in the system of co-ordinates selected.

Thus we write the functions α' and $\delta T/T$ determining the deviation of the distribution for phonons moving in a given direction from the equilibrium value in the form

$$\alpha' = \alpha P_2(\cos\theta), \quad \frac{\delta T}{T} = \beta P_2(\cos\theta), \quad (8.10)$$

where α and β are coefficients independent of the angles. With the aid of (8.5) and (8.10) the collision integral $J_I(n)$ becomes:

$$J_I(n) = N_r \int n_0(n_0 + 1) c d\sigma(p, \psi) \left(\alpha - \frac{pc}{kT} \beta \right) [P_2(\cos\theta')]. \quad (8.11)$$

Let us now use the following result of Legendre's theorem on the addition of these functions. Let there be two directions in space fixed at their polar angles θ and ψ and azimuths φ and φ' ; then the Legendre function of zero order (depending on the cosine of angle θ' between these directions) satisfies the following integral relationship:

$$\int_0^{2\pi} P_n(\cos\theta') d\varphi = 2\pi P_n(\cos\theta) P_n(\cos\psi). \quad (8.12)$$

Expression (8.11) contains an integration over the solid angle of the scattered phonons, and therefore applying relationship (8.12) we rewrite the collision integral (8.11) as:

$$J_I(n) = P_2(\cos\theta) N_r \int c d\sigma(p, \psi) [1 - P_2(\cos\psi)] \left(\alpha - \frac{pc}{kT} \beta \right) n_0(n_0 + 1).$$

From this, after integration over the elements of solid angle dO of the scattered phonons, we obtain:

$$J_I(n) = \frac{1}{\tau} P_2(\cos\theta) n_0(n_0 + 1) \left(\alpha - \frac{pc}{kT} \beta \right) \left(\frac{pc}{kT} \right)^4, \quad (8.13)$$

where we introduce the symbol

$$\frac{1}{\tau} = \frac{N_r}{4\pi c} \left[\frac{P_0(kT/c)^2}{\hbar^2 \rho_0} \right]^2 \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right]. \quad (8.14)$$

The quantity τ has the dimension of time.

Next we calculate the integrals on the right-hand sides of (8.2) and (8.3),

$\int_0^\infty J_I(n) p^2 dp$ and $\int_0^\infty J_I(n) \epsilon p^2 dp$. The required integration is carried out very simply by taking as the distribution function n_0 the Planck equilibrium function. Thus we find

$$\int_0^\infty J_I(n) p^2 dp = P_2(\cos\theta) (6! \alpha - 7! \beta) \left(\frac{kT}{c} \right)^3 \frac{1}{\tau}, \quad (8.15)$$

$$\int_0^\infty J_I(n) \epsilon p^2 dp = P_2(\cos\theta) (7! \alpha - 8! \beta) \left(\frac{kT}{c} \right)^3 \frac{kT}{\tau}. \quad (8.16)$$

We now carry out a transformation of the collision integral J_{II} analogous to that given above, taking into consideration the process of scattering of phonons by phonons. The integral concerned is written as follows:

$$J_{II}(n) = -(2\pi\hbar)^{-3} \int \int c d\sigma(p, p_1, p', p'_1) \{ (n n_1 (n' + 1) (n'_1 + 1) - n' n'_1 (n + 1) (n_1 + 1)) \} dp_1 \quad (8.17)$$

where the differential effective cross-section $d\sigma(p, p_1, p', p'_1)$ is determined by the relation (2.9).

Let us substitute in (8.17) the phonon distribution functions in the form of the sums of the equilibrium functions and small additions such as those

of (8.5). Evaluation of (8.10) then gives

$$J_{II}(n) = \alpha(2\pi\hbar)^{-3} \int \int c \, d\sigma(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}'_1) n_0 n_{10} (n'_0 + 1) (n'_{10} + 1) \cdot \\ \{P_2(\cos\theta) + P_2(\cos\theta_1) - P_2(\cos\theta') - P_2(\cos\theta'_1)\} d\mathbf{p}_1, \\ - (2\pi\hbar)^{-3} \beta \left(\frac{c}{kT} \right) \int \int c \, d\sigma(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}'_1) n_0 n_{10} (n'_0 + 1) (n'_{10} + 1) \\ \{p P_2(\cos\theta) + p_1 P_2(\cos\theta_1) - p' P_2(\cos\theta') - p'_1 P_2(\cos\theta'_1)\} d\mathbf{p}_1. \quad (8.18)$$

Here \mathbf{p} and \mathbf{p}_1 are the momenta of the colliding phonons, \mathbf{p}' and \mathbf{p}'_1 are the momenta of scattered phonons, θ is the angle formed by the path of the phonon with the corresponding index and that of the phonon with the given direction. We introduce angles ψ_1 , ψ' and ψ'_1 between the directions of the momenta \mathbf{p}_1 , \mathbf{p}' and \mathbf{p}'_1 and the momentum \mathbf{p} , and apply the result (8.12) of the addition theorem for Legendre functions. As a result (8.18) becomes:

$$J_{II}(n) = P_2(\cos\theta) \alpha(2\pi\hbar)^{-3} \int \int c \, d\sigma(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}'_1) n_0 n_{10} (n'_0 + 1) (n'_{10} + 1) \times \\ [1 + P_2(\cos\psi_1) - P_2(\cos\psi') - P_2(\cos\psi'_1)] d\mathbf{p}_1 \\ - P_2(\cos\theta) \beta \frac{c}{kT} (2\pi\hbar)^{-3} \int \int c \, d\sigma(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}'_1) n_0 n_{10} (n'_0 + 1) (n'_{10} + 1) \\ \{p + p_1 P_2(\cos\psi_1) - p' P_2(\cos\psi') - p'_1 P_2(\cos\psi'_1)\} d\mathbf{p}_1. \quad (8.19)$$

Let us show that the second integral in (8.19) appears even without allowance for scattering. Making the obvious substitution for $P_2(\cos\psi)$, the bracketed expression in the integral may be written as follows:

$$\frac{3}{2} (p' \sin^2\psi' + p'_1 \sin^2\psi'_1 - p_1 \sin^2\psi_1). \quad (8.20)$$

We deduct from the equation for conservation of energy $p + p_1 - p' - p'_1 = 0$, the projection of the equation of momentum conservation in the direction of the momentum \mathbf{p}

$$p + p_1 \cos\psi_1 - p' \cos\psi' - p'_1 \cos\psi'_1 = 0.$$

Our result is

$$p_1(1 - \cos\psi_1) - p'(1 - \cos\psi') - p'_1(1 - \cos\psi'_1) = 0. \quad (8.21)$$

Multiplying the expression obtained by 3 adding the result to expression (8.20), we obtain

$$\frac{3}{2} [p_1(1 - \cos\psi_1)^2 - p'(1 - \cos\psi')^2 - p'_1(1 - \cos\psi'_1)^2]. \quad (8.22)$$

It is easy to see from equation (8.21) that the expression (8.22) is proportional to $(1 - \cos\psi_1)^2$ when the angles between the momenta of colliding phonons are small. But in accordance with (2.7) the differential effective cross-section for the scattering process in question diverges as $(1 - \cos\psi)^{-2}$ in this case, if

dispersion is neglected. Thus the second integral in (8.19) remains finite for all angles between the momenta of colliding phonons.

If dispersion is neglected the first integral in this same expression (8.19) diverges logarithmically for small angles between the momenta of colliding phonons.

The calculation of the integrals $\int_0^\infty J_{II}(n) p^2 dp$ and $\int_0^\infty J_{II}(n) \varepsilon p^2 dp$, which we will not reproduce here due to its extreme unwieldiness, shows that at temperatures above 0.9° K these integrals may be neglected in comparison with the corresponding integrals on the right-hand sides of equations (8.2) and (8.3) involving $J_I(n)$. Equality of the corresponding integrals involving $J_I(n)$ and $J_{II}(n)$ occurs only at 0.7° K. Therefore when considering the phenomenon of viscosity in the temperature region above 0.9° K, we will make allowance on the right-hand sides of the equations only for effects due to scattering of phonons by rotons, these being described by the collision integral $J_I(n)$.

As shown above, at lower temperatures, the parameter α may be assumed equal to zero. Equation (8.2) is used to determine the only unknown parameter β .

According to (8.19), in this case integral $J_{II}(n)$ simply equals

$$J_{II}(n) = -P_2(\cos\theta) \frac{c\beta}{kT} (2\pi\hbar)^{-3} \iiint c d\sigma(p, p_1, p', p'_1) n_0 n_{10} (n'_0 + 1) (n'_{10} + 1) \\ \times [p + p_1 P_2(\cos\psi) - p' P_2(\cos\psi') - p'_1 P_2(\cos\psi'_1)] dp_1. \quad (8.23)$$

The calculation of the integral $\int_0^\infty J_{II}(n) \varepsilon p^2 dp$, appearing on the right-hand side of equation (8.2), which we will not discuss here, gives the following result:

$$\int_0^\infty J_{II}(n) \varepsilon p^2 dp = -P_2(\cos\theta) \beta (kT) \left(\frac{kT}{c}\right)^3 \frac{1}{\tau_p} \quad (8.24)$$

where τ_p is a quantity with the dimension of time being determined by the relationship:

$$\frac{1}{\tau_p} = \frac{3 \cdot 13! (u+2)^4}{5 \times 2^{13} (2\pi)^3 \hbar^7 \rho_0^2 c} \left(\frac{kT}{c}\right)^9. \quad (8.25)$$

In view of the complete convergence of the integral (8.23), expression (8.25) naturally does not contain the dispersion parameter.

The integrals appearing on the left-hand sides of the equations (8.2) and (8.3) can be calculated by elementary methods, and their values are

$$\int_0^\infty n_0 (n_0 + 1) \varepsilon p^3 dp = \frac{4}{15} \pi^4 kT \left\{ \frac{kT}{c} \right\}^4, \quad (8.26)$$

$$\int_0^\infty n_0 (n_0 + 1) p^3 dp \simeq \frac{36}{5} \left(\frac{kT}{c}\right)^4. \quad (8.27)$$

We now substitute in equations (8.2) and (8.3) the values for all the integrals involved. Eliminating immaterial factors and dropping angular functions, we obtain the following pair of equations for the determination of the parameters α and β valid in the temperature region above 0.9°K

$$\frac{\partial u}{\partial x} \frac{4\pi^4}{15} = (7!\alpha - 8!\beta) \frac{1}{\tau}, \quad \frac{\partial u}{\partial x} \frac{36}{5} = (6!\alpha - 7!\beta) \frac{1}{\tau}. \quad (8.28)$$

For temperatures below 0.9°K, by applying the same method to equation (8.2) we obtain the equation for the determination of parameter β :

$$\frac{4\pi^4}{15} \frac{\partial u}{\partial x} = -\beta \left(\frac{8!}{\tau} + \frac{1}{\tau_\varphi} \right). \quad (8.29)$$

9. LENGTH OF THE MEAN FREE PATH CHARACTERISING VISCOSITY

We will now calculate the length of the mean free path characterising viscosity. The parameters α and β , which determine the deviation of the distribution function from its equilibrium value, are characterised to a considerable extent by their equilibrium values. With simultaneous change in the values of these parameters, the viscosity coefficient changes correspondingly. We will consider α and β as functions of the time t . Then in complete analogy to equation (8.28) we may obtain two equations

$$\begin{aligned} \frac{\pi^3}{3} \frac{dx}{dt} - \frac{36}{5} \frac{d\beta}{dt} &= -(6!\alpha - 7!\beta) \frac{1}{\tau}, \\ \frac{36}{5} \frac{d\alpha}{dt} - \frac{4\pi^4}{15} \frac{d\beta}{dt} &= -(7!\alpha - 8!\beta) \frac{1}{\tau}, \end{aligned} \quad (9.1)$$

determining the law governing change in these parameters in the temperature region above 0.9°K. We seek a solution of this system of linear differential equations which will be in a form proportional to e^{-t/θ_φ} . The quantity θ_φ is the natural choice for the characteristic time determining viscosity. Multiplying by the phonon velocity c gives us the corresponding phonon mean free path λ_φ . Determining θ_φ from the set (9.1) we obtain a quadratic characteristic equation, with roots equal to

$$\theta_\varphi \approx \frac{16\tau}{6!}, \quad \frac{1}{3} \frac{\tau}{6!}. \quad (9.2)$$

In comparing the appearance of viscosity under macroscopic examination, the largest absolute magnitude of these roots must be chosen. Therefore, taking the largest values θ_φ obtained for the free path of phonons, and applying (9.2) and (8.14), we obtain the following expression:

$$\frac{1}{\lambda_\varphi} = \frac{45N_r}{4\pi c^2} \left[\frac{P_0(kT/c)^2}{\hbar^2 \varrho_0} \right]^2 \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right]. \quad (9.3)$$

To determine the characteristic time θ_φ in the temperature region below 0.9°K, we obtain in analogy with (9.1) the linear differential equation

$$\frac{4\pi^4}{15} \frac{d\beta}{dT} = -\beta \left(\frac{8!}{\tau} + \frac{1}{\tau_\varphi} \right). \quad (9.4)$$

In this case λ_φ is determined by the expression:

$$\frac{1}{\lambda_\varphi} = \left(\frac{8!}{\tau} + \frac{1}{\tau_\varphi} \right) \frac{15}{4\pi^4 c} = \frac{15 \cdot 7!}{2\pi^5 c^2} \left[\frac{P_0 (kT/c)^2}{\hbar^2 \varrho_0} \right]^2 \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right] N_7 + \frac{9 \cdot 13! (u+2)^4}{(2\pi \hbar)^7 2^{17} \varrho_0^2 c^2} \left(\frac{kT}{c} \right)^9. \quad (9.5)$$

Substituting in the expressions (9.3) and (9.5) the numerical values of all the parameters, we finally obtain

$$\frac{1}{\lambda_\varphi} = 3.4 \times 10^6 T^{9/2} e^{-A/kT}, \quad (T > 1^\circ\text{K}) \quad (9.6)$$

$$\frac{1}{\lambda_\varphi} = 1.1 \times 10^8 T^{9/2} e^{-A/kT} + 5.6 \times 10^2 T^9 \quad (T < 0.8^\circ\text{K}).$$

Figure 1 is a graphic representation of the dependence of $1/\lambda_\varphi$ on temperature†. The points along the curve of $1/\lambda_\varphi$ were obtained by interpolation for temperatures between 0.8 and 1.0°K.

Let us now apply the results of the present paragraph and confirm the validity of the assumptions we made about the velocity of the establishment of energetic equilibrium. The expression (7.15) determines the time of establishment of energetic equilibrium for phonons, t_φ . From the assumptions made, it follows that the process of establishment of energetic equilibrium occurs faster than the viscous processes, so that the time t_φ for the phonons, which plays a substantial role in viscosity, should be less than the smallest of the times θ_φ determining the relationship (9.2). It can easily be seen from (9.1) that the phonon component of the viscosity coefficient depends fundamentally on phonons with an energy of the order 7–8 kT . Comparing the times t_φ and θ_φ for phonons possessing such energy, we can confirm that at all temperatures below 1.3–1.4°K the assumption made is fulfilled.

In examining the question of the velocity of establishment of energetic equilibrium it must be kept in mind that the basic parameters, and above all the dispersion parameter γ determining the quantity t_φ , are known only extremely approximately. For this reason the most convincing verification of the assumptions made can only be by experiment. The agreement between the computed and measured values for the viscosity coefficient obtained through

† In all known measurements of the viscosity coefficient of helium II, the size of the system has been considerably greater than the length of the phonon free path. However Andronikashvili has pointed out in experiments on heat conduction in helium II that in narrow slits the mean free path at temperature 1.3°K is already equal to the dimensions of the slits. This circumstance considerably complicates the appearance of heat conduction in helium II in narrow apertures.

the comparisons carried out below convinces us of the correctness of the assumptions made. We are inclined to attribute the mutual approach of the estimated times t_p and θ_p at temperatures above 1.4°K to the approximate nature of the calculations made. This is indirectly confirmed by an analysis of the data on sound absorption in helium II. It should be kept in mind that at temperatures above 1.4°K , the role of the phonon component of the viscosity coefficient becomes relatively slight in comparison to the roton component, and for this reason this question is of no consequence for the temperature region above 1.4°K .

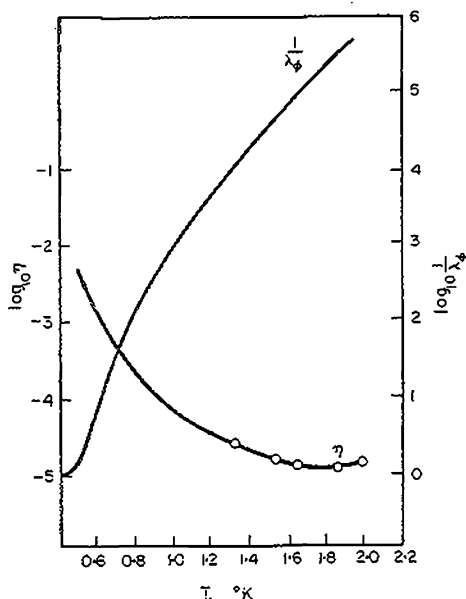


FIG. 1.

10. TEMPERATURE DEPENDENCE OF THE PHONON COMPONENT OF THE VISCOSITY COEFFICIENT

Let us now turn to the calculation of the phonon component of the viscosity coefficient. First we examine the temperature region above 0.9°K . Let us solve the system (8.28) with respect to α and β , and substitute the values obtained in expression (8.5) for the deviation δn of the distribution function from its equilibrium value. Then we obtain

$$\delta n = -n_0(n_0 + 1) \cos \theta \sin \theta \cos \varphi \frac{\tau}{6!} \left(31.2 - 3.44 \frac{pc}{kT} \right) \frac{\partial u}{\partial x}. \quad (10.1)$$

In doing this we have replaced the function $P_2(\cos \theta)$ by $P_{2,1}(\cos \theta) \cos \varphi$. We proceed in a similar way for the temperature region below 0.9°K . From equation (8.19) we find the magnitude of the parameter β , and further, using (8.7)

and (8.10), we find δn :

$$\delta n = -n_0(n_0 + 1) \cos \theta \sin \theta \cos \varphi \frac{4\pi^2 \tau \tau_\varphi}{15(8! \tau_\varphi + \tau)} \frac{p c}{kT} \frac{\partial u}{\partial x}. \quad (10.2)$$

The z -component of momentum, differing from zero in our case and directed along x , equals

$$\sigma_{xz} = -(2\pi \hbar)^{-3} \int p c \delta n \cos \theta \sin \theta \cos \varphi p^2 dp dO. \quad (10.3)$$

Calculating in turn the value σ_{xz} which δn gives by (10.1) and (10.2), and comparing the result with the expression for the appropriate components of the pressure tensor $\sigma_{xx} = \eta_\varphi \partial u / \partial x$, we find after simple reductions the phonon component of the viscosity coefficient:

$$\begin{aligned} \eta_\varphi &= \frac{0.157 (\hbar^2 \varrho_0 c)^2 e^{\Delta/kT}}{(2\pi \mu kT^{1/2}) P_0^4} \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right]^{-1} (T > 1^\circ \text{K}) \\ \eta_\varphi &= \left(\frac{4\pi}{15} \right)^3 (\hbar^2 \varrho_0 c) \left[8! (2\pi \mu kT)^{1/2} P_0^4 e^{-\Delta/kT} \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 \right. \right. \\ &\quad \left. \left. + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right] + \frac{3 \cdot 13! (u+2)^4}{5 \times 2^{19}} \left(\frac{kT}{c} \right)^5 \right]^{-1} (T < 0.8^\circ \text{K}). \end{aligned} \quad (10.4)$$

Substituting the numerical values of all the parameters in (10.4), we obtain as the final result

$$\left. \begin{aligned} \eta &= 8.7 \times 10^{-9} T^{-1/2} e^{\Delta/kT}, & (T > 1^\circ \text{K}) \\ \eta_\varphi &= 7.8 \times 10^{-10} (T^{1/2} e^{-\Delta/kT} + 4.8 \times 10^{-6} T^5)^{-1} & (T < 0.8^\circ \text{K}). \end{aligned} \right\} \quad (10.5)$$

Thus the phonon component of the viscosity coefficient increases as temperature falls. At temperatures above 0.9°K , it increases in accordance with the law $e^{\Delta/kT} T^{-1/2}$. For temperatures below 0.7°K , where only the effect of scattering phonons by phonons is significant, this law is replaced by the law T^{-5} . A more complex law, determined by formula (10.5) governs the intermediate area.

11. TEMPERATURE DEPENDENCE OF THE VISCOSITY COEFFICIENT OF HELIUM II

In section 6 we calculated the roton component of the viscosity coefficient η_r , which turned out to be independent of temperature. In section 10 we set forth formulae for the phonon component of the viscosity coefficient, disclosing a sharp rise of η_φ with temperature drop. The actual viscosity coefficient of helium II is equal to the sum of the roton and phonon components. Since the coefficient for helium is the sum of a part that is constant and one that increases with fall in temperature, the viscosity coefficient ought to rise sharply with fall in temperature. No mention has previously been made of this interesting fact.

According to (6.6) and (10.4) the actual coefficient of viscosity of helium II is given as follows:

$$\eta = \frac{P_0^2 \tau_r}{15\mu} + \frac{0.157(\hbar^2 \varrho_0 c)^2 e^{A/kT}}{(2\pi\mu kT)^{1/2} P_0^4} \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right]^{-1},$$

($T > 1^\circ\text{K}$) (11.1)

$$\eta = \frac{P_0^2 \tau_r}{15\mu} + \left(\frac{4\pi^3}{15} \right)^3 (\hbar^2 \varrho_0 c)^3 \left\{ 8! (2\pi\mu kT)^{1/2} P_0^4 e^{-A/kT} \left[\frac{2}{15} + \frac{33}{35^2} \left(\frac{P_0}{\mu c} \right)^2 + \frac{14A}{75} \left(\frac{P_0}{\mu c} \right) + A^2 \right] + \frac{3 \times 13! (u+2)^4}{5 \times 2^{19}} \left(\frac{kT}{c} \right)^5 \right\}^{-1} \quad (T < 0.8^\circ\text{K}).$$

Since the roton component of the viscosity coefficient cannot be accurately calculated, we determine this quantity by experiment. For this purpose we employ Andronikashvili's¹ data on measurements of the viscosity coefficient down to a temperature of 1.35°K . Subtracting the phonon component calculated from formula (10.5) from the experimental values for the viscosity coefficient, we obtain for the temperature interval between 1.35° and 1.9°K an almost constant magnitude of $1 \cdot 10^{-5}$ viscosity units of measurement. This we identify with the roton component of the viscosity coefficient. Thus we have

$$\eta_r = \frac{P_0^2 \tau_r}{15\mu} = 1 \times 10^{-5} \text{ poise.} \quad (11.2)$$

The values for the constants τ_r and V_0 may be obtained from the resulting value of η_r . These constants are roughly equal to

$$\tau_r \simeq 1.5 \times 10^{10} \text{ sec/cm}^3; \quad V_0 \simeq 0.5 \times 10^{-33} \text{ erg/cm}^3.$$

Using now the numerical values of η_r and the expression (10.5) determining the numerical value of η_p , we write for the final expression of the viscosity coefficient (in poise)

$$\left. \begin{aligned} \eta \times 10^5 &= 1 + 8.7 \times 10^{-4} T^{-1/2} e^{A/kT}, & (T > 1^\circ\text{K}) \\ \eta \times 10^5 &= 7.8 \times 10^{-5} (T^{1/2} e^{-A/kT} + 4.8 \times 10^{-6} T^5)^{-1} & (T < 0.8^\circ\text{K}). \end{aligned} \right\} \quad (11.3)$$

The formulae (11.3) determine the values of η for both temperature regions—i.e. above and below 0.9°K .

Neither of these two formulae is valid in the neighbourhood of 0.9°K . This is explained by the fact that at this temperature, the times characterising the processes of emission and absorption of phonons become equal to the times characterising the viscous processes of scattering. It might have been possible to take this circumstance into account in equation (8.3). However, since this effect is found to be significant only in a small interval of temperature of the order of 0.2°K in the neighbourhood of 0.9°K , we will not give it special consideration.

For purposes of calculation, we use the first expressions of (11.3) in the temperature regions above 1°K, and the second for temperatures below 0.8°K. We find the values for the viscosity coefficient in the intermediate area by interpolation between these calculated values.

In Fig. 1 a comparison is made between the values of the viscosity coefficient obtained from (11.3) and those measured experimentally by Andronikashvili¹. The experimental values for the viscosity coefficient disclose some rise with temperature in a small region near the lambda point. However, it should be remembered that the results that we obtained must not be extended to the temperature region in the neighbourhood of the lambda point. In all our discussions it has been assumed that the phonon and roton gases are practically ideal. Since this assumption is not valid in the region near the lambda point, our results are naturally not applicable in this region.

In conclusion, it must be noted that all the results obtained in the present paper are of extremely limited accuracy owing to the extraordinarily crude state of our knowledge of such theoretical parameters as $\partial c/\partial \rho$, $\partial^2 c/\partial \rho^2$, $\partial^2 \Delta/\partial \rho^2$ and $\partial P_0/\partial \rho$. Of course this remark does not apply to the values obtained for the temperature dependence, which has been determined sufficiently accurately.

In his earlier papers, Tisza obtained the law $\eta \sim T^5$ for the viscosity coefficient from entirely incorrect considerations. In his last work Tisza² rejected this result and obtained another temperature variation $\eta \sim T^{\frac{1}{2}}$ using an arbitrary system of postulates.

Turning to Andronikashvili's¹ latest experiment, and in the same connection to the results of the present paper, one may be satisfied that in reality there is nothing resembling the law $\eta \sim T^{\frac{1}{2}}$ for helium II. In spite of Tisza's assertions to the contrary, helium II possesses a viscosity coefficient η with a derivative $d\eta/dT$ negative for all temperatures except for a small region near the lambda point. In this respect it does not differ from ordinary fluids.

The authors express their gratitude to E. Andronikashvili, E. Lifshitz, and V. Peshkov, for participation in discussions of the questions studied in the present paper.

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71. ON THE ELECTRON-POSITRON INTERACTION

The wave equation for the electron-positron system is established up to terms of order v^2/c^2 with exchange interactions taken into account.

1. It is well known that it is possible in classical electrodynamics to describe a system of interacting charges up to terms of order v^2/c^2 (where v is the velocity of the charge and c is the velocity of light) with the help of the Hamiltonian function depending on the co-ordinates and momenta of the particles and not containing the radiation field. In quantum mechanics the Hamiltonian of the system of electrons was obtained by Breit¹ up to terms of order of v^2/c^2 .

The aim of the present paper is to establish the wave equation describing the electron-positron system (also up to terms of order v^2/c^2). It is necessary to consider here, besides the interactions appearing in the Breit equations, the particular exchange interaction between the electron and the positron. This will be our main problem. As was already noted², the result of Breit was obtained by an unsatisfactory method, in which some terms were artificially neglected. Because of this we shall also consider that part of the electron-positron interaction which is equivalent to the part considered by Breit.

Let us begin with the following form for the Hamiltonian operator of the system of electrons in the second quantised representation:

$$H = \sum_n E_n a_n^+ a_n + \frac{1}{2} \sum_{nn's's'} V_{ns's'}^{n's} a_n^+ a_s^+ a_n a_{s'}. \quad (1)$$

Here E_n is the energy of the electron in the n th state; $V_{ns's'}^{n's}$ are the coefficients (the matrix elements) corresponding to the transition of one electron from state n to state n' with a simultaneous transition of another electron from s to s' ; a_n and a_n^+ are operators satisfying the commutation relations

$$\begin{aligned} a_n^+ a_{n'} + a_{n'} a_n^+ &= \delta_{nn'}, \\ a_n a_{n'} + a_{n'} a_n &= a_n^+ a_{n'}^+ + a_{n'}^+ a_n^+ = 0. \end{aligned} \quad (2)$$

These operators act on functions of the occupation numbers, which in our case (one electron and one positron) have the form

$$c(0, \dots, 0, 1_n, 0, \dots; 1, 1, \dots, 1, 0_{-s}, 1, \dots),$$

i.e. one state having positive energy and all but one of the states having negative energy are filled. Because of this, in (1), one of the upper and one of the

В. Б. Берестетский и Л. Д. Ландау, О взаимодействии между электроном и позитроном, *Журнал Экспериментальной и Теоретической Физики*, 19, 673, 1949.

lower indices of $V_{n_s}^{n's'}$ are related to states with positive energy and the others to states with negative energy (we shall denote the latter by a minus sign $-$). Using (2) we can write (1) in the following form

$$H = \sum_n E_n a_n^+ a_n + \sum_s E_{-s} a_{-s}^+ a_{-s} + \sum_{n s n' s'} (V_{n_s, -s'}^{n', -s} - V_{n_s, -s'}^{-s, n'}) a_n^+ a_{-s'}^+ a_{n'} a_{-s} \quad (3)$$

(Here a summation over positive values of the indices is implied).

In order to introduce the positron into these considerations it is necessary to introduce the operators:

$$b_s = a_{-s}^+; \quad b_s^+ = a_{-s}, \quad (4)$$

with a simultaneous transformation of the functions

$$c(0 \dots 1_n \dots; 1, 1 \dots 0_{-s} \dots) = \gamma(\dots 1_n \dots; 0 \dots 0, 1_s, 0 \dots).$$

Then

$$H = \sum_n E_n (a_n^+ a_n + b_n^+ b_n - 1) - \sum_{n n' s s'} (V_{n_s, -s'}^{n', -s} - V_{n_s, -s'}^{-s, n'}) a_n^+ b_{s'}^+ a_{n'} b_s \quad (5)$$

(where $E_{-n} = -E_n$).

Our problem now consists of finding the coefficients $V_{n_s, -s'}^{n', -s}$ and $V_{n_s, -s'}^{-s, n'}$. If they can be represented as the matrix elements of some operator V acting on the co-ordinates (both spatial and spin) of the electron and the positron

$$-(V_{n_s, -s'}^{n', -s} - V_{n_s, -s'}^{-s, n'}) = \int u_n^*(r_1) v_{s'}^*(r_2) V u_{n'}(r_1) v_s(r_2) d\tau_1 d\tau_2, \quad (6)$$

this would mean that the electron-positron system is described by the wave equation

$$i \hbar \frac{\partial}{\partial t} \Psi(r_1, \sigma_1; r_2, \sigma_2) = [H_0(1) + H_0(2) + V] \Psi, \quad (7)$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hamiltonians of the electron and positron in the absence of their interaction, without any restrictions on the symmetry properties of the wave function Ψ .

The explicit form of the functions u_n and v_s is not important here. In particular, we can mean by them the wave functions for free particles. This, of course, will by no means imply the use of perturbation theory.

2. The free state of the electron is described by two spinor functions $\varphi(\varphi^\lambda; \lambda = 1, 2)$ and $\chi(\chi^\lambda)$ satisfying the Dirac equations

$$c(\sigma \cdot p) \varphi = (E + m c^2) \chi, \quad c(\sigma \cdot p) \chi = (E - m c^2) \varphi, \quad (8)$$

where σ is the Pauli operator, p is the momentum operator and m is the electron mass. In the case of positive values of E when $E - m c^2 \ll m c^2$ up to terms of order of $1/c^2$ one has

$$c \chi = \frac{(\sigma \cdot p)}{2m} \left(1 - \frac{p^2}{4m^2 c^2} \right) \varphi. \quad (9)$$

It is possible to consider, in place of equation (8), one spinor function φ which satisfies the equation

$$\frac{p^2}{2m} \left(1 - \frac{p^2}{4m^2 c^2} \right) \varphi = (E - m c^2) \varphi. \quad (10)$$

The wave function φ is not normalised, since

$$\int (\varphi^* \varphi + \chi^* \chi) d\tau = 1$$

or up to terms of order $1/c^2$:

$$\int \varphi^* \left(1 + \frac{p^2}{4m^2 c^2} \right) \varphi = 1. \quad (11)$$

If one introduce instead of φ the normalised function u

$$\int u^* u d\tau = 1,$$

then with the same accuracy

$$\varphi = \left(1 - \frac{p^2}{8m^2 c^2} \right) u; \quad u = \left(1 + \frac{p^2}{8m^2 c^2} \right) \varphi. \quad (12)$$

Equation (10) also retains its form for the normalised function

$$H_0 u = (E - m c^2) u,$$

where

$$H_0 = (p^2/2m) - (p^4/8m^3 c^2). \quad (13)$$

In the case of states with negative energy ($E < -m c^2$) there are analogous relations but the roles of φ and χ are interchanged. Now

$$c \varphi = - \frac{(\sigma \cdot p)}{2m} \left(1 - \frac{p^2}{4m^2 c^2} \right) \chi, \quad (14)$$

and the normalised function

$$u = \left(1 + \frac{p^2}{8m^2 c^2} \right) \chi \quad (15)$$

satisfies the equation

$$-H_0 u = (E + m c^2) u, \quad (16)$$

where H_0 is the operator (13).

We wish to introduce the wave function for a positron, in place of the wave function for an electron in a state of negative energy. To do this we write out one of equations (8)

$$c(p \cdot \sigma_\mu^\lambda) \chi^\mu = (E - m c^2) \varphi^\lambda,$$

and go to the complex-conjugate equation. Since $p^* = -p$ and $(\sigma_\mu^1)^* = \sigma_\mu^1$, then

$$c(p \cdot \sigma_\mu^1) \bar{\chi}_\mu = (-E + m c^2) \bar{\varphi}_\lambda, \quad (17)$$

where, in accordance with the properties of the spinor transformation, the complex-conjugate spinors

$$\bar{\chi} = (\chi^\mu)^*; \quad \bar{\varphi} = (\varphi^\lambda), \quad (\chi^1 = -\chi_2; \chi^2 = \chi_1) \quad (18)$$

are introduced. (Notice that here $\chi^\mu = -(\chi_\mu)^*$).

From (17) it will be seen that the spinors $\bar{\chi}$ and $\bar{\varphi}$ satisfy the same equation as φ and χ in the case of positive E and thus they describe a particle with positive energy and with its momentum in the opposite direction (because of the change $e^{i(p \cdot r)/\hbar}$ to $e^{-i(p \cdot r)/\hbar}$) to that of the momentum corresponding to the state with negative energy, i.e. they describe a positron. The formula (18) also defines a relation between the directions of the spins.

So we define the wave function of a positron in the following manner

$$v_s^i = u_{-s}^\lambda \quad (19)$$

where

$$\bar{v}_{s,i} = -u_{-s,\lambda}. \quad (19a)$$

This function satisfies the equation

$$H_0 v_s = (E_s - m c^2) v_s, \quad (20)$$

where H_0 is the operator (13).

3. Let us find, to begin with, the matrix element $V_{n,-s}^{n',-s'}$ in formula (5), corresponding to transitions without a change in the sign of the energy (the Breit interaction). In doing this we may start from the classical expression for the interaction energy of the charges (see, e.g. ref. 3.).

$$\int \{r^{-1} \varrho(r_1) \varrho(r_2) - (1/2c)[(r^{-1} \mathbf{j}(r_1) \cdot \mathbf{j}(r_2))] + r^{-3}[\mathbf{j}(r_1) \wedge \mathbf{r}][\mathbf{j}(r_2) \wedge \mathbf{r}]\} d\tau_1 d\tau_2 \quad (21)$$

and substitute in this as densities of charge and current the appropriate ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$) transition densities

$$\varrho(r_1) = e \psi_n^*(r_1) \psi_n(r_1); \quad \varrho(r_2) = e \psi_{-s}^*(r_1) \psi_{-s}(r_2);$$

$$\mathbf{j}(r_1) = ec \psi_n^*(r_1) \boldsymbol{\alpha}_1 \psi_n(r_1); \quad \mathbf{j}(r_2) = e \psi_{-s}^*(r_2) \boldsymbol{\alpha} \psi_{-s}(r_2),$$

where ψ are four component Dirac functions and $\boldsymbol{\alpha}$ are Dirac matrices. Then

$$\begin{aligned} V_{n,-s}^{n',-s'} &= e^2 \int \psi_n^*(r_1) \psi_{-s}(r_2) [r^{-1} - \frac{1}{2} r^{-1} (\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2) \\ &\quad - \frac{1}{2} r^{-3} (\boldsymbol{\alpha}_1 \cdot \mathbf{r})(\boldsymbol{\alpha}_2 \cdot \mathbf{r})] \psi_n(r_1) \psi_{-s}(r_2) d\tau_1 d\tau_2. \end{aligned} \quad (22)$$

This result can also be obtained by the method of Møller potentials⁴.

The expression (22) is meaningful only up to terms of order $1/c^2$. Therefore, we express the small components of the Dirac functions in terms of the large ones by the formulae (9) and (15), in which it will be sufficient to restrict

ourselves to the first terms. Then

$$\begin{aligned}
 V_{n,-s}^{n',-s'} &= e^2 \int \varphi_n^* \chi_{-s}^* \varphi_n \chi_{-s'} \frac{d\tau_1 d\tau_2}{r} \\
 &+ \frac{e^2}{4m^2 c^2} \int [(\sigma_1 \cdot p_1 \varphi_n)^* (\sigma_1 \cdot p_1 \varphi_n) \chi_{-s}^* \chi_{-s'}^* \\
 &+ \varphi_n^* \varphi_n (\sigma_2 \cdot p_2 \chi_{-s})^* (\sigma_2 \cdot p_2 \chi_{-s'})] \frac{d\tau_1 d\tau_2}{r} + \frac{e^2}{8m^2 c^2} \int ([\varphi_n^* \sigma_1 (\sigma_1 \cdot p_1 \varphi_n) \\
 &+ (\sigma_1 \cdot p_1 \varphi_n)^* \sigma_1 \varphi_n] \cdot [\chi_{-s}^* \sigma_2 (\sigma_2 \cdot p_2 \chi_{-s'}) + (\sigma_2 \cdot p_2 \chi_{-s})^* \sigma_2 \chi_{-s'}]) \frac{d\tau_1 d\tau_2}{r} \\
 &+ \frac{e^2}{8m^2 c^2} \int [\varphi_n^* (\sigma_1 \cdot r) (\sigma_1 \cdot p_1 \varphi_n) + (\sigma_1 \cdot p_1 \varphi_n)^* (\sigma_1 \cdot r_1) \varphi_n] \\
 &\times [\chi_{-s}^* (\sigma_2 \cdot r) (\sigma_2 \cdot p_2 \chi_{-s'}) + (\sigma_2 \cdot p_2 \chi_{-s})^* (\sigma_2 \cdot r) \chi_{-s'}] \frac{d\tau_1 d\tau_2}{r^3}. \quad (23)
 \end{aligned}$$

We will not give here the calculations, which are in general simple, connected with the transformation (23), and will restrict ourselves only to the most important details. In order to reduce this expression to the form which interests us, it is necessary to perform an integration by parts. Since, when doing this, higher powers of $1/r$ appear, it is necessary to separate out the region near $r = 0$, and the surface integral over the surface surrounding this region can give a finite value when $r \rightarrow 0$.

Further, in all terms but one, one can change φ_n to u_n and χ_{-s} to u_{-s} , since they already contain factors $1/c^2$. The first term must be expressed in terms of the normalised functions with the help of the formulae (12) and (16). Lastly, the wave functions of the negative energy states are replaced by the positron wave functions, according to (19) so that

$$u_{-s}^* \sigma u_{-s'} = -v_s^* \sigma v_s. \quad (24)$$

Finally we obtain

$$V_{n,-s}^{n',-s'} = \int u_n^*(r_1) v_s^*(r_2) V_1 u_n v_s d\tau_1 d\tau_2, \quad (25)$$

where

$$\begin{aligned}
 V_1 &= \frac{e^2}{r} - \pi \left(\frac{e\hbar}{mc} \right)^2 \delta(r) - \frac{e^2 \hbar}{4m^2 c^2} \frac{1}{r^3} \left[([r \wedge p_1] \cdot \sigma_1) - ([r \wedge p_2] \cdot \sigma_2) \right. \\
 &+ 2([r \wedge p_1] \cdot \sigma_2) - 2([r \wedge p] \cdot \sigma_1) \left. \right] - \frac{e^2}{2m^2 c^2} \left(\frac{1}{r} (p_1 \cdot p_2) + \frac{1}{r^3} x_i x_k p_{1i} p_{2k} \right) \\
 &+ \left(\frac{e\hbar}{2mc} \right)^2 \left[\frac{(\sigma_1 \cdot \sigma_2)}{r^3} - \frac{3(\sigma_1 \cdot r)(\sigma_2 \cdot r)}{r^5} \right] - \frac{8\pi}{3} \left(\frac{e\hbar}{2mc} \right)^2 (\sigma_1 \cdot \sigma_2) \delta(r). \quad (26)
 \end{aligned}$$

The last term, which contains a δ -function, arose from the surface integral mentioned above. In this formula one may ignore the apparent divergence

of the integrals in the last but one term, and, for example, carry out first the angular integrations.† The other δ -functions contained in (26) appeared in an analogous way by the substitution of the normalised functions in place of the non-Hermitean terms of the type $i r^{-3}(\mathbf{r} \cdot \mathbf{p})$.‡ (The operator V_1 is Hermitean.)

4. Let us go now to the calculation of the second matrix element in (5), which expresses the exchange interaction. We shall write it in the form

$$V_{n,-s}^{-\epsilon, n'} = e \int \psi_{n'}^*(\mathbf{r}) [\Phi_{n,-s}(\mathbf{r}) - (\boldsymbol{\alpha} \cdot \mathbf{A}_{n,-s}(\mathbf{r}))] \psi_{-s'}(\mathbf{r}) d\tau e^{i(E_n - E_{-s'})t/\hbar}, \quad (27)$$

where $\Phi_{n,-s}$ and $\mathbf{A}_{n,-s}$ are the potentials induced by the transition of the electron from the state n to the state $-s$.

It is not convenient to use for the potentials $\Phi_{n,-s}$ and $\mathbf{A}_{n,-s}$ the integrals for the retarded potentials, since the retardation factor $\exp[(i/\hbar c)(E_{-s'} - E_s) r]$ contains in the exponent a large factor of order $2m c^2$ which makes it impossible to use its expansion. We shall, therefore, begin with the equations for the potentials.

$$\begin{aligned} \nabla^2 \Phi_{n,-s} - \frac{1}{c^2} \frac{\partial^2 \Phi_{n,-s}}{\partial t^2} &= 4\pi e \psi_{-s}^* \psi_n e^{-i(E_n - E_{-s})t/\hbar}, \\ \nabla^2 \mathbf{A}_{n,-s} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{n,-s}}{\partial t^2} &= -4\pi e \psi_{-s}^* \boldsymbol{\alpha} \psi e^{-i(E_n - E_{-s})t/\hbar}. \end{aligned} \quad (28)$$

Since we consider particle velocities which are small in comparison with the velocity of light, only wave numbers of order $p/\hbar \ll m c/\hbar$ play a part in these equations, whilst the frequencies are approximately equal to

$$(E_n - E_{-s})/\hbar \approx 2m c^2/\hbar. \quad (29)$$

Therefore we can neglect the spatial derivatives in (28) in comparison with the time derivatives. One may, with the same degree of accuracy, use (29) on the right-hand sides of these equations. Then

$$\begin{aligned} \Phi_{n,-s} &= -\pi \left(\frac{\hbar}{m c} \right)^2 e \psi_{-s}^* \psi_n e^{-i(2m c^2/\hbar)t} = -\pi \left(\frac{\hbar}{m c} \right)^2 e (\varphi_{-s}^* \varphi_n + \chi_{-s}^* \chi_n) e^{-i(2m c^2/\hbar)t}, \\ \mathbf{A}_{n,-s} &= -e\pi \left(\frac{\hbar}{m c} \right)^2 \psi_{-s}^* \boldsymbol{\alpha} \psi e^{-i(2m c^2/\hbar)t} \\ &= -\pi \left(\frac{\hbar}{m c} \right)^2 e (\varphi_{-s}^* \boldsymbol{\sigma} \chi_n + \chi_{-s}^* \boldsymbol{\sigma} \varphi_n) e^{-i(2m c^2/\hbar)t}. \end{aligned} \quad (30)$$

† For a discussion of similar questions see ref. 5.

‡ Let us note that the Hamiltonian function of the electron in an external electric field $\mathbf{E} = -\nabla\Phi$ up to terms of order $1/c^2$ has the following form.

$$H = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + e\Phi - \frac{e\hbar}{4m^2 c^2} ([\mathbf{E} \wedge \mathbf{p}] \cdot \boldsymbol{\sigma}) + \frac{e\hbar^2}{8m^2 c^2} \nabla^2 \Phi$$

On the basis of (9) and (14) we can conclude $\Phi_{n,-s}$ has the order of magnitude $1/c^3$ and $A_{n,-s}$ also contains a term of order $1/c^2$ (with the product of χ_{-s} by φ_n).

Further, when substituted into (27), the term containing $\Phi_{n,-s}$ again contains the small components of the Dirac functions and will therefore have the order of magnitude $1/c^4$. The term which contains $A_{n,-s}$ will retain the order of magnitude $1/c^2$. Thus to the accuracy in which we are interested we have

$$\Phi_{n,-s} = 0, \quad A_{n,-s}(r, t) = -\pi \left(\frac{\hbar}{mc} \right)^2 e \chi_{-s}^*(r) \sigma \varphi_n(r) e^{-i(2mc^2/\hbar)t} \quad (31)$$

and

$$V_{n,-s}^{-e,n'} = \pi e^2 \left(\frac{\hbar}{mc} \right)^2 \int ([u_{-s}^*(r) \sigma u_n(r)] \cdot [u_n^*(r) \sigma u_{-s}(r)]) d\tau. \quad (32)$$

In the latter expression we have replaced the non-normalised functions by the normalised ones since the expression is a quantity of order $1/c^2$.

It remains now for us to express (32) in terms of the positron wave functions. To do this let us express the scalar product of two vectors constructed from our spinor functions with the help of the operator σ , in terms of the scalars which can be directly constructed from the four spinors $\bar{u}_{n'}$, u_n , \bar{u}_{-s} , and u_{-s} . It is not difficult to verify that

$$((u_{-s}^* \sigma u_n) \cdot (u_n^* \sigma u_{-s})) = (\bar{u}_{n'} u_n^{\mu}) (\bar{u}_{-s\mu} u_{-s}^{\mu}) - (\bar{u}_{n'\lambda} \bar{u}_{-s}^{\lambda}) (u_n^{\mu} u_{-s\mu}). \quad (33)$$

Introduce now the positron functions according to the definitions (19) and (19a). Then

$$\begin{aligned} \bar{u}_{-s\mu} u_{-s}^{\mu} &= -v_{s\mu} \bar{v}_s^{\mu} = \bar{v}_{s'\mu} v_s^{\mu} = v_s^* v_s, \\ \bar{u}_{n'\lambda} \bar{u}_{-s}^{\lambda} &= \bar{u}_{n'\lambda} v_s^{\lambda} = u_n^* v_s, \\ u_n^{\mu} u_{-s\mu} &= -u_n^{\mu} \bar{v}_{s'\mu} = -v_s^* u_n, \\ ((u_{-s}^* \sigma u_n) \cdot (u_n^* \sigma u_{-s})) &= (u_n^* u_n) (v_s^* v_s) + (u_n^* v_s) (v_s^* u_n). \end{aligned} \quad (34)$$

The second term in (34) differs from the first one by the exchange of the electron and positron functions. It can be written down using the well-known "exchange operator"

$$\begin{aligned} (u_n^* v_s) (v_s^* u_n) &\equiv u_n^*(1) v_s^*(2) v_s(1) u_n(2) \\ &= \frac{1}{2} u_n^*(1) v_s^*(2) [1 + (\sigma_1 \cdot \sigma_2)] u_n(1) v_s(2), \end{aligned} \quad (35)$$

where the indices in brackets denote the corresponding spin variables (the operator σ_1 acts on the spin variables (1) and so on). Now we can write (32) in the final form

$$V_{n,s}^{-e,n'} = \int u_n^*(r_1) v_s^*(r_2) V_2 u_n(r_1) v_s(r_2) d\tau_1 d\tau_2, \quad (36)$$

where

$$V_2 = \frac{1}{2} \pi \left(\frac{e\hbar}{mc} \right)^2 [3 + (\sigma_1 \cdot \sigma_2)] \delta(r_1 - r_2). \quad (37)$$

Thus the wave equation of the electron-positron system can be represented in the form (7), where

$$V = -V_1 + V_2. \quad (38)$$

Let us note that the δ -function appearing in the expression for the interaction energy does not imply, of course, the presence of an especially strong interaction. In reality, having the same integrated value, it is spread over a region of order \hbar/mc . Therefore, terms containing the δ -function can be considered, as for the other terms of order $1/c^2$ as a perturbation on the Coulomb interaction.

We express our gratitude to I. Ya. Pomeranchuk for a discussion about the questions considered.

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72. THE EQUILIBRIUM FORM OF CRYSTALS

A discussion is given of the way in which the surface tension of a crystal depends on the direction of the face. The results obtained are used to examine the nature of the equilibrium surface of crystals.

It is generally supposed that the surface tension α of a crystal is different for different faces (crystal planes). Thus two adjoining faces could have entirely different values of α . However, in 1915 Ehrenfest¹ indicated the correct approach to the problem, and this has been further developed by Yamada², but many of their conclusions are invalid. The purpose of the present note is to attempt a complete discussion of the problem.

To simplify the treatment, we shall consider a two-dimensional "crystal lattice" consisting of a plane square grid. The crystal planes are here represented by straight lines through the lattice points.

Let α_0 be the "surface tension" for a face with indices (01). Let us consider another face meeting the first at a small angle φ and having indices (1 n), where n is large. The surface of the crystal bounded by this face has the form of "steps" whose length n (in units of the lattice parameter a) is large and whose height is small[†], as shown in Fig. 1, where the broken line is the face (16).

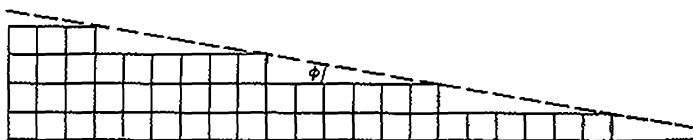


FIG. 1.

The presence of each "step" leads to the appearance of some additional surface energy, which we denote by β . When n is sufficiently large, the steps are so far apart that their "interaction" may be neglected, and the addition to the surface tension α_0 is given simply by the product of β and the number of steps per unit length $1/n a$.

If we use the angle φ between the faces (1 n) and (01), then for sufficiently large n we have $\varphi = 1/n$, and so the surface tension of the face (1 n) may be

Л. Д. Ландау, О равновесной форме кристаллов, *Сборник в честь 70-летия А. Ф. Иоффе*, Москва 1950, p. 44.

[†] It may be noted that the assumption that the steps have a height of unity is made merely for definiteness. The actual form of the steps (and, in particular, their height) is determined by the condition of minimum energy and may be different in different cases. It is, however, important that for faces close to some given face of a crystal of a given substance the steps have a single definite form.

written

$$\alpha = \alpha_0 + \beta \frac{\varphi}{a} \quad (\varphi > 0). \quad (1)$$

As φ tends to zero (i.e. as n tends to infinity), the ratio $(\alpha - \alpha_0)/\varphi$ therefore tends to a finite limit, which may be regarded as the derivative $d\alpha/d\varphi$.

Let us now consider the face with indices $(1n)$ inclined in the opposite direction to the face (01) . The same arguments again give $\beta/n a$ for the change in surface tension, but now $\varphi \cong -1/n$, so that

$$\alpha = \alpha_0 - \beta \frac{\varphi}{a} \quad (\varphi < 0), \quad (2)$$

and the derivative $d\alpha/d\varphi = -\beta/a$.

Thus we see that the surface tension can be written as a continuous function of the direction of the face, but the function is a very peculiar one: for every value of φ it has two different derivatives, one in the direction of the argument (angle) increasing, and the other in the direction of the argument decreasing.

The difference $\Delta(d\alpha/d\varphi)$ between the two values of the derivative at each point is an important characteristic of this function. We shall show that $\Delta(d\alpha/d\varphi)$ has the property of decreasing as the indices of the face increase.

We have seen that for the face (01) $\Delta(d\alpha/d\varphi)$ is equal to the relatively large quantity $2\beta/a$. It is also easy to show that for the face $(1n)$ with n large this difference is zero in the approximation where the "interaction" between the "steps" is neglected, and the validity of this increases with n . In this approximation the surface tension α of the face (i.e. the energy per unit length) consists of the energy $\alpha_0 \cos \varphi$ of the "horizontal" sections of the crystal surface (parallel to the face (01)) and the energy $(\beta/a) \sin \varphi$ due to the presence of $1/a \sqrt{n^2 + 1} = (1/a) \sin \varphi$ "independent" steps per unit length. But the function

$$\alpha(\varphi) = \alpha_0 \cos \varphi + \frac{\beta}{a} \sin \varphi \quad (\varphi > 0) \quad (3)$$

is an ordinary analytic function having only one derivative at each point. Thus to calculate $\Delta(d\alpha/d\varphi)$ in this case we must take into account the relatively small "interaction" effect of the steps.

First of all, it is evident that, in order that the face $(1n)$ should be stable and capable of existence, it is necessary that the steps should "repel" one another; if not, they would "coalesce"†. It may be shown that such a repulsion must actually occur. Figure 2 shows two steps. Their interaction energy is related to the finiteness of the distance cd , i.e. to the absence of interaction of atoms on ab with the atoms which would be on de if the second step did not exist. The interaction between two electrically neutral groups of atoms at a distance is a van der Waals attraction, and so its absence results in a repulsion.

† It must be noted that the proof given here applies only to large distances between the steps, and does exclude the (unlikely) possibility that every face $(1n)$ is metastable and breaks up, for example, into two faces (01) and (10) (if $\beta/a > \alpha_0$).

It is easy to determine the manner of dependence of this repulsion on the distance (i.e. on the length n). The Van der Waals interaction of two atoms at a distance r is proportional to $1/r^6$; summation over all pairs of atoms (which may be replaced by integration over ab and de) evidently gives a quantity proportional to $1/n^4$. This proof, however, depends essentially on the fact that the model considered is two-dimensional. In the actual three-dimensional case a similar treatment would lead to a $1/n^3$ law (for a face of the form $(01n)$).

Let the interaction energy of two steps be denoted by

$$U = \frac{A}{n^3} \quad (A > 0). \quad (4)$$

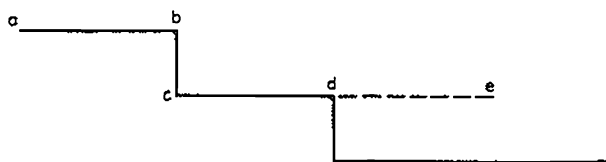


FIG. 1.

The additional surface tension due to the interaction of the steps is evidently

$$\begin{aligned} \alpha_1 &= \frac{1}{a n} \left(\frac{A}{n^3} + \frac{A}{(2n)^3} + \frac{A}{(3n)^3} + \dots \right) \\ &= 1.2 \frac{A}{a n^4}. \end{aligned} \quad (5)$$

We shall show that the repulsion of the steps results in stability not only of $(1n)$ faces but also, for example, of faces of the type $(2n)$ with n an odd number. Such a face consists of steps at alternate distances of $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n-1)$, and such an alternation is energetically more favourable than a breakdown into two faces $(1, \frac{1}{2}(n+1))$ and $(1, \frac{1}{2}(n-1))$: in the former case

$$\begin{aligned} \alpha_1 &= \frac{A}{a n} \left[\frac{1}{[\frac{1}{2}(n-1)]^3} + \frac{1}{[\frac{1}{2}(n+1)]^3} + \frac{2}{n^3} \right. \\ &\quad \left. + \frac{1}{[\frac{1}{2}(3n-1)]^3} + \frac{1}{[\frac{1}{2}(3n+1)]^3} + \frac{2}{(2n)^3} + \dots \right]; \end{aligned}$$

in the latter case we have

$$\begin{aligned} \alpha_1 &= \frac{A}{a n} \left[\frac{1}{[\frac{1}{2}(n-1)]^3} + \frac{1}{(n-1)^3} + \dots \right] \\ &\quad + \frac{A}{a n} \left[\frac{1}{[\frac{1}{2}(n+1)]^3} + \frac{1}{(n+1)^3} + \dots \right]. \end{aligned}$$

It is easily seen that the second expression is greater than the first, and this proves that the face $(2n)$ is stable.

Let us now calculate the difference between the two values of the derivative $d\alpha/d\varphi$ for the face $(1n)$ which results from the interaction between the steps. A face arbitrarily close to the face $(1n)$ is $(N, Nn \pm 1)$, where N is a sufficiently large number. Such a face has N steps in a (horizontal) distance $Nn \pm 1$. In other words, this distance has $N - 1$ intervals of length n and 1 interval of length $n \pm 1$. The interaction energy of the steps in this distance is equal to the interaction energy of N steps at intervals of n (i.e. $Nn\alpha_1$, with α_1 given by (5)) plus

$$A \left[\frac{1}{(n \pm 1)^3} - \frac{1}{n^3} + \frac{2}{(2n \pm 1)^3} - \frac{2}{(2n)^3} + \frac{3}{(3n \pm 1)^3} - \frac{3}{(3n)^3} + \dots \right].$$

To determine α , this energy must be divided by the length $(Nn \pm 1)a$. This gives up to terms of order $1/N$.

$$\alpha_1 + \delta\alpha_1 = \alpha_1 \mp \frac{\alpha_1}{Nn} + \frac{A}{Nna} \left[\frac{1}{(n \pm 1)^3} - \frac{1}{n^3} + \frac{2}{(2n \pm 1)^3} - \frac{2}{(2n)^3} + \dots \right].$$

The angle between the faces $(1n)$ and $(N, Nn \pm 1)$ is $\delta\varphi = \mp 1/Nn^2$. Hence

$$\frac{d\alpha_1}{d\varphi} = \alpha_1 n \mp \frac{An}{a} \left[\frac{1}{(n \pm 1)^3} - \frac{1}{n^3} + \frac{2}{(2n \pm 1)^3} - \frac{2}{(2n)^3} + \dots \right].$$

Finally, the difference between the two values of $d\alpha_1/d\varphi$ is

$$\Delta \left(\frac{d\alpha}{d\varphi} \right) = \frac{An}{a} \left[\frac{1}{(n+1)^3} + \frac{1}{(n-1)^3} - \frac{2}{n^3} + \frac{2}{(2n+1)^3} + \frac{2}{(2n-1)^3} - \frac{4}{(2n)^3} + \dots \right].$$

This is in fact non-zero. For large n it tends to zero:

$$\Delta \left(\frac{d\alpha}{d\varphi} \right) = \frac{12A}{an^4} \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots \right) = \frac{2\pi^4}{15} \cdot \frac{A}{an^4}, \quad (6)$$

i.e. as $1/n^4$. It may be remembered that this formula has been derived essentially for the three-dimensional case, since we have used formula (4).

The above properties of the function $\alpha(\varphi)$ make it possible to draw certain conclusions concerning the form of the equilibrium surface of a crystal. The problem of determining this from the surface tension was first solved by Wulff. We use a co-ordinate system x, y in the plane of the "crystal" considered; let $y = y(x)$ be the equation of its "surface" (i.e. its boundary line). The function $y(x)$ is determined by the condition of minimum "surface energy", i.e. minimum value of the line integral $\int \alpha ds$ or $\int \alpha \sqrt{1 + y'^2} dx$ for a given "volume" of the crystal, i.e. given area $\int y dx$. Here α is a function of the direction of the tangent to the required curve, i.e. a function of y' .

Using Lagrange's method, we seek a maximum of the integral

$$\int [\alpha \sqrt{1 + y'^2} + \lambda y] dx$$

with an undetermined parameter λ . Since the integrand does not involve x explicitly, Euler's equation gives

$$\frac{\alpha}{\sqrt{1 + y'^2}} - y' \sqrt{1 + y'^2} \frac{d\alpha}{dy'} + \lambda y = \text{const.}$$

The constant may be made to vanish by a suitable choice of the origin of y . Using as a new independent variable the angle of slope φ of the tangent ($\tan \varphi = y'$), we obtain

$$y = -\frac{1}{\lambda} \left(\alpha \cos \varphi - \frac{d\alpha}{d\varphi} \sin \varphi \right). \quad (7)$$

We also have

$$dx = \cot \varphi dy = \frac{\cos \varphi}{\lambda} \left(\alpha + \frac{d^2 \alpha}{d\varphi^2} \right),$$

and hence, integrating (and putting the constant of integration equal to zero, as can always be achieved by suitable choice of the origin of x),

$$x = \frac{1}{\lambda} \left(\alpha \sin \varphi + \frac{d\alpha}{d\varphi} \cos \varphi \right). \quad (8)$$

Equations (7) and (8) give in parametric form the shape of the crystal boundary. Since $d\alpha/d\varphi$ has in general, as shown above, two different values, for each value of φ equations (7) and (8) determine two pairs of values of x and y , i.e. two different points. Each such pair of points defines the two ends of a straight segment of boundary at a given angle φ to the axis of x . The length of this segment is

$$\Delta s = (1/\lambda) \Delta \left(\frac{d\alpha}{d\varphi} \right), \quad (9)$$

and is proportional to the difference of the values of the derivative $d\alpha/d\varphi$ for a given value of φ .

Thus the equilibrium boundary of the crystal must have a very peculiar form. It consists of segments of faces in all directions, each segment having a length proportional to the corresponding value of $\Delta(d\alpha/d\varphi)$. On account of the above-mentioned property of this quantity, therefore, the higher the indices of the face the shorter is the segment corresponding to it.

All the results described can, of course, be generalised immediately to a three-dimensional crystal. Its equilibrium surface must consist of plane areas of all the faces, these areas being very small for faces with high indices. The quantity $\Delta(d\alpha/d\varphi)$ decreases very rapidly with increasing n , according to formula (6); bearing in mind also the weakness of van der Waals forces, we can suppose that faces with indices of the order of 10 and higher will not be found in practice.

The above discussion shows that the equilibrium form of crystals must in practice consist of a small number of plane areas (faces with low indices), but these do not meet at an angle; they are joined by rounded areas.† This has in fact been observed in rock salt by Lukirski², to whom the author's thanks are due for drawing attention to the problems here discussed.

I am glad to have the opportunity of contributing this paper to the symposium in honour of the seventieth birthday of Abram Fedorovich Ioffe, whose name is so well known to every Soviet physicist.

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† The previous footnote shows that we cannot exclude the possibility that some crystals do have faces meeting at an angle.

73. ON THE THEORY OF SUPERCONDUCTIVITY

The existing phenomenological theory of superconductivity is unsatisfactory since it does not allow us to determine the surface tension at the boundary between the normal and the superconducting phases and does not allow for the possibility to describe correctly the destruction of superconductivity by a magnetic field or current. In the present paper a theory is constructed which is free from these faults. We find equations for the Ψ -function of the "superconducting electrons" which we introduced and for the vector potential. We have solved these equations for the one-dimensional case (a superconducting half-space and flat plates).

The theory makes it possible to express the surface tension in terms of the critical magnetic field and the penetration depth of the magnetic field in superconductors. The penetration depth depends in a strong field on the field strength and this effect will be especially evident in the case of small size superconductors. The destruction of superconductivity in thin plates by a magnetic field is through a second-order phase transition and it only becomes a first-order transition starting with plates of a thickness more than a certain critical thickness. While the critical external magnetic field increases with decreasing thickness of the plates, the critical current for destroying the superconductivity of plates decreases with decreasing thickness.

1. INTRODUCTION

It is well known that there exists at present no properly developed microscopic theory of superconductivity. At the same time there is a fairly widespread view that the phenomenological theory of superconductivity is in a much more satisfactory state and is reliably based on the equation of F. London and H. London^{1, 2}:

$$\text{curl } \Lambda j_s = -\frac{1}{c} \mathbf{H}, \quad (1)$$

where Λ is a quantity depending only on temperature, j_s is the supercurrent density, c is the velocity of light and \mathbf{H} is the magnetic field strength, here identical with the magnetic induction. Equation (1) in combination with Maxwell's equation, $\text{curl } \mathbf{H} = 4\pi j_s/c$, and the equations $\text{div } \mathbf{H} = 0$ and $\text{div } j_s = 0$, leads under stationary conditions to the equations:

$$\nabla^2 \mathbf{H} - \mathbf{H}/\delta^2 = 0 \quad \text{and} \quad \nabla^2 j_s - j_s/\delta^2 = 0, \quad \text{where} \quad \delta^2 = \Lambda c^2/4\pi. \quad (2)$$

For a plane boundary between the superconductor and vacuum or a non-superconductor these equations have solutions:

$$\mathbf{H} = H_0 e^{-z/\delta} \quad \text{and} \quad j_s = \frac{c}{4\pi\delta} \mathbf{H}, \quad (3)$$

Л. Д. Ландау и Б. П. Гинзбург, К теории сверхпроводимости, *Журнал Экспериментальной и Теоретической Физики*, 20, 1064 (1950).

in which the external field H_0 is taken as parallel to the boundary, which is normal to the z -axis. For a film of thickness $2d$ in a parallel field we get:

$$H = H_0 \cosh(z/\delta) / \cosh(d/\delta),$$

$$j_z = -\frac{cH_0}{4\pi\delta} \sinh(z/\delta) / \cosh(d/\delta), \quad (4)$$

if $z = 0$ at the centre of the film.

For a superconductor of arbitrary shape it follows from (2) that the field penetrates only to a depth of the order of δ , which is according to experimental data about 10^{-5} cm. Qualitatively, this result is of course in agreement with the fact that a magnetic field does not penetrate into the body of a superconductor; quantitatively, however, there is no certainty that equations (1) to (4) are always correct. Moreover, this theory throws no light on the question of the surface energy at a boundary between superconducting and normal phases of the same metal, and also leads to a contradiction with experiment concerning the destruction of superconductivity of a thin film by a magnetic field.

The thermodynamic treatment of the transition of a film of thickness $2d$ from the superconducting to the normal state leads^{2,3} to the following expression for the critical field, H_c :

$$\left(\frac{H_c}{H_{cb}}\right)^2 = \left(1 - \frac{\delta}{d} \tanh \frac{d}{\delta}\right)^{-1} \quad (5)$$

in which H_{cb} is the critical field of the bulk material. This expression is not in agreement with experiment. Thus, if at a given temperature the constant δ is determined from measured values of $(H_c/H_{cb})^2$ for various values of d , according to (5), then this "constant" δ depends markedly on d ; for example, if $T = 4^\circ$ then for $d = 0.3 \times 10^{-5}$ cm, $\delta = 3.4 \times 10^{-5}$ cm, while for $d = 1.2 \times 10^{-5}$ cm, $\delta = 2 \times 10^{-5}$ cm.

It has been pointed out³ that the position may be improved by taking into account the difference of the surface energy at the boundary of the metal with a vacuum according as the metal is in the superconducting or the normal state; the difference of surface energies, $\sigma_s - \sigma_n$, introduced for this purpose must be of the order of $\delta H_{cb}^2/8\pi$. Now the surface energy is usually equal to the bulk free energy per unit volume times a length of the order of atomic dimensions. Thus here, where the difference of free energies is $H^2/8\pi$, one might expect $\sigma_s - \sigma_n$ to be of the order of 10^{-7} to 10^{-8} times $H_{cb}^2/8\pi$ and not $10^{-5} H_{cb}^2/8\pi$. An even more contradictory situation arises at the boundary separating the normal and superconducting phases of the metal; the surface energy connected with the field and supercurrent here as predicted from the solution of equation (3), is equal^{5,3} to $-\delta H_{cb}^2/8\pi$, i.e. is negative. Thus in order to obtain the observed positive surface energy σ_{ns} , it is necessary, to introduce a surface energy, σ'_{ns} , of non-magnetic origin, which is given by the equation:

$$\sigma'_{ns} = \sigma_{ns} + \frac{\delta H^2}{8\pi}$$

and which is greater than $\delta H_{cb}^2/8\pi$. There is no justification for introducing such a relatively enormous energy σ'_{ns} not connected with the field distribution. On the contrary one would expect any rational theory of superconductivity to lead automatically to an expression for σ_{ns} in terms of the ordinary parameters characterising the superconductor.

The theory based on equation (1), even with the additional surface energy, does not enable the destruction of superconductivity in thin films by a current⁶ to be considered, since this problem is not of a thermodynamic nature.

The aim of the present work is the construction of a theory free from these defects. Incidentally, as we shall see, the theory leads also to a number of new qualitative conclusions which may be checked experimentally.

2. BASIC EQUATIONS

In the absence of a magnetic field the transition into the superconducting state at the critical temperature T_c is a phase transition of the second kind. In the general theory of such transitions⁷ there always enters some parameter η which differs from zero in the ordered phase and which equals zero in the disordered phase. For example, in ferroelectrics the spontaneous polarisation plays the role of η and in ferromagnetics the spontaneous magnetisation⁸. In the phenomenon of superconductivity, in which it is the superconducting phase that is ordered, we shall use Ψ to denote this characteristic parameter. For temperatures above T_c , $\Psi = 0$ in the state of thermodynamic equilibrium, while for temperatures below T_c , $\Psi \neq 0$. We shall start from the idea that Ψ represents some "effective" wave function of the "superconducting electrons". Consequently Ψ may be precisely determined only apart from a phase constant. Thus all the observable quantities must depend on Ψ and Ψ^* in such a way that they are unchanged when Ψ is multiplied by a constant of the type $e^{i\alpha}$. We may note also that since the quantum mechanical connection between Ψ and the observable quantities has not yet been determined we may normalise Ψ in an arbitrary manner. We shall see below how we must carry out this normalisation in such a way that $|\Psi|^2$ shall equal the concentration, n_s , of "superconducting electrons" introduced in the usual way.

Consider first a uniform superconductor in the absence of a magnetic field, and suppose that Ψ is independent of position. The free energy of the superconductor is then in accordance with the general theory of second-order phase transitions, dependent only on $|\Psi|^2$ and may be expanded in series form in the neighbourhood of T_c . Thus near T_c we may write for the free energy F_{so} ,

$$F_{so} = F_{no} + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4. \quad (6)$$

In equilibrium $\partial F_{so}/\partial |\Psi|^2 = 0$, $\partial^2 F_{so}/\partial^2 |\Psi|^2 > 0$, and in addition we must have that $|\Psi|^2 = 0$ for $T \geq T_c$ and $|\Psi|^2 > 0$ for $T < T_c$. It follows therefore that $\alpha_c = 0$, $\beta_c > 0$, and for $T < T_c$, $\alpha < 0$. Thus in equilibrium,

for $T \leq T_c$,

$$|\Psi|^2 \equiv |\Psi_\infty|^2 = -\frac{\alpha}{\beta} = \frac{T_c - T}{\beta_c} \left(\frac{d\alpha}{dT} \right)_c,$$

and

$$F_{so} = F_{no} - \frac{\alpha^2}{\alpha\beta} = F_{no} - \frac{(T_c - T)^2}{2\beta_c} \left(\frac{d\alpha}{dT} \right)_c^2, \quad (7)$$

in which it is taken into account that, within the limits of validity of the expansion (6), $\alpha(T) = (d\alpha/dT)_c (T_c - T)$ and $\beta(T) = \beta_c$; the choice of the subscript ∞ for Ψ is determined by considerations of convenience which will become evident from what follows. The quantity F_{no} in (6) and (7) is evidently the free energy of the normal phase. Well-known thermodynamic considerations show (see also below) that $F_{so} - F_{no} = H_{cb}^2/8\pi$, where H_{cb} is the critical magnetic field for a bulk specimen and the free energies, as everywhere in this paper, relate to unit volume. Thus from (7),

$$H_{cb}^2 = \frac{4\pi\alpha^2}{\beta} = \frac{4\pi(T_c - T)^2}{\beta_c} \left(\frac{d\alpha}{dT} \right)_c^2. \quad (8)$$

The form of this expression is well known to be completely confirmed by experiment, which therefore provides a foundation for the assumptions made above.

Consider now a superconductor in a time-independent magnetic field. In order to obtain the density of total free energy F_{sH} , it is now necessary to add to F_{so} the field energy $H^2/8\pi$ and the energy connected with the possible appearance of a gradient of Ψ in the presence of a field. This last energy, at least for small values of $|\text{grad } \Psi|^2$, can as a result of series expansion with respect to $|\text{grad } \Psi|^2$ be expressed in the form $\text{const } |\text{grad } \Psi|^2$, i.e. it looks like the density of kinetic energy in quantum mechanics. Thus we shall write the corresponding expression in the form

$$\left(\frac{\hbar^2}{2m} \right) |\text{grad } \Psi|^2 = \frac{1}{2m} | -i\hbar \text{grad } \Psi |^2$$

in which $\hbar (= 1.05 \times 10^{-27})$ is Dirac's constant and m is a certain coefficient. We have not, however, taken into account as yet the interaction between the magnetic field and the current connected with the presence of $\text{grad } \Psi$. In view of what has been said, and the requirement that the whole scheme shall be gauge-invariant, we must allow for the influence of the field by making the usual change of $i\hbar \text{grad}$ to $[-i\hbar \text{grad} - (e/c)\mathcal{A}]$, where \mathcal{A} is the vector potential of the field and e is a charge, which there is no reason to consider as different from the electronic charge. Thus the energy density connected with the presence of $\text{grad } \Psi$ and the field H takes the form

$$\frac{H^2}{8\pi} + \frac{1}{2m} \left| -i\hbar \text{grad } \Psi - \frac{e}{c} \mathcal{A} \Psi \right|^2.$$

Consequently

$$F_{sH} = F_{sn} + \frac{H^2}{8\pi} + \frac{1}{2m} \left| -i\hbar \text{grad } \Psi - \frac{e}{c} \mathbf{A} \Psi \right|^2. \quad (9)$$

The equation for Ψ may now be found from the requirement that the total free energy of the body, $\int F_{sH} dV$, shall be as small as possible. Thus, varying with respect to Ψ^* , we find that

$$\frac{1}{2m} \left(-i\hbar \text{grad} - \frac{e}{c} \mathbf{A} \right)^2 \Psi + \frac{\partial F_{so}}{\partial \Psi^*} = 0 \quad (10)$$

and moreover, at the boundary of the superconductor, in view of the arbitrariness of the variation $\delta\Psi^*$, the following condition must hold:

$$\left(\mathbf{n} \cdot \left[-i\hbar \text{grad } \Psi - \frac{e}{c} \mathbf{A} \Psi \right] \right) = 0 \quad (11)$$

where \mathbf{n} is the unit vector normal to the boundary.

The condition (11) is obtained if no supplementary requirements are imposed on Ψ (natural boundary conditions); if however it is demanded from the start that at the boundary with a vacuum $\Psi = 0$ then (11) is not obtained. But the condition $\Psi = 0$ or const is not admissible in the present scheme, since then there would be no solution to the problem of the superconducting plate except for particular values of the thickness $2d$. We therefore impose no further conditions on Ψ at the boundary with a vacuum, and are thus led to (11). At first sight this result may appear unacceptable, since it is natural to demand that the wave function at the boundary of a metal should vanish. The essence of the matter, however, lies in the fact that the Ψ -function introduced above is in no way a true wave function of the electrons in the metal, but is a certain average quantity.

We may suppose that our function $\Psi(\mathbf{r})$ is directly connected with the density-matrix $\varrho(\mathbf{r}, \mathbf{r}') = \int \Psi^*(\mathbf{r}, \mathbf{r}'_i) \Psi(\mathbf{r}, \mathbf{r}'_i) d\mathbf{r}'_i$, where $\Psi(\mathbf{r}, \mathbf{r}'_i)$ is the true wave-function of the electrons in the metal, depending on the co-ordinates of all electrons, \mathbf{r}_i ($i = 1, 2, \dots, N$); the \mathbf{r}'_i are the co-ordinates of all the electrons except the one considered, whose co-ordinates at two points are taken as \mathbf{r} and \mathbf{r}' . It might be thought that when $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$, $\varrho = 0$ for a non-superconducting body having no long-range order, while in the superconducting state $\varrho = \varrho_0 (\neq 0)$. It is reasonable to suppose now that the density-matrix is connected with our Ψ -function by the relation $\varrho(\mathbf{r}, \mathbf{r}') = \Psi^*(\mathbf{r}) \Psi(\mathbf{r}')$.

So far as the equation for \mathbf{A} is concerned, if we assume that $\text{div } \mathbf{A} = 0$ and vary the free energy with respect to \mathbf{A} we obtain the usual expression:

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} = \frac{2\pi i e \hbar}{2m} (\Psi^* \text{grad } \Psi - \Psi \text{grad } \Psi^*) + \frac{4\pi e^2}{m c^2} |\Psi|^2 \mathbf{A}, \quad (12)$$

in which the right-hand side contains the expression for the supercurrent:

$$\mathbf{j} = -\frac{ie\hbar}{2m}(\Psi^* \text{grad } \Psi - \Psi \text{grad } \Psi^*) - \frac{e^2}{mc} \Psi^* \Psi \mathbf{A}.$$

It should be noticed that an expression analogous to (11) is obtained for the quantity in brackets, from which it is evident that at the boundary $(\mathbf{j} \cdot \mathbf{n}) = 0$, as required. The solution of the problem of the distribution of field and current in a superconductor is now reduced to an appropriate integration of equations (10) and (12).

We shall examine below only the one-dimensional problem, with the z -axis normal to the boundary separating the superconducting phase ($z > 0$) from the normal phase or vacuum; we shall take the field \mathbf{H} as directed along the y -axis and the current \mathbf{j} and vector potential \mathbf{A} along the x -axis (thus $H_y = dA_x/dz$, or simply $H = dA/dz$). In the one-dimensional solution it is natural to consider $|\Psi|^2$ as dependent only z , so that $\Psi = e^{i\tau(x,y)} \Psi(z)$. However, bearing in mind the gauge-invariance of the equations, we may by a suitable choice of \mathbf{A} arrange that $\Psi = \Psi(z)$ and hence $\mathbf{j} = -e^2/mc |\Psi|^2 \mathbf{A}$ (from the conditions that $\text{div } \mathbf{j} = dj_z/dz = 0$ and $(\mathbf{j} \cdot \mathbf{n}) = 0$ it follows that $j_z = 0$). Moreover the equations do not now contain the imaginary i (since $(\mathbf{A} \cdot \text{grad } \Psi) = (\mathbf{A} \cdot i \cdot (d\Psi/dz) \mathbf{k}) = 0$), and we may therefore consider Ψ as real. Consequently equations (10) and (12) take the form:

$$\begin{aligned} \frac{d^2 \Psi}{dz^2} + \frac{2m}{\hbar^2} |\alpha| \left(1 - \frac{e^2}{2m c^2 |\alpha|} A^2 \right) \Psi - \frac{2m}{\hbar^2} \beta \Psi^3 &= 0, \\ \frac{d^2 A}{dz^2} - \frac{4\pi e^2}{m c^2} \Psi^2 A &= 0 \end{aligned} \quad (13)$$

in which equation (6) has been used, with the additional fact that $\alpha < 0$.

Let us now determine the surface energy at a plane boundary between the normal and superconducting phases. In the normal phase, the total free energy, including field energy, is $F_{n0} + H_{cb}^2/8\pi$. In the region where $\Psi \neq 0$ and there is superconductivity the energy density is F_{sH} (equation 9), and in addition we must take account of the energy density due to the "magnetization" of a superconductor in a field parallel to the boundary with the non-superconducting phase, in the form:

$$-M H_{cb} = -\frac{H(z) - H_{cb}}{4\pi} \cdot H_{cb},$$

where M plays the role of the magnetisation. Thus the surface energy may be written:

$$\sigma_{ns} = \int \left(F_{sH}(z) - \frac{H(z) H_{cb}}{4\pi} + \frac{H_{cb}^2}{4\pi} - F_{n0} - \frac{H_{cb}^2}{8\pi} \right) dz, \quad (14)$$

in which the integration is extended over the transition layer between the phases (the z -axis is normal to this layer). It is readily verified that the inte-

grand vanishes at great distances from the transition layer, for in the superconducting phase $H = 0$ and $F_{sH} = F_{s0} = F_{n0} - \alpha^2/2\beta$ (see equation (7)), while in the normal phase $\Psi = 0$, $F_{sH} = F_{n0} + H_{cb}^2/8\pi$ and $H_0 = H_{cb}$. From equations (7-9)

$$\sigma_{ns} = \int \left\{ \alpha \Psi^2 + \frac{\beta \Psi^4}{2} + \frac{\alpha^2}{2\beta} + \frac{\hbar^2}{2m} \left(\frac{d\Psi}{dz} \right)^2 + \frac{e^2}{2m c^2} A^2 \Psi^2 + \frac{H^2}{8\pi} - \frac{H_{cb} H}{4\pi} \right\} dz. \quad (15)$$

From the minimum condition for σ_{ns} , which is the free energy per unit area, we may of course obtain both the first of equations (13), by variation of (15) with respect to Ψ , and the second of equations (13), by variation with respect to A .

At the boundary of a superconductor with a vacuum in the one-dimensional case the condition (11) assumes the form

$$\frac{d\Psi}{dz} = 0. \quad (16)$$

We shall now introduce the following parameters, H_{cb} , δ_0 and κ and in addition new variables, z' , Ψ' , A' and H' :

$$\begin{aligned} z' &= z/\delta_0, \quad \Psi'^2 = \frac{\Psi^2}{\Psi_\infty^2} = \frac{\Psi^2}{|\alpha|/\beta}, \quad A' = \sqrt{\frac{e^2}{2m c^2 |\alpha|}} \cdot A = \frac{A}{\sqrt{2} H_{cb} \delta_0}, \\ H' &= \frac{dA'}{dz'} = \frac{1}{\sqrt{2}} \cdot \frac{H}{H_{cb}}, \quad \delta_0^2 = \frac{m c^2 \beta}{4\pi e^2 |\alpha|} = \frac{m c^2}{4\pi e^2 \Psi_\infty^2}, \\ H_{cb}^2 &= \frac{4\pi \alpha^2}{\beta}, \quad \kappa^2 = \frac{1}{2\pi} \left(\frac{m c}{e \hbar} \right)^2 \beta = \frac{2e^2}{\hbar^2 c^2} H_{cb}^2 \delta_0^4. \end{aligned} \quad (17)$$

Equations (13) now take the form:

$$\begin{aligned} \frac{d^2 \Psi'}{dz'^2} &= \kappa^2 [-(1 - A'^2) \Psi' + \Psi'^3], \\ \frac{d^2 A'}{dz'^2} &= \Psi'^2 A'. \end{aligned} \quad (18)$$

The primes have been omitted from these equations since in what follows, unless we explicitly state the contrary, only the new variables will be used. With these variables, (15) must be written in the form:

$$\begin{aligned} \sigma_{ns} &= \frac{H_{cb}^2}{4\pi} \delta_0 \int \left\{ \frac{1}{2} - (1 - A'^2) \Psi'^2 + \frac{1}{2} \Psi'^4 + \frac{1}{\kappa^2} \left(\frac{d\Psi'}{dz'} \right)^2 \right. \\ &\quad \left. + \left(\frac{dA'}{dz'} \right)^2 - 2 \left(\frac{dA'}{dz'} \right) \left(\frac{dA'}{dz'} \right) \right\} dz'. \end{aligned} \quad (19)$$

If $\kappa = 0$, then from (18) and (16) $\Psi^2 = n_s = \text{constant}$, and our equations go over into equations (2) with $\delta^2 = \delta_0^2 = m c^2 / 4\pi e^2 n_s$ (compare equation (2) with the second of equations (13)). This result is true in general; if we put $\text{grad } \Psi = 0$ in equation (12) it becomes equivalent to equation (2), or, more directly, $j = -(e^2/mc)|\Psi|^2 A$, which leads to equation (1). Although for $\kappa = 0$ our scheme becomes formally identical with the usual theory it is substantially different even in this limiting case. For in equations (1) and (2) the parameter $A (= 4\pi \delta^2/c^2 = m/n_s e^2)$ is a constant, independent of field, at a given temperature, while in our theory, even for $\kappa = 0$, the value of Ψ^2 which is the same as n_s and which determines, as in equation 20, the value of δ , is such as to minimise the free energy, and this results in a variation of the penetration depth δ with H in superconductors of small dimensions.

From the limiting case, $\kappa = 0$; and from the following discussion it is clear that the experimentally determined quantity is the parameter $\delta_0^2 (= m c^2 / 4\pi e^2 \Psi_\infty^2)$, δ_0 being the penetration depth for a weak field into a bulk superconductor. It is just this quantity which enters also into the expression for the dielectric constant $\epsilon (= \epsilon_0 - 4\pi e^2 \Psi_\infty^2 / m \omega^2)$ of a superconductor in an alternating field of not too high a frequency ω (ϵ_0 is a certain constant contribution to ϵ from all particles other than "superconducting electrons"). The parameter $\Psi_\infty^2 (= n_s)$, which evidently corresponds to the concentration of "superconductive electrons", does not appear as a measurable quantity, resembling in this way the number of free electrons in the ordinary quantum theory of metals. Thus in both expressions we may talk only of the effective number of electrons, which may be determined from the values of ϵ or δ_0^2 by attributing to m the value appropriate to a free electron. Proceeding in this way we relate the concentration of "superconducting electrons" $n_s (= \Psi_\infty^2)$ with the observable quantity δ_0 (putting $e = 4.8 \times 10^{-10}$ e.s.u., $m = 9.1 \times 10^{-28}$ g) by the equation:

$$\delta_0^2 = \frac{m c^2 \beta}{4\pi e^2 |\alpha|} = 2.84 \times 10^{11} \frac{\beta}{|\alpha|} = \frac{2.84 \times 10^{11}}{\Psi_\infty^2} \text{ cm}^2. \quad (20)$$

From (20) and from measurements of the critical field $H_{cb} = (\sqrt{4\pi \alpha^2/\beta})$ we may determine α and β . Besides H_{cb} and δ_0 (or α and β) there enters also into the theory the dimensionless parameter κ :

$$\kappa^2 = \left(\frac{2e^2}{\hbar^2 c^2} \right) H_{cb}^2 \delta_0^4, \quad (21)$$

which, with $e = 4.8 \times 10^{-10}$ e.s.u., becomes:

$$\kappa^2 = 4.64 \times 10^{14} H_{cb}^2 \delta_0^4, \quad (22)$$

where δ_0 is measured in cm and H_{cb} in gauss. From the experimental data discussed in section 4 it follows that for mercury

$$\kappa^2 \doteq 0.027; \quad \kappa \doteq 0.165; \quad \sqrt{\kappa} \doteq 0.406. \quad (23)$$

3. THE SUPERCONDUCTING HALF-SPACE

We shall consider first the case of a superconducting half-space bounded by a vacuum (superconducting for $z > 0$, boundary at $z = 0$). The solution will of course refer also to a sufficiently thick plate whose half thickness $d \gg 1$ (or in the usual units $d \gg \delta_0$). For $z = 0$, $H = H_0$, and for $z = \infty$, $H = A = 0$ (the present choice of $A(\infty) = 0$ is perfectly natural and moreover possible). Further, for $z = \infty$ we are dealing with a superconductor in the absence of a field and far from any boundaries, and consequently solution (7) must apply, i.e. in the new variables $\Psi_\infty^2 = 1$, $d\Psi/dz = 0$. Thus for $z = \infty$,

$$\Psi_\infty^2 = 1, \quad \frac{d\Psi}{dz} = H = A = 0. \quad (24)$$

This solution naturally satisfies the equations (18). As regards the boundary with the vacuum at $z = 0$, condition (16) must be satisfied there; substituting (18) into (24) we see that in the absence of a magnetic field the presence of the boundary has no influence on the function Ψ which therefore has the same value everywhere:

$$\Psi^2 = \Psi_\infty^2 = 1 \quad \text{if} \quad H \equiv A \equiv 0. \quad (25)$$

In the presence of a magnetic field solution (25) of course does not apply and we must integrate (18) with the boundary conditions (24) for $z = \infty$ and the conditions

$$H = \frac{dA}{dz} = H_0, \quad \frac{d\Psi}{dz} = 0 \quad \text{for} \quad z = 0. \quad (26)$$

The values of A_0 and Ψ_0 are not known beforehand.

The equations (18) unfortunately cannot be integrated exactly and we can indicate only one of their integrals:

$$(1 - A^2) \Psi^2 - \frac{1}{2} \Psi^4 + \left(\frac{dA}{dz} \right)^2 + \frac{1}{\kappa^2} \left(\frac{d\Psi}{dz} \right)^2 = \text{const.} \quad (27)$$

For the case which interests us the constant $= \frac{1}{2}$ because of (24), and thus

$$H^2 = \left(\frac{dA}{dz} \right)^2 = \frac{1}{2} - \frac{1}{\kappa^2} \left(\frac{d\Psi}{dz} \right)^2 - (1 - A^2) \Psi^2 + \frac{1}{2} \Psi^4. \quad (28)$$

Turning instead to the approximate solution of equations (18), we now give the solution valid for small values of κ (more precisely the solution will be valid for small values of the product κH_0^2). In order to find this solution we substitute

$$\Psi = \Psi_\infty + \varphi = 1 + \varphi \quad \text{for} \quad |\varphi| \ll 1. \quad (29)$$

Then in the first approximation up to terms of order φA and φ^2 , the system (18) assumes the form

$$\frac{d^2 \varphi}{dz^2} = \kappa^2 (2\varphi + A^2), \quad \frac{d^2 A}{dz^2} = A. \quad (30)$$

This system may be integrated at once and its solution may be used for finding the next approximation and so on. The corresponding solution, with the conditions (24) and (26), up to and including terms in H_0^3 has the form

$$\begin{aligned}\Psi &= 1 + \frac{\kappa H_0^2}{\sqrt{2}(2 - \kappa^2)} \left(\frac{\kappa}{\sqrt{2}} e^{-2z} - e^{-\sqrt{2}\kappa z} \right), \\ A &= -H_0 e^{-z} - \frac{\kappa H_0^3}{\sqrt{2}(2 - \kappa^2)} \left\{ \frac{\kappa}{4\sqrt{2}} e^{-3z} - \frac{e^{-(\sqrt{2}\kappa+1)z}}{\kappa(\kappa + \sqrt{2})} \right. \\ &\quad \left. - \frac{3\kappa^3 + 3\sqrt{2}\kappa^2 - 8\kappa - 4\sqrt{2}}{4\sqrt{2}\kappa(\kappa + \sqrt{2})} e^{-z} \right\}.\end{aligned}\quad (31)$$

For $z = 0$, naturally, $d\Psi/dz = 0$, $H = H_0$ and

$$\begin{aligned}\Psi_0 &= 1 - \frac{\kappa H_0^2}{2(\kappa + \sqrt{2})}, \\ A_0 &= -H_0 - \frac{\kappa(\kappa + \sqrt{2})}{4(\kappa + \sqrt{2})^2} H_0^3.\end{aligned}\quad (32)$$

The biggest of the terms in Ψ neglected in (32) are of the order $\kappa^2 H_0^4$ and in A of the order $\kappa^2 H_0^5$. The field H_0 in the equilibrium state is less than or equal to the critical field H_{cb} for the superconductor, which in the new variables is $1/\sqrt{2}$ (see equation (17)). According to (32), for $\kappa = 0.165$ (see equation (23)) $\Psi_0 \geq 0.974$ (the equality applies when $H_0 = 1/\sqrt{2}$), and thus the application of equation (31) here is completely justified if it is sufficient to determine $(\Psi - 1)$ to a few per cent. At the present time such an accuracy in the measurement of δ_0 is far from having been reached.

Since from the experimental data it follows that $\kappa \ll 1$, and also for a reason indicated below the solution of equations (18) possible for another limiting case when $\kappa \rightarrow \infty$ does not offer any intrinsic interest, we shall not discuss it.

If $\kappa = 0$, then in the problem under discussion $\Psi \equiv 1$ for any H —this corresponds to the usual theory based on equations (1) with $A = \text{const}$. If $\kappa > 0$, the solution exists only up to a certain "second critical field" H_{c2} . The range of fields $H_{cb} (= 1/\sqrt{2}) < H < H_{c2}$ represents a metastable (superheated) state in which the superconducting phase can exist since it represents a relative minimum of the free energy but the absolute minimum of free energy is already that corresponding to the normal phase. The more detailed investigation of this question and a calculation of the dependence of the field H_{c2} on κ has not yet been carried through.

Let us now note that for $\kappa \geq 1/\sqrt{2}$ a peculiar instability of the normal phase of the metal occurs. Indeed, suppose the whole metal is in equilibrium, and in the normal state, i.e. $H_0 = 1/\sqrt{2}$. Then it can be shown that for $\kappa \geq 1/\sqrt{2}$ an instability appears with respect to the formation of thin layers of supercon-

ducting phase in the sense that solution of (18) appear with $\Psi \neq 0$. In fact, assuming that $\Psi \ll 1$, we can take $H = H_0 = \text{const}$ and the first equation (18) assumes the form

$$\frac{d^2 \Psi}{dz^2} = -\kappa^2(1 - H_0^2 Z^2) \Psi. \quad (33)$$

This equation in its form coincides with Schrödinger's equation for the harmonic oscillator and is well known to have solutions for Ψ which vanish for $z = \pm \infty$ if $\kappa = 2H_0(n + \frac{1}{2})$, where $n = 0, 1, 2, \dots$

Since for the normal phase $H_0 \geq 1/\sqrt{2}$ the minimum value of κ for which solutions can appear is $1/\sqrt{2}$. The point $z = 0$ chosen in (33) is quite arbitrary, i.e. a "parasitic" solution can appear anywhere, and indeed there occurs a certain instability of the normal phase connected with the fact that when $\kappa > 1/\sqrt{2}$ the surface energy $\sigma_{ns} < 0$ (see end of section 4).

It has not been necessary to investigate the nature of the state which occurs when $\kappa > \kappa_0$ since from the experimental data, it is true somewhat preliminary and worked out on the basis of equation (22), it follows that $\kappa \ll 1$. Leaving on one side the question of the true value of κ , we must in any case, because of the indicated instability of the solution, note that all results obtained by us are valid only for the case

$$H < H_0 \left(= \frac{1}{\sqrt{2}} \right)$$

We may use the solution (31) to investigate the dependence on field strength of the penetration depth of a magnetic field in a bulk superconductor^{9,10}. In agreement with the experimental method of measurement^{11,12} we define the penetration depth of a magnetic field in a bulk superconductor in the following way:

$$\delta = \frac{1}{H_0} \int_0^\infty H dz = \frac{\delta_0}{H_0} \int_0^\infty H dz = \delta_0 \frac{|A_0|}{H_0}, \quad (35)$$

where H_0 is the external field (field at $z = 0$) and in the first expression we use the usual and in the second and third the reduced units for H , H_0 , A_0 and z . Substituting the field (31) into (35) we have (in the usual units)

$$\begin{aligned} \delta &= \delta_0 \left\{ 1 + \frac{\kappa(\kappa + 2\sqrt{2})}{8(\kappa + \sqrt{2})^2} \left(\frac{H_0}{H_{cb}} \right)^2 \right\} = \delta_0 \left\{ 1 + f(\kappa) \left(\frac{H_0}{H_{cb}} \right)^2 \right\}, \\ \frac{d\delta}{dT} &= \frac{d\delta_0}{dT} + f(\kappa) \left(\frac{H_0}{H_{cb}} \right)^2 \left\{ \frac{d\delta_0}{dT} - \frac{2(dH_{cb}/dT)}{H_{cb}} \delta_0 \right\}. \end{aligned} \quad (36)$$

From this it is clear that the quantity δ_0 , as already mentioned, represents the penetration depth in a weak field. The function $f(\kappa)$ grows monotonically with κ in such a way that $f(0) = 0$, $f(\infty) = 1/8$, and for $\kappa \ll 1$ $f(\kappa) \sim \kappa/4\sqrt{2}$. Thus for $H_0 = H_{cb}$, even for $\kappa = 1/\sqrt{2}$, $\delta = 1.07 \delta_0$, and for $\kappa = 0.165$, $\delta = 1.028 \delta_0$. If, as was the case in (12), measurements of δ are carried out using a weak

and slowly varying field H_1 in the presence of a strong field H_0 , then $(\delta - \delta_0)/\delta_0 = 3f(\kappa)(H_0/H_{cb})^2$, i.e. the effect is tripled. We see that the expected change of δ with H for mercury, for which according to our estimate $\kappa = 0.165$, is very small and lies outside the limits of accuracy of measurements achieved in (12) (the data of ref. 11 as regards the dependence of δ on H are probably for reasons indicated in ref. 12 not true; this is also evident from the fact that in ref. 11 for a number of cases δ varies as H_0 rather than as H_0^2 † since it is an even function of H_0). As we shall see in section 4 for thin superconductors the dependence of δ on H_0 is much bigger than for bulk ones and may be observed in experiments of the type described in ref. 10 (thus it is possible that the dependence of δ on H in ref. 10 is real, which does not contradict the absence of a noticeable effect in ref. 12).

4. THE SURFACE ENERGY AT THE BOUNDARY OF THE SUPERCONDUCTING AND NORMAL PHASES

For the calculation of σ_{ns} we must find the solution of the equations (18) for a superconducting half-space limited by a half-space consisting of the normal phase of the same metal. Since the only difference between two phases is that in the one $\Psi \neq 0$ and in the other $\Psi = 0$, it is reasonable to suppose that the transition between the two phases takes place continuously in some transition layer. It can be shown that our equations have just such a continuous smooth solution, and do not for instance lead to a solution satisfying the conditions of the problem in which the function Ψ vanishes suddenly at some point. Thus the transition from the superconducting phase to the normal takes place in a transition layer in which for $z = \infty$ we have the superconducting phase and for $z = -\infty$ the normal phase. This means that we must seek a solution of the equations (18) with the boundary conditions

$$\begin{aligned} \Psi = \Psi_\infty = 1, \quad H = A = \frac{d\Psi}{dz} = 0 \quad \text{when } z = \infty; \\ \Psi = \frac{d\Psi}{dz} = 0, \quad H = H_0 = \frac{1}{\sqrt{2}}, \quad A = H_0 z + \text{const} \quad \text{when } z = -\infty. \end{aligned} \quad (37)$$

In fact, of course, the transition layer has a breadth of the order of δ_0 (more precisely, as we shall see below, of the order of δ_0/κ), just as the magnetic field in a superconductor falls to zero in a distance of the order of δ_0 although strictly speaking it vanishes only at $z = \infty$.

Substituting (28) into (19) we obtain an expression for the surface energy σ_{ns} :

$$\begin{aligned} \sigma_{ns} &= \frac{H_{cb}^2}{2\pi} \delta_0 \int_{-\infty}^{\infty} \left\{ \frac{1}{2} - (1 - A^2) \Psi^2 + \frac{1}{2} \Psi^2 - H_0 H \right\} dz \\ &= \frac{H_{cb}^2}{2\pi} \delta_0 \int_{-\infty}^{\infty} \left\{ \frac{1}{\kappa^2} \left(\frac{d\Psi}{dz} \right)^2 + H^2 - H_0 H \right\} dz, \end{aligned} \quad (38)$$

† Even in weak fields, where it must necessarily vary as H_0^2 .

where the relation (28) has been used, H_0 has been put equal to $1/\sqrt{2}$ and all quantities under the integral sign are expressed in reduced units.

In view of the fact that the general case involves the solution of equations (18) we can give an analytical expression for σ_n , only for sufficiently small κ . In this case in the superconducting phase for large z (far from the transition region) $\Psi = 1 - \text{const } e^{-\sqrt{2}\kappa z}$ (see equations (30) and (31)), i.e. changes only slowly with z . Consequently we shall seek a solution of the second equation (18) in the form

$$A = \exp \left\{ - \int \Psi dz \right\}. \quad (39)$$

It is easy to see that this solution is valid for

$$\left| \frac{d}{dz} \left(\frac{1}{\Psi} \right) \right| \ll 1. \quad (40)$$

Substituting (39) in (28) we find that $d\Psi/dz = \kappa(1 - \Psi^2)/\sqrt{2}$; i.e.

$$\Psi = \tanh \left(\frac{\kappa z}{\sqrt{2}} \right), \quad (41)$$

provided that the origin of z is suitably chosen. It should be noted that the solution (41) is at the same time the strict solution of equation (18) for $\Psi(z)$ in the absence of an external field and subject to the condition that $\Psi(\infty) = 1$ and $\Psi(0) = 0$. From (40) it is evident that the solution (41) in the presence of a field applies as long as the inequality

$$\kappa \ll \sqrt{2} \sinh^2 \left(\frac{\kappa z}{\sqrt{2}} \right) \quad (42)$$

is satisfied. With this condition, and taking into account (39) and (41), we find that

$$A = \exp \left\{ - \int \Psi dz \right\} = C \exp \left[- \frac{\sqrt{2}}{\kappa} \ln \cosh \left(\frac{\kappa z}{\sqrt{2}} \right) \right], \quad (43)$$

$$\text{and } H = \frac{dA}{dz} = -\Psi A = -A \tanh \left(\frac{\kappa z}{\sqrt{2}} \right).$$

For $\kappa z \ll 1$,

$$\Psi = \frac{\kappa z}{\sqrt{2}}, \quad A = C \exp \left[- \frac{\kappa z^2}{2\sqrt{2}} \right], \quad \kappa z^2 \gg 1, \quad (44)$$

where the inequality $\kappa z^2 \gg 1$ is obtained from (42). It is evident that the approximation (44) is valid if $1/\kappa \gg z \gg 1/\sqrt{\kappa}$; these inequalities may be satisfied if κ is sufficiently small. For an estimate of the constant C in (43) and (44) we take account of the fact that $H \leq H_0 (= 1/\sqrt{2})$, and consequently $|A| \leq 1/\sqrt{2} \tanh(\kappa z/\sqrt{2})$, or, if $\kappa z \ll 1$, $|A| \sim 1/\kappa z$ in the region where

$H \sim 1/\sqrt{2}$. From this, taking into account that equation (44) still applies as regards order of magnitude for $z \sim \kappa^{-1/2}$, we find that $C \sim \kappa^{-1/2}$. Thus at the boundary of the region of validity of the solutions (41) to (43) $A \gg 1$ (since $\kappa \ll 1$). But if $A \gg 1$ the equations (18) simplify and assume the form

$$\frac{d^2 \Psi}{dz^2} = \kappa^2 A^2 \Psi; \quad \frac{d^2 A}{dz^2} = \Psi^2 A. \quad (45)$$

Introducing the variables $\zeta = z \sqrt{\kappa}$, $\varphi = \Psi/\sqrt{\kappa}$, and $B = A \sqrt{\kappa}$, we obtain from (45) universal equations

$$\frac{d^2 \varphi}{d\zeta^2} = \varphi B^2; \quad \frac{d^2 B}{d\zeta^2} = \varphi^2 B, \quad (46)$$

which likewise cannot be integrated analytically but must be solved numerically once and for all. However, there is no need even to do this since it is easy to see that the contribution to σ_{ns} from the region where equations (45) and (46) are valid, i.e. the region $-\infty < z \lesssim \kappa^{-1/2}$ is of the order $\kappa^{-1/2}$. Similarly the contribution from the region $\kappa^{-1/2} < z < \infty$ is of the order of κ^{-1} . Indeed, substituting into (38) the solution (41) and (43) it is easy to see that the contribution to σ_{ns} from the terms $H^2 - H_0 H$ under the integral sign is a quantity

of order $\kappa^{-1/2}$ while $\int_{1/\sqrt{\kappa}}^{\infty} \frac{1}{\kappa^2} (d\Psi/dz)^2 dz = \sqrt{2}/3\kappa + \text{terms of the order } \kappa^{-1/2}$, connected with the lower limit of the integral. In this way, apart from terms of the order of $\kappa^{-1/2}$, we have

$$\sigma_{ns} = \frac{\delta_0 H_{cb}^2}{3\sqrt{2}\pi\kappa}, \quad \Delta = \frac{\sigma_{ns}}{H_{cb}^2/8\pi} = \frac{1.89\delta_0}{\kappa}, \quad \text{if } \sqrt{\kappa} \ll 1 \quad (47)$$

It is especially important to emphasise that, for small values of κ , $\sigma_{ns} > 0$ which is absolutely necessary, and the attainment of which was our main aim. For sufficiently large κ , on the other hand, $\sigma_{ns} < 0$ (this is apparent immediately from (38) since $H^2 < H_0 H$), which indicates that such large values of κ do not correspond to the usually observed state of affairs (as a result of a numerical integration it turns out that $\sigma_{ns} = 0$ when $\kappa = 1/\sqrt{2}$). The value (23) assumed by us for mercury is very small from all other points of view but insufficiently small for the applicability of (47), since in this case $\sqrt{\kappa} = 0.407$. The numerical integration for $\kappa = 0.165$ leads to a value of about $6\delta_0$ for Δ , while according to (47) $\Delta = 11.4\delta_0$.

The thickness of the transition layer is evidently of the order of δ_0/κ , i.e. about $10\delta_0$.

5. SUPERCONDUCTING PLATES (FILMS)

The solution is one dimensional for plane plates and films as well as for a half space. Here it is of interest to calculate the critical magnetic field, H_c , for destruction of superconductivity in films and the magnetic moment of the film in an arbitrary field H_0 ; moreover when there is a total current J flowing

through the film we have to find the critical value of the current J_c to destroy superconductivity, and also the dependence of J_c on a superimposed field H_0 .

The critical field H_c , as is shown by thermodynamic considerations^{2,3} is determined by the relations

$$\frac{H_c^2}{8\pi} = \frac{H_{cb}^2}{8\pi} - \frac{\sigma}{d},$$

$$\sigma = \int_0^d \left(\frac{H^2(z)}{8\pi} - \frac{H_c H(z)}{4\pi} + \Delta F \right) dz, \quad (48)$$

in which the thickness of the plate is $2d$, the z -axis is perpendicular to the plate with $z = 0$ at its centre, H_{cb} is the critical field for a bulk superconductor and ΔF is the contribution to the free energy density of the superconductor resulting from penetration of the magnetic field. In the theory based on equation (1), $\Delta F = A j_s^2/2$, and substitution of equation (4) into (48) leads to equation (5), if surface energy is ignored. In our case we obtain from (9) and (7)

$$\Delta F = F_{n0} + \alpha \Psi^2 + \beta \Psi^4/2 + \frac{\hbar^2}{2m} \left(\frac{d\Psi}{dz} \right)^2 + \frac{e^2}{2mc^2} A^2 \Psi^2 - \left(F_{n0} - \frac{\alpha^2}{2\beta} \right),$$

and thus

$$\sigma = \frac{H_{cb}^2 \delta_0}{4\pi} \int_0^d \left\{ \frac{1}{2} - (1 - A^2) \Psi^2 + \frac{1}{2} \Psi^4 + \frac{1}{\kappa^2} \left(\frac{d\Psi}{dz} \right)^2 + H^2 - 2H_c H \right\} dz, \quad (48a)$$

in which the new units are used in the integrand. The quantity (48a) is denoted by σ since it is clear from (15) that it is equivalent to the surface energy integrated with the proper limits.

The magnetic moment of the film per unit area in an external field H_0 , parallel to the film, is given by

$$\mu = \int_{-d}^d \frac{H(z) - H_0}{4\pi} dz = \frac{1}{2\pi} (A(d) - H_0(d)) \quad (49)$$

where in the transition to the second expression it has been taken into account that for a film without a total current in an external field $H(z) = H(-z)$.

$A(d)$ has been substituted for $\int_0^d H(z) dz$, since the potential A will be chosen below in such a way that $A(0) = 0$. Equation (49) can be obtained either from the fact that the work of magnetisation of the film is given by

$$-\mu H_0 = \frac{H_0}{4\pi} \int (H_0 - H(z)) dz$$

or directly, from the fact that the field $H(z)$ plays the role of magnetic induction $B(z)$, and thus the expression $(H(z) - H_0)/4\pi$ is equivalent to the magnetisation $M = (B - H)/4\pi$.

For the determination of H_c , μ and J_c we must find the solution of equations (18) with the boundary conditions

$$\frac{d\Psi}{dz} = 0, \quad H = H_0 \pm H_J, \quad H_J = \frac{2\pi}{c} J, \quad \text{when } z = \pm d. \quad (50)$$

Here H_0 is the external magnetic field directed along the y -axis, J is the total current $\left(J = \int_0^d j \, dz, \text{ where } j \text{ is the current density}\right)$ flowing along the film in the direction of the negative x -axis, and $2H_J (= 4\pi J/c)$ is the difference between the values of the total field on both sides of the film due to the current J . If the current J and the field H_0 are not mutually perpendicular then there are two non-vanishing components of the potential A (in fact A_x and A_y) instead of the single component A_x in the case considered above. We should then have instead of (18) a system of initial equations of the form

$$\begin{aligned} \frac{d^3\Psi}{dz^2} &= \kappa^2 \{-(1 - A_x^2 - A_y^2) \Psi + \Psi^3\}, \\ \frac{d^2 A_x}{dz^2} &= \Psi^2 A_x, \quad \frac{d^2 A_y}{dz^2} = \Psi^2 A_y. \end{aligned} \quad (51)$$

These equations have to be solved for the conditions

$$\begin{aligned} H_x &= H_{x0}, \quad H_y = H_{y0} \pm H_J, \\ H_J &= \frac{2\pi J}{c}, \quad \frac{d\Psi}{dz} = 0, \quad \text{when } z = \pm d. \end{aligned} \quad (52)$$

The axes have now been chosen in such a way that the total current has a component only along the x -axis and consequently the field H_J is directed along the y -axis; H_{x0} and H_{y0} are the components of the external field along the x and y -axes.

For sufficiently thick plates, i.e. when $d \gg \delta_0$, the value H_c may be immediately obtained from the results of section 3 by allowing d to tend to infinity in (48). Thus substituting the solution (31) into (48) we have, for $d \gg \delta_0$,

$$\frac{H_c}{H_{cb}} = 1 + \frac{\delta_0}{2d} \left(1 + \frac{1}{2} f(\kappa)\right). \quad (53)$$

Here $f(\kappa) = \kappa(\kappa + 2\sqrt{2})/8(\kappa + \sqrt{2})^2$, the same function as in (36); equation (53) is valid up to terms of the order $(\delta_0/d)^2$. To the same approximation in the usual theory^{1,2} we should obtain the expression (53) with $\kappa = 0$ (see

equation (5)). Taking (36) into account, equation (53) may be written in the form

$$\frac{H_c}{H_{cb}} = 1 + \frac{\delta_0}{2d} + \frac{\Delta\delta}{4d}, \quad (54)$$

where $\Delta\delta = \delta(H_{cb}) - \delta_0$.

For films of arbitrary thickness the solution of (18) must be carried out again. The solution of (31) suggests that for thin plates as well as for thick ones the function Ψ changes only slowly with z , if κ is small. Starting from this supposition, which is subsequently justified, we suppose that

$$\Psi = \Psi_0 + \varphi, \quad |\varphi| \ll \Psi_0, \quad \text{and} \quad \varphi = 0 \quad \text{when} \quad z = 0. \quad (55)$$

Then equations (18) in the first approximation assume the form

$$\left. \begin{aligned} \frac{d^2\varphi}{dz^2} &= \kappa^2 \{ \Psi_0^2 - \Psi_0 + (3\Psi_0^2 - 1)\varphi + A^2\Psi_0 \}, \\ \frac{d^2A}{dz^2} &= \Psi_0^2 A. \end{aligned} \right\} \quad (56)$$

From the second of the equations (56), taking account of the boundary conditions (50), we find the values of A and H to be

$$\left. \begin{aligned} A &= \frac{H_0 \sinh \Psi_0 z}{\Psi_0 \cosh \Psi_0 d} + \frac{H_J \cosh \Psi_0 z}{\Psi_0 \sinh \Psi_0 d}, \\ H &= \frac{dA}{dz} = \frac{H_0 \cosh \Psi_0 z}{\cosh \Psi_0 d} + \frac{H_J \sinh \Psi_0 z}{\sinh \Psi_0 d}. \end{aligned} \right\} \quad (57)$$

Substituting (57) in the first of the equations (56) we find φ , and from the requirement that for $z = \pm d$, $d\varphi/dz = 0$, we obtain a transcendental equation determining Ψ_0 .

As we shall see in practice we may with sufficient accuracy put $\kappa = 0$. We shall therefore give the expression for φ and the equations for Ψ_0 for the case $\kappa \neq 0$ only when $H_J = 0$, i.e. for a film in an external field. In this special case

$$\begin{aligned} \varphi &= -\frac{\Psi_0(\Psi_0^2 - 1)}{3\Psi_0^2 - 1} \left\{ 1 - \cosh \kappa z \sqrt{3\Psi_0^2 - 1} \right\} \\ &+ \frac{\kappa H_0^2}{2\Psi_0^2 \sqrt{3\Psi_0^2 - 1} \cosh^2 \Psi_0 d} \left\{ \frac{1 - \cosh \kappa z \sqrt{3\Psi_0^2 - 1}}{\kappa \sqrt{3\Psi_0^2 - 1}} \right. \\ &\quad \left. - \frac{\kappa \sqrt{3\Psi_0^2 - 1} (\cosh \kappa \sqrt{3\Psi_0^2 - 1} z - \cosh 2\Psi_0 z)}{4\Psi_0^2 - \kappa^2 (3\Psi_0^2 - 1)} \right\} \end{aligned} \quad (58)$$

$$\Psi_0^2 - 1 = \frac{2H_0^2 \left\{ 1 - \frac{\sinh 2\Psi_0 d}{2\Psi_0 d} - \frac{\kappa d \sqrt{3\Psi_0^2 - 1}}{\sinh \kappa d \sqrt{3\Psi_0^2 - 1}} \right\}}{\cosh^2 \Psi_0 d \{ 4\Psi_0^2 - \kappa^2 (3\Psi_0^2 - 1) \}}. \quad (59)$$

In the limiting case $\kappa = 0$, for arbitrary H_0 and H_J , naturally $\varphi = 0$ and

$$\Psi_0^2(\Psi_0^2 - 1) = \frac{H_0^2 \left(1 - \frac{\sinh 2\Psi_0 d}{2\Psi_0 d}\right)}{2 \cosh^2 \Psi_0 d} - \frac{H_J^2 \left(1 + \frac{\sinh 2\Psi_0 d}{2\Psi_0 d}\right)}{2 \sinh^2 \Psi_0 d}. \quad (60)$$

Let us note that for $\kappa = 0$ the equation for $\Psi (= \Psi_0 = \text{const})$ may be immediately obtained from the condition of minimum free energy, i.e. from the condition $d\sigma/d\Psi = 0$. It is clear from (48), that this condition gives

$$\Psi_0^2 - 1 = -\frac{1}{d} \int_0^d A^2 dz,$$

which leads to (60).

Let us now discuss in somewhat more detail the destruction of superconductivity in a film by an external field in the absence of a total current. If $\kappa = 0$, then $\Psi = \Psi_0 = \text{const}$ and the solution (57) applies with $H_J = 0$. Substituting this solution in (48) we easily find (in the usual units)

$$\left(\frac{H_c}{H_{cb}}\right)^2 = \frac{\Psi_0^2(2 - \Psi_0^2)}{1 - \frac{1}{\eta} \tanh \eta}, \quad \text{where} \quad \eta = \frac{\Psi_0 d}{\delta_0}. \quad (61)$$

In this case ($\kappa = 0$, $H_J = 0$) equation (60) becomes, when $H_0 = H_c$, or in the usual units when $H_0 = H_c/\sqrt{2} H_{cb}$,

$$\left(\frac{H_c}{H_{cb}}\right)^2 = \frac{4\Psi_0^2(\Psi_0^2 - 1) \cosh^2 \eta}{1 - (\sinh 2\eta)/2\eta}, \quad \text{where} \quad \eta = \frac{\Psi_0 d}{\delta_0}. \quad (62)$$

From (61) and (62), from the measured values of H_c/H_{cb} and from d , we can determine Ψ_0 and δ_0 . It is easy to see that for small values of η and for $H = H_c$, $\Psi_0 = 0$ and

$$\frac{H_c}{H_{cb}} = \sqrt{6} \frac{\delta_0}{d}. \quad (63)$$

Thus in this case we have a phase change of the second kind; with growth of the field Ψ_0 decreases and at the transition point $\Psi_0 = 0$. As is evident from (60), for $H_J = 0$, up to terms of order d^2 (taking into account that $H_0^2 d^2$ may be of the order unity) we have

$$\Psi_0^2 = \frac{1 - (H_0/H_{cb})^2 (d^2/6\delta_0^2)}{1 - \frac{2}{15} (H_0/H_{cb})^2 (d^4/\delta_0^4)}.$$

The transition to the normal state is one of the second kind for $d \leq d_c$, where it is easily shown from (61) and (62) that

$$d_c = \sqrt{5} \delta_0/2. \quad (64)$$

The point $d = d_c$ is a kind of critical Curie point⁷, and for $d > d_c$ we have a transition of the first kind; i.e. for $H_0 = H_c$, $\Psi_0 > 0$ and there is a latent heat of transition (for $d < d_c$ and for $H_0 = H_c$ we have a jump in the specific heat; the specific heat of thin plates evidently depends on H_0).

The penetration depth of the field is clearly from (57) the quantity

$$\delta = \frac{\delta_0}{\Psi_0}, \quad (65)$$

and we see that for sufficiently thin specimens the penetration depth may be appreciably larger than for the bulk metal when $H_0 \sim H_c$. Here (see equations (49) and (57) with $H_J = 0$)

$$\mu = -\frac{H_0 d}{2\pi} \left(1 - \frac{\delta}{d} \tanh \frac{d}{\delta} \right) = -\left(\frac{H_0 d}{2\pi} \right) \left\{ \frac{1}{3} \left(\frac{d}{\delta} \right)^2 - \frac{2}{15} \left(\frac{d}{\delta} \right)^4 + \dots \right\} \quad (66)$$

From measurements of μ we may find the penetration depth δ which according to (65) and (60) depends on H_0 .

For $\kappa \neq 0$ all the expressions become exceedingly complicated in the general case. However, for small values of κ , which are the only ones which interest us, and for not too large values of d , we may expand all the expressions as series in κd . The result is, that in the range of thicknesses for which the transition is of the second kind, equation (63) must be replaced by the expression:

$$\left(\frac{H_c}{H_{cb}} \right)^2 = 6 \left(\frac{\delta_0}{d} \right)^2 - \frac{7}{10} \kappa^2 + \frac{11}{1400} \kappa^4 \left(\frac{d}{\delta_0} \right)^2 \dots \quad (67)$$

The value of d_c is then given by

$$d_c^2 = \frac{5}{4} \left(1 - \frac{7}{24} \kappa^2 + \dots \right) \delta_0^2. \quad (68)$$

If we take for κ^2 the value (23) then in practice it is hardly necessary to take into account the term in κ^2 in (67), (68) and in the analogous expressions.

The only experimental data on destruction of superconductivity in films by an external field suitable for a quantitative discussion are those given in ref. 4 and refer to mercury. The scatter of points, however, even in these measurements was rather large, and moreover in the absence of tables the values of H_c/H_{cb} had to be taken from graphs; nevertheless the chief source of error is due to the fact the thickness of the films indicated in ref. 4 is some sort of average value and may, especially for the thin films, differ considerably from the thickness d entering in our formulae in which it is assumed of course that the film is ideally uniform.

In Table 1 we reproduce the values of δ_0 obtained with the aid of (63) on the basis of the data for H_c/H_{cb} as a function of d given in ref. 4; the values shown in brackets are those for which the calculation from equation (63) is already invalid since $d > d_c$. Underneath these values in brackets are put the values of δ_0 obtained directly from equations (61) and (62).

TABLE I. Values of δ_0 for mercury (δ_0 and d in units of 10^{-5} cm)

$\frac{2d}{T^\circ\text{K}}$	0.596	0.840	1.178	1.423	1.690	2.400	4.390	10.880	$2d_c = \sqrt{5}\delta_{0\min}$
4.13	5.13	4.61	4.07	4.17	3.80	3.37	3.08	(3.72) 3.56	6.9
4.12	4.12	4.06	3.47	3.36	3.27	3.11	2.72	(3.52) 3.30	6.1
4.10	3.47	3.38	2.87	3.02	2.79	2.53	2.23	(3.21) 2.50	5.1
4.05	2.66	2.62	2.32	2.27	2.08	1.86	(1.80) 1.80	(2.86) 1.95	4.0
4.00	2.28	2.31	1.92	1.82	1.76	1.56	(1.57) 1.57	(2.72) 1.70	3.5
3.80	1.69	1.62	1.40	1.28	1.24	1.10	(1.31) 1.15	(2.63) 1.39	2.5
3.60	1.27	1.24	1.08	0.99	0.98	(0.87) 0.87	(1.23) 0.99	(2.50) 1.16	1.95
3.00	1.10	1.10	0.92	0.84	(0.83) 0.83	(0.77) 0.72	(1.16) 0.84	—	1.61
2.50	0.92	0.94	0.86	0.80	(0.75) 0.75	(0.73) 0.66	(1.13) 0.78	(2.45) 1.0	1.48

In the last column are shown the values of $2d_c$ obtained from equation (64) with the help of the minimum values of δ_0 in the corresponding line. From Table I, as also directly from the graph given in ref. 4, showing the dependence of $\ln(H_c/H_{cb})$ on $\ln 2d$, it is clear that there is a sharp break in the course of this dependence which sets in as d passes through d_c (in Table I the single values of δ_0 and the values in brackets according to (63) are simply quantities proportional to $(H_c/H_{cb})d$; this product falls as d rises to d_c and for $d > d_c$ the sharp rise begins). We are inclined to regard this behaviour as confirmation of the conclusion that the character of the transition is different for $d < d_c$ and $d > d_c$. The fall of the values of δ_0 with rise of d , clearly evident from Table I for $d > d_c$, may be completely explained by the already mentioned difference between the values of d indicated in ref. 4 and the effective values d_{eff} . Alternatively it is evident that the thinner the film the more will d_{eff} depart from d , and that $d_{\text{eff}} < d$. The observed dependence of δ_0 on d for $d < d_c$ is in agreement with this picture; but we can see no reason for the increase with d of the values of δ_0 calculated according to (61) and (62) when $d > d_c$.

We must however bear in mind two considerations. First, the whole of our scheme based on the expansion of F_{s0} and (10) in powers Ψ^2 up to the terms in Ψ^4 is generally speaking valid only in the region close to T_c in which the relation (8) and the equation

$$\delta_0^2 = \frac{\text{const}}{T_c - T} = \frac{\delta_{00}^2}{1 - T/T_c}, \quad (69)$$

are valid, where δ_{00} is a certain constant (see equations (20) and (7)). For mercury the region, where (8) is valid and therefore (69) should be applicable lies between

T_c and $T \sim 3.80-4.0^\circ\text{K}$. For smaller values of T we must in general take into account higher terms in the series expansion of F_{s_0} (i.e. terms in Ψ^5 etc. in (18)) and the application of all the formulae obtained without the substitution of $|\alpha|/\beta$ by $(d\alpha/dT)_c (T_c - T)/\beta_c$ is possible only if the non-linear dependence of $|\alpha|/\beta$ on $(T_c - T)$ is more important than the influence of terms in Ψ^5 etc. Such a situation is possible, but it could not be assumed to occur unless it were demonstrated by an analysis of sufficiently extensive experimental data; this is not possible at present owing to the absence of the latter. In view of what has been said, the data of Table 1 for $T < 3.80^\circ\text{K}$ may be distorted.

The second consideration which we must bear in mind is that T_c varies considerably from film to film; in ref. 4 all the data were reduced to $T_c = 4.167^\circ\text{K}$ and this operation, evidently inaccurate for $T = 4.12^\circ\text{K}$ and $T = 4.13^\circ\text{K}$, may also influence the data in Table 1 at lower temperatures. The whole question clearly requires a more detailed experimental investigation; for the moment we shall take for δ_0 the lowest of the values in Table 1 and compare them with the data obtained by other methods^{10,12}. In doing this we must consider the fact that in ref. 12 the quantity directly measured was only $\delta_0 - \delta_0(2.5^\circ)$, and that δ_0 was calculated by means of an extrapolation which does not appear a priori valid. The values of δ_0 obtained in ref. 10 are based on the previous measurements with the colloids and are likewise inaccurate; here also the measured quantity was $\delta_0 - \delta_0(2.5^\circ)$. As can be seen from Table 2, in which all the quantities must be multiplied by 10^{-5} cm within the limits of the accuracy achieved up to the present time the data of Table 1 coincide with those obtained by other methods (we must especially emphasise that the data of ref. 12 relate to bulk specimens).

TABLE 2.

$T^\circ\text{K}$	δ_0 from Table 1	$\delta_0 - \delta_0(2.5^\circ)$ from Table 1	δ_0 from ref. 10	$\delta_0 - \delta_0(2.5^\circ)$ from ref. 10	δ_0 from ref. 12	$\delta_0 - \delta_0(2.5^\circ)$ from ref. 12
4.13	3.08	2.42	4.08	3.28	2.28	1.82
4.12	2.72	2.06	3.57	2.77	2.04	1.58
4.10	2.28	1.62	2.80	2.00	1.72	1.26
4.05	1.80	1.14	2.34	1.54	1.31	0.85
4.00	1.56	0.90	1.95	1.15	1.10	0.64
3.80	1.10	0.44	1.38	0.58	0.77	0.31
3.50	0.87	0.21	—	—	0.61	0.15
3.00	0.72	0.06	—	—	0.50	0.04
2.50	0.66	0.00	0.80	0.00	0.46	0.00

Assuming for δ_0 the values indicated in the second column of Table 1 we may calculate κ with the help of (22) taking also into account the fact that for mercury close to T_c , $H_{cb} = 187 (T_c - T)$. Thus if we use the most reliable value of δ_0 at 4°K we obtain the result (23). Using the value of δ_0 indicated in ref. 12 for mercury and for tin we obtain $\kappa \sim 0.15$.

Let us now turn to the question of the destruction of superconductivity of a film by a current. For $\kappa = 0$ the function Ψ_0 in the presence of a current

is given by equation (60), which for $d \ll 1$ takes the form

$$\Psi_0^2 = 1 - \frac{H_0^2 d^2}{3} - \frac{H_J^2}{\Psi_0^4 d^2}. \quad (70)$$

The field H_J as a function of Ψ_0 becomes zero for $\Psi_0 = 0$ and for some non-vanishing value of Ψ_0 (if $H_0 = 0$ then $H_J = 0$ for $\Psi_0 = 1$); between these two values of Ψ_0 , H_J exhibits a maximum. In other words the function Ψ_0 for given H_J may according to (70) have two values. It is easy to see that the superconductivity of a film is stable only so long as the field H_J of the current grows with decrease of Ψ_0 (in this case the free energy is less than that corresponding to the same H_J but a lower value of Ψ_0). The critical field H_{Jc} is determined from the condition $dH_J/d\Psi_0 = 0$, which leads to the relation

$$\frac{H_c}{H_{cb}} = \frac{2\sqrt{2}}{3\sqrt{3}} \cdot \frac{d}{\delta_0} \left[1 - \left(\frac{H_0}{H_c} \right)^2 \right]^{3/2}, \quad (71)$$

where H_c is the critical field of a given film in the absence of a current, H_0 is the external field and J_c is the critical current ($H_{Jc} = (2\pi/c) J_c$). In the absence of a field H_0 we have

$$\frac{H_{Jc}}{H_{cb}} = \frac{2\sqrt{2}}{3\sqrt{3}} \cdot \frac{d}{\delta_0}. \quad (72)$$

For the case of arbitrary relative orientations of H_0 and H_J , where we must use equations (51) with the boundary conditions (52), it is easy to see that we obtain the previous equations (60), (70) and (71) with $H_0^2 = H_{x0}^2 + H_{y0}^2$ (the current J is directed along the negative x -axis, the field H_J along the y -axis). It should be noted that it follows from (63) and (72) for sufficiently thin films that

$$H_c H_{Jc} = \frac{4}{3} H_{cb}^2. \quad (73)$$

Thus although the values of H_c and H_{Jc} for thin films may be greatly different from H_{cb} , the product $H_c H_{Jc}$, which is equal for a massive specimen to H_{cb}^2 , is multiplied by a factor $4/3$ for the very thinnest films. Relations (71) and (72) are in qualitative agreement with experiments from which, however, it is impossible to draw quantitative conclusions.

Summarising we may indicate that for an experimental verification of the theory there is a whole number of possibilities: measurement of the critical field and current for films (equations (61), (62), (63) and (72)); measurement of the influence of field on the critical current (see equation (71)); measurement of the magnetic moment (see equations (65) and (66)); measurement of σ_{ns} and, finally, measurement of $\delta(H_0)$ for bulk superconductors (see equations (36) and (53)). However, working with films a direct determination of κ (if it is really small) is in practice apparently not possible. Thus for the determination of κ not using (23) we must either determine σ_{ns} —a quantity which is particularly sensitive to κ —or carry out exact (~ 1 per cent) measurements with bulk superconductors of the influence on δ of fields of the order of H_{cb} .

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74. ON MULTIPLE PRODUCTION OF PARTICLES DURING COLLISIONS OF FAST PARTICLES

1. GENERAL RELATIONS

Collisions of ultra-fast nuclear particles can be accompanied by the appearance of a large number of new particles (many-pronged stars in cosmic radiation). Fermi¹ propounded the ingenious idea of the possibility of applying statistical methods for studying this process. However, the quantitative calculation given by him appears unconvincing to us and incorrect at several points (in particular, in regard to distribution in energy and angle).

Qualitatively the whole process of collision has the following appearance. At the moment of collision there appear a large number of particles[†] concentrated in a volume whose linear dimensions are determined by the range of the nuclear forces and by the energies of the colliding particles (concerning this, see below); it must be emphasised that we can speak of the number of particles at this moment only in a limited sense, since for a system with such a high density of strongly interacting particles (mesons and nucleons) the concept of the number of particles has in general no precise meaning. The "mean free path" of particles in such a system is clearly very small compared to its dimensions. In the course of time, the system expands, but the aforementioned property of the free path must be valid also for a significant part of the process of expansion. This part of the expansion process must have a hydrodynamic character, since the smallness of the mean free path permits us to consider the motion of the matter in the system in a macroscopic hydrodynamical fashion as the motion of an ideal (non-viscous and non-heat-conducting) liquid. Since the velocities in the system are comparable to the velocity of light, we are dealing, not with ordinary, but rather with relativistic hydrodynamics.

The total "number of particles" in the system is not at all constant during the course of the hydrodynamic stage of the expansion. Therefore, the number of particles in the resulting star is determined, not by the number of particles which appear at the very moment of collision (as Fermi mistakenly assumes) but rather by the number of particles in the system at the moment of transition to the second stage of the expansion—the stage of free separation of the particles. This essential point was first made by I. Ya. Pomeranchuk².

Л. Д. Ландау, О множественном образовании частиц при столкновениях быстрых частиц, *Известия Академии Наук СССР, Серия Физическая*, 17, 51 (1963).

[†] In fact, the appearance of a large number of particles is the condition for the applicability of the method for treating the problem which is presented below, and of the associated formulas.

The transition from the first stage to the second occurs when the free path of particles in the system becomes equal to its linear dimensions. A very essential point is that at that moment the order of magnitude of the temperature of the system is

$$T_c \sim \mu c^2 \quad (1)$$

(μ is the meson mass; the temperature is always given in energy units), practically independent of the properties of the system, i.e. of the energy of the colliding particles. In fact, for values of the temperature substantially lower than μc^2 , the density of the equilibrium number of particles falls exponentially with cooling (as $e^{-\mu c^2/T}$) so that the mean free path rapidly becomes equal to the dimensions of the expanding system, even when the latter are relatively large. Formula (1) for T_c (with the π -meson mass substituted for μ) is also valid when, in addition to mesons, other heavier particles are formed, since in order for the free path of all particles to be small, it is already sufficient that there be a high density of π -mesons in the system.

For the hydrodynamic considerations, it is necessary to have an equation of state for the matter in the system. As equation of state of highly compressed matter for temperatures $T \gg \mu c^2$ we use:

$$p = \frac{\varepsilon}{3}, \quad (2)$$

where p is the pressure and ε is the energy density. Although we have not at present any rigorous proof that this must be the equation of state for arbitrary matter in the ultrarelativistic case, nevertheless in our opinion this assumption is highly plausible.

Since the number of particles in the system is not fixed, but is rather determined from the conditions of statistical equilibrium, its chemical potential (just as for black-body radiation) is

$$\zeta = \varepsilon - Ts + p = 0$$

(s is the entropy per unit volume). Then

$$Ts = \varepsilon + p = \frac{4\varepsilon}{3}$$

so if we take into account also that for fixed volume (equal to unity) $d\varepsilon = Tds$ we find the relations:

$$S \sim \varepsilon^{3/4}; \quad T \sim \varepsilon^{1/4} \quad (3)$$

which, as expected, coincide with the relations for black-body radiation.

The computation of the total number of particles appearing during the break-up is greatly simplified if we consider the motion of the ideal fluid to be adiabatic. The only thing that could destroy the adiabaticity would be shock waves, and it is hard to imagine how they could be formed during the expansion process. Therefore, the entropy of each of the individual regions of the system remains unchanged during the expansion.

Let us break up the system into a set of regions which are macroscopically small, i.e. practically uniform, but which still contain a sufficiently large number of particles; let s_α be the entropies of these regions. Also let n_α be the number of particles in the α -th region which have been produced at the time of the start of its free separation. This time may not be the same for the various regions, since the system as a whole is highly non-uniform. The quantities s_α and n_α individually depend strongly on the temperature (for $T \ll \mu c^2$, they vary as $e^{-\mu c^2/T}$), but the ratio s_α/n_α depends only slightly on temperature, so that, since T_c in turn depends little on the properties of the system, we may consider that

$$n_\alpha = \text{const} \cdot s_\alpha,$$

where the constant ratio is a universal constant [if we measure entropy in dimensionless units, dimensional arguments show that the constant is of order $(\mu c/\hbar)^3$]. Summing this equality over all domains, we find that

$$N = \text{const} \cdot S, \quad (4)$$

where N is the total number of particles in the star, and S is the total entropy of the system. Since the entropy stays constant during the whole course of the hydrodynamic stage of the expansion, we may consider S to be the entropy of the system at the initial time—the time of the collision.† Formula (4) enables us to determine the total number of particles appearing during the collision, without a detailed examination of the motion of the system.

2. TOTAL NUMBER OF PARTICLES

Let us first consider “head-on” collisions in which the particles pass each other at distances comparable to the range of interaction, as distinguished from peripheral collisions where the impact parameter is large compared to the range of force.

We start with head-on collisions of two protons, and determine the energy dependence of the total number of particles formed. Let E' be the energy of each of the protons in the centre of mass system (c.m.s.). The total entropy of the system, S , is proportional to $\epsilon^{3/4} V$ where V is the volume over which the energy is distributed. In the c.m.s. the matter is at rest at the moment immediately following the collision. Therefore, $\epsilon = E'/V$, and so the entropy, and consequently the number of particles, is proportional to $E'^{3/4} V^{1/4}$.

The transverse dimension of the system, a , is of order of magnitude of the range of nuclear force, i.e. $a \sim \hbar/\mu c$. The longitudinal dimension (in the c.m.s.) is shortened by the Lorentz contraction in the ratio $\sim Mc^2/E'$ (M is the

† More precisely, after the passage of shock waves, which can arise at the moment of collision; the passage of a shock wave is accompanied by a compression of the matter, after which the expansion stage begins and proceeds adiabatically from then on.

proton mass). Thus the system is in the form of a highly flattened disk, and its volume is

$$V \sim a^3 M c^2 / E'.$$

So the number of particles is

$$N \sim E'^{3/4} V^{1/4} \sim \sqrt{E'}$$

or, going over to the energy E in the laboratory system in which one of the protons is at rest, using the formula $EMc^2 = 2E'^2$, we finally get:

$$N \sim E^{1/4}.$$

This formula coincides with the one obtained by Fermi, but his reasoning appears to us to be completely unconvincing. From dimensional arguments (and taking account of the fact that the ratio of masses of proton and π -meson is fairly close to the unity) we may write:

$$N = K \left(\frac{E}{2Mc^2} \right)^{1/4} \quad (5)$$

where K is a constant of the order of unity.

Now let us consider the collision of two identical nuclei of atomic weight A . It would be completely erroneous to treat such a collision as a series of collisions of nuclear protons and neutrons. In fact, since the distance between nucleons in the nuclei is precisely of the order of their range of interaction, we must look upon the result of the collision as a process of meson formation involving as a unit the whole space occupied by the nuclei.

Suppose that the speed of the incident nucleus is equal to that of the proton in the preceding problem. Then its energy will be A times as large. Since the mass density in a nucleus is approximately the same as that of the proton, referred to its sphere of interaction, the energy density immediately after collision is the same as in the previous case. Since the Lorentz contraction is unchanged, the number of particles formed is simply proportional to the volume of the nucleus, i.e. to A . Thus we finally obtain:

$$N = KA \left(\frac{E}{2AMc^2} \right)^{1/4} = KA^{3/4} \left(\frac{E}{2Mc^2} \right)^{1/4}. \quad (6)$$

For a given energy, the number of particles is proportional to $A^{3/4}$. We note that according to this formula, heavy nuclei are much more effective in particle formation than protons: two nuclei with energy E give as many particles as two protons with energy EA^2 .

When the two nuclei have different weights the problem becomes more complicated, but elementary considerations related to the fact that in collision the lighter nucleus pulls out only a part of the heavier one, show that the number of particles is determined essentially by the mass of the lighter nucleus, and depends only slightly on the mass of the heavier one.

If we are dealing with collisions of a meson with a nucleon or nucleus, it follows that we should expect relatively little difference from the case of a nucleon.

Determination of the constant K from existing experimental data gives the value

$$K \sim 2.$$

As for peripheral collisions of the nucleons, at first glance one might conclude that the average number of particles produced should decrease rapidly with increasing impact parameter. A basis for this conclusion might be the fact that the rest energy of the matter concentrated in each individual region of the meson field of the colliding nucleons decreases rapidly (exponentially) with increasing distance from their "centre". However, the incorrectness of this derivation is clear from the fact that it leads to a contradiction with the quantum uncertainty relations; the rest energy of a portion of the system would turn out to small compared to the uncertainty

$$\Delta E \sim \hbar c/\Delta,$$

where Δ is the thickness of the region, compressed by the Lorentz contraction just as for central collisions. In fact, this relation means only that the quantity which is small is not the actual energy of the system (in those cases where such a system occurs at all) but rather its mathematical expectation. In other words, it is not the number of particles appearing that decreases, but only the probability that such a collision shall occur.

Thus for collisions of two nucleons it is in general meaningless to distinguish between central and peripheral collisions; the effective cross-section for collision with production of a many-pronged star is determined by the "radius" of the nucleon, $\hbar/\mu c$. The picture is somewhat changed in the case of a collision of two nuclei. It is clear that as we vary the impact parameter from zero to the sum of the radii of the nuclei, the number of particles formed must decrease from the maximum value given by formula (6) to the value given by (5) and corresponding to the collision of two nucleons.

3. DISTRIBUTION OF PARTICLES PRODUCED IN ENERGY AND DIRECTION

A study of the angular distribution of the particles formed, and their distribution in energy, requires a detailed consideration of the hydrodynamical motion of the matter in the system.

The relativistic hydrodynamic equations are contained in the relations

$$\frac{\partial T^{ik}}{\partial x^k} = 0, \quad (7)$$

† This result was clarified in discussions with E. L. Feinberg.

where T^{ik} is the energy-momentum tensor of the matter:

$$T^{ik} = p g^{ik} + (\varepsilon + p) u^i u^k \quad (8)$$

(u^i is the four-velocity; $g^{11} = g^{22} = g^{33} = 1$, $g^{00} = -1$; from now on we set $c = 1$).

As we have already indicated, at the moment of collision the system has the form of a highly flattened disk. This shape is maintained throughout a significant part of the hydrodynamical stage of the expansion. During this stage, the motion of the matter can be considered to be one-dimensional, along the short axis of the disk (x -axis). Then the equations of motion are:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x} = 0, \quad \frac{\partial T^{01}}{\partial t} + \frac{\partial T^{11}}{\partial x} = 0, \quad (9)$$

where

$$T^{00} = \varepsilon(u^0)^2 + p(u^1)^2, \quad T^{01} = (\varepsilon + p) u^0 u^1, \quad T^{11} = \varepsilon(u^1)^2 + p(u^0)^2, \quad (10)$$

and u^0 and u^1 are related by the equation:

$$(u^0)^2 - (u^1)^2 = 1. \quad (11)$$

In the c.m.s. the "disk" expands symmetrically to both sides. We choose our co-ordinate origin in the median plane and shall consider the motion in the half-space expanding along the positive x -axis (so that $x > 0$, $u^1 > 0$).

Let us call the initial thickness of the "disk" Δ . We consider some instant of time $t \gg \Delta$, when the expansion has already progressed significantly. Neglecting the initial thickness of the disk we can assert that all the matter will be in the region $0 < x < t$, since the velocity cannot exceed that of light. Most of this space will contain matter which, though moving with a speed comparable to the light velocity, is not ultra-relativistic; only in a thin layer $t - x \ll t$ will there be matter moving with a velocity close to that of light. As we shall see later, in this last region there is concentrated only a *small* part of the entropy, but a *large* part of the energy of the system. Therefore, the examination of this small-sized ultra-relativistic region is very essential. To do this we replace the variable x by $\xi = t - x$. Then the first of equations (9) takes the form:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial (T^{00} - T^{11})}{\partial \xi} = 0, \quad (12)$$

and, subtracting (12) from the second equation of (9) we find:

$$\frac{\partial}{\partial t} (T^{00} - T^{11}) + \frac{\partial}{\partial \xi} (T^{00} - 2T^{01} + T^{11}) = 0. \quad (13)$$

In the ultra-relativistic case both components u^0 , u^1 of the four-velocity are large compared to unity and almost equal (we recall that $u^0 = 1/\sqrt{1-v^2}$, $u^1 = v/\sqrt{1-v^2}$, where v is the ordinary velocity (in units $c = 1$). Later we shall denote by u (in first approximation) either of the quantities u^0 and u^1 .

According to (10):

$$u^0 \approx u^1 = u \gg 1, \quad u^0 - u^1 \approx \frac{1}{2u}.$$

Using these equalities and the equation of state (2), we obtain from (10):

$$\left. \begin{aligned} T^{00} &\approx (\varepsilon + p) u^2 = \frac{4}{3} \varepsilon u^2, \\ T^{00} - T^{01} &= (\varepsilon u^0 - p u^1)(u^0 - u^1) \approx \frac{\varepsilon}{3}, \\ T^{00} - 2T^{01} + T^{11} &= (\varepsilon + p)(u^0 - u^1)^2 \approx \frac{\varepsilon}{3u^2}, \end{aligned} \right\} \quad (14)$$

after which equations (12-13) take the form:

$$\left. \begin{aligned} \frac{\partial}{\partial t}(\varepsilon u^2) &= -\frac{1}{4} \frac{\partial \varepsilon}{\partial \xi}, \\ \frac{\partial \varepsilon}{\partial t} &= -\frac{\partial}{\partial \xi} \left(\frac{\varepsilon}{u^2} \right). \end{aligned} \right\} \quad (15)$$

We shall look for solutions of these equations in the domain of values $t \gg \xi \gg 1$.

A solution satisfying all the necessary requirements can be obtained as follows. Let us make the assumption, which we later show to be valid, that the function $u(\xi, t)$ is such that

$$u^2 = f \frac{t}{\xi}, \quad (16)$$

where f is a slowly (logarithmically) varying function of ξ and t . Neglecting the derivatives of f , we then obtain from (15)

$$f \frac{\partial}{\partial t}(\varepsilon t) = -\frac{\xi}{4} \frac{\partial \varepsilon}{\partial \xi}, \quad f t \frac{\partial \varepsilon}{\partial t} = -\frac{\partial}{\partial \xi}(\varepsilon \xi).$$

Next we introduce the new variables

$$\tau = \ln \frac{t}{1}, \quad \eta = \ln \frac{\xi}{1}, \quad (17)$$

and in place of ε , a new unknown function φ according to the relation

$$\varepsilon = e^\varphi. \quad (18)$$

From the two equations thus obtained,

$$f \left(1 + \frac{\partial \varphi}{\partial \tau} \right) = -\frac{1}{4} \frac{\partial \varphi}{\partial \eta} \quad \text{and} \quad f \frac{\partial \varphi}{\partial \tau} = - \left(1 + \frac{\partial \varphi}{\partial \eta} \right), \quad (19)$$

we eliminate f and get:

$$1 + \frac{\partial \varphi}{\partial \tau} + \frac{\partial \varphi}{\partial \eta} + \frac{3}{4} \cdot \frac{\partial \varphi}{\partial t} \cdot \frac{\partial \varphi}{\partial \eta} = 0. \quad (20)$$

Following the general procedure for obtaining the general integral of a partial differential equation of first order, we first form the complete integral:

$$\varphi = A \eta - \frac{4(1+A)}{4+3A} \tau + B, \quad (21)$$

containing two constants A and B . The general integral (containing one arbitrary function) is obtained from the complete integral if we consider B to be a function of A , determined by equation (21) and the equation

$$\eta - \frac{4}{(4+3A)^2} \tau + \frac{dB}{dA} = 0 \quad (22)$$

obtained by setting the derivative $\partial \varphi / \partial A$ equal to zero.

Since we are looking for a solution of the equation of motion in the region of values $t \gg 1$, $\xi \gg 1$, the "initial moment" of the motion corresponds to values $\tau \sim 1$, $\eta \sim 1$. At this "moment", the system in the domain under consideration can be regarded as still uniform, so that the function φ is practically constant and equal to some value φ_0 (the logarithm of the initial energy density ε_0). Thus the initial condition for our problem, to within the logarithmic accuracy we are using, is:

$$\varphi - \varphi_0 \sim 1 \quad \text{for} \quad \eta \sim 1, \quad \tau \sim 1. \quad (23)$$

A solution satisfying this condition is obtained from (21-22), if the arbitrary function $B(A)$ is chosen so that

$$B - \varphi_0 \sim 1, \quad B' \sim 1.$$

Then we can omit B' altogether in (22), (since $\eta \gg 1$, $\tau \gg 1$), and can set $B \approx \varphi_0$ in (21). We then have from (22)

$$\frac{4+3A}{2} = \sqrt{\frac{\tau}{\eta}}$$

(we choose the positive root, since in the other case the function f in (16) would turn out to be negative, which is clearly impossible), after which (21) gives:

$$\left. \begin{aligned} \varphi &= \varphi_0 - \frac{4}{3} (\eta + \tau - \sqrt{\tau \eta}), \\ \varepsilon &= \varepsilon_0 \exp \left[-\frac{4}{3} (\eta + \tau - \sqrt{\tau \eta}) \right] \end{aligned} \right\} \quad (24)$$

When ξ becomes of the order of t , formula (16) as expected gives $u \sim 1$. From formula (24) it follows that in this region ($\eta \approx t$):

$$\varepsilon = \varepsilon_0 e^{-4\tau/3} = \varepsilon_0 \left(\frac{\Delta}{t}\right)^{4/3}.$$

Even though the domain $\xi \sim t$ is outside the region of ultra-relativistic motion, this result should be correct as to order of magnitude.

The function f is found from φ using either of the equations (19):

$$f = \frac{1}{2} \sqrt{\frac{\tau}{\eta}}.$$

In accordance with our assumptions, it is a slowly varying function of t and ξ , of order unity.

Using the formulas we have obtained, let us see how the energy and entropy are distributed throughout the thickness of the "disk". The energy density is given by the component $T^{00} \sim \varepsilon u^2$ of the energy-momentum tensor (we recall that for each element of the matter ε is the energy density in the "proper" frame of reference, in which that element is at rest). So for the energy dE located in a slab of thickness $d\xi$ we have:

$$dE \sim \varepsilon a^2 u^2 d\xi = \varepsilon a^2 u^2 \xi d\eta,$$

where a is the radius of the disk. Setting $u^2 \sim t/\xi$, in accord with (16) and using equation (24) we obtain:

$$dE \sim \exp \left[-\frac{1}{3} (\sqrt{\tau} - 2\sqrt{\eta})^2 \right] d\eta. \quad (25)$$

From this it is clear that the energy distribution has a maximum at $\eta = \tau/4$; this means that the energy is concentrated mainly in the region

$$\xi \sim \sqrt[4]{t\Delta^3}.$$

For $t \gg \Delta$ we get $\xi \ll t$, so that this region is at the limit of applicability of the one-dimensional solution we are considering.

The entropy density is given by the fourth component s^0 of the four-vector of entropy current density $s^i = s u^i$. Since $s \sim \varepsilon^{3/4}$ (according to (4)), $s^0 \sim u \varepsilon^{3/4}$, and we find for the entropy associated with a slab of thickness $d\xi$:

$$dS \sim s u a^2 d\xi \sim a^2 s u \xi d\eta,$$

or, using formula (24):

$$dS \sim \exp \left[-\frac{1}{2} (\sqrt{\tau} - \sqrt{\eta})^2 \right] d\eta. \quad (26)$$

This distribution has a maximum for $\eta = t$; i.e. the entropy unlike the energy is concentrated mainly in the region $\xi \sim t$.

The solution of the equation of motion which we have obtained is applicable so long as the angle of flight θ (the angle which the trajectory of a given element of the matter makes with the x -axis) is sufficiently small. This is necessary in order that the distance $t\theta$, which the element travels during the time t in the transverse direction, be small compared to the transverse dimensions of the system, a :

$$t\theta \ll a. \quad (27)$$

To evaluate the small angle θ , we use the transverse components of equation (7), which we have as yet not considered. Thus we get:

$$\frac{\partial T^{02}}{\partial t} \sim \frac{\partial T^{22}}{\partial y},$$

or, to order of magnitude,

$$\frac{T^{02}}{t} \sim \frac{T^{22}}{a};$$

so that substituting $T^{02} \sim \varepsilon u^2 \theta$ and $T^{22} \sim \varepsilon$ (the transverse component of the four-velocity is $u^y \sim u \theta$), we get:

$$u^2 \theta \sim \frac{t}{a}.$$

Finally, noting that $u^2 \sim t/\xi$, we find:

$$\theta \sim \frac{\xi}{a}. \quad (28)$$

Combining this formula (27) we see that the condition for applicability of the one-dimensional solution is:

$$t\xi \ll a^2. \quad (29)$$

We note that the limiting time for the one-dimensional solution is the greater, the smaller the value of ξ . For the central region, $\xi \sim t$, and the limiting time is $t \sim a$.

Starting at the moment

$$t_1 = \frac{a^2}{\xi}, \quad (30)$$

a significant sideways motion appears in the hydrodynamical motion; we shall call the resulting motion of the matter conical hydrodynamic flight. As we shall see later, in this stage of the motion the velocity approaches that of light so quickly, that for each element of matter the quantity ξ remains practically constant in time. In addition, one can show that all derivatives of hydrodynamic quantities, both with respect to the direction of ξ as well as with respect to the transverse direction, can be neglected in the equations. Thus, in particular, it follows that, because of the smallness of the sidewise forces, the direc-

tion of motion will remain unchanged, i.e. the flight will proceed radially (conically).

Furthermore, in view of the smallness of the forces during conical flight, the energy flux traveling within any cone, $\theta = \text{const}$, must remain constant; the same is true for the entropy flow. The cross sectional area of such a cone is proportional to t^2 , so the conditions of constancy of flow of energy and entropy are:

$$\varepsilon u^2 t^2 = \text{const}, \quad s u t^2 \sim \varepsilon^{3/4} u t^2 = \text{const}. \quad (31)$$

From these two relations we find:

$$u \sim t, \quad \varepsilon \sim \frac{1}{t^4}, \quad (32)$$

which give the law of variation with time of u and ε during conical flight. From (32) we see that in this case the velocity actually approaches that of light faster than during the preceding stage. The change in the co-ordinate ξ of the moving element of matter is given by the formula:

$$\frac{d\xi}{dt} = 1 - v_x = (u^0 - u^1) \sqrt{1 - v^2} \approx \frac{1}{2u^2} \sim \frac{1}{t^2},$$

from which it is clear that during this stage of the flight, the quantity approaches a constant value more rapidly.

For $t \sim t_1$, the solution (32) must agree to order of magnitude with the one-dimensional solution considered earlier. For the "joining" of the two solutions, it is convenient to introduce the symbols λ and L , according to the equations:

$$\frac{\xi}{a} = e^{-\lambda}, \quad \frac{\Delta}{a} = e^{-L}. \quad (33)$$

Then

$$\eta = \ln \frac{\xi}{\Delta} = L - \lambda, \quad (34)$$

while the value of the variable τ corresponding to the moment t_1 is

$$\tau_1 = \ln \frac{t_1}{\Delta} = \ln \frac{a^2}{\xi \Delta} = L + \lambda. \quad (35)$$

Substituting this value in (26), we find that the entropy distribution is given by

$$dS \sim e^{\sqrt{L^2 - \lambda^2}} d\lambda.$$

Since each element of the matter now moves with $\xi = \text{const}$ while its entropy, by virtue of the adiabaticity of the motion, remains constant, the same formula gives the entropy at the moment of break-up of the matter into individual freely moving particles. The number of particles produced will be distributed according to this same law:

$$dN = C e^{\sqrt{L^2 - \lambda^2}} d\lambda, \quad (36)$$

where C is a normalising factor. The angle of flight

$$\theta \sim \frac{\xi}{a} = e^{-\lambda} \quad (37)$$

remains constant along with ξ for each element of the matter, and consequently for each particle. Consequently, formulae (36) and (37) determine in parametric form (parameter λ) the angular distribution of the produced particles (in the c.m.s.). The constant L which appears in the formula is related simply to the energy of the colliding particles. In fact, the ratio Δ/a is the Lorentz contraction of the system and is equal, in the notation of section 2 to

$$\frac{MAc^2}{E'} = \sqrt{\frac{2MAc^2}{E}}$$

(where MA is the mass of the particles). Therefore

$$L = \frac{1}{2} \ln \frac{E}{2MAc^2}. \quad (38)$$

The distribution (36) shows that, although the angle of departure in the c.m.s. is of the order of unity for most of the particles, there also occur much smaller angles. It is easy to see that the angular distribution does not at all show spherical symmetry, as Fermi assumed, but that $dN/d\phi$, referred to unit solid angle, increases rapidly with decreasing θ .

Formula (36) is easily written in explicit form. In order to take into account angles of the order of unity, we define λ as

$$\lambda = -\ln \tan \frac{\theta}{2}. \quad (39)$$

With this definition, the smallest value $\lambda = 0$, corresponds to the largest possible value $\theta = \pi/2$. Formula (36) then becomes

$$dN \sim \exp \left[\sqrt{L^2 - \ln^2 \tan \frac{\theta}{2}} \right] \frac{d\phi}{\sin^2 \theta}. \quad (40)$$

This formula agrees well with the experimental data³. For practical purposes, formula (36) can be written to sufficient accuracy in the form:

$$dN \sim e^{-\frac{\lambda^2}{2L}} d\lambda. \quad (41)$$

Thus the angular distribution can be written as a Gaussian distribution, if we choose as variable the quantity $\lambda = -\ln \tan \theta/2$. In view of the logarithmic dependence of λ on θ , the actual distribution curve of the particles with respect to the angle θ itself must have relatively very straight tails on both sides of the maximum.

We note that the largest value of λ which it is still meaningful to consider must correspond to the condition:

$$\int_{\lambda=\lambda_{\max}}^{\lambda=L} dN \sim 1,$$

or, substituting (36), to logarithmic accuracy,

$$C e^{\sqrt{L^2 - \lambda_{\max}^2}} \sim 1.$$

According to formulae (5) and (36) the total number of particles is $N \sim e^{L/2}$; therefore

$$\int_{\lambda=0}^{\lambda=L} dN \sim C e^L \sim e^{L/2}.$$

Thus $C \sim e^{L/2}$, and we obtain for λ_{\max} ,

$$\lambda_{\max} = \frac{\sqrt{3}}{2} L. \quad (42)$$

For determining the energy distribution of the particles, we consider the quantity u , which is proportional to the energy of the particles (the energy of a particle is the time component of the four-vector μu^i : $\mu u^0 \approx \mu u$). During the stage of one-dimensional motion $u \sim \sqrt{t/\xi}$, and at the moment $t = t_1$ it reaches the value $u \sim \sqrt{t_1/\xi}$. Therefore, "tacking on" the one-dimensional motion to the solution (32), we find that during the stage of conical motion:

$$u \sim \sqrt{\frac{t_1}{\xi}} \cdot \frac{t}{t_1} \sim \frac{t}{a}. \quad (43)$$

In similar fashion we match the laws (24) and (32) of variation of the "proper" energy density ε . For $t \sim t_1$ the quantity ε reaches the value:

$$\varepsilon = \varepsilon_0 \exp \left[-\frac{4}{3} (2L - \sqrt{L^2 - \lambda^2}) \right].$$

Determining from this the coefficient of proportionality in (32), we find:

$$\varepsilon = \varepsilon_0 \left(\frac{t_1}{t} \right)^4 \exp \left[-\frac{4}{3} (2L - \sqrt{L^2 - \lambda^2}) \right]. \quad (44)$$

The start of the free separation of the particles corresponds to the moment t_0 when ε , decreasing, reaches the value ε_c corresponding to the criterion (1). From (44) we find:

$$t_c \sim t_1 \left(\frac{\varepsilon_0}{\varepsilon_c} \right)^{1/4} \exp \left[-\frac{1}{3} (2L - \sqrt{L^2 - \lambda^2}) \right].$$

Setting $t \sim t_c$ in (43) and substituting for t_1 from (35), we find the following expression for the energy $\mu u'$ of the particles at the moment of their free separation:

$$\mu u' \sim \mu \frac{t_c}{a} = \text{const} \exp \left[\lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right].$$

We note that the energy of the outgoing particles is measured by the ratio of the time (or the distance from the origin) at the moment of decay into particles to the characteristic time a/c of the system. The constant coefficient in the expression for $\mu u'$ is determined from the obvious relation:

$$\int \mu u' dN = E' \sim \sqrt{E M A} \sim M A e^L,$$

and we get finally:

$$\mu u' \sim M \exp \left\{ -\frac{L}{6} + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right\}. \quad (45)$$

Formulae (36) and (45) give in parametric form the energy distribution (in the c.m.s.) of the particles produced. From (45) we see that most of the particles ($\lambda \sim 0$) have energies $\mu u' \sim M e^{L/6} \sim M (E'/A M^{1/6})$ only slightly exceeding M .

We must still go over from the c.m.s. to the original laboratory frame of reference in which one of the nucleons was at rest before the collision. The angle χ of the outgoing particle in the laboratory system is related to the angle θ in the c.m.s. by the transformation formula:

$$\tan \chi \approx \chi = \frac{v' \sqrt{1 - V^2} \sin \theta}{v' \cos \theta + V},$$

where v' is the velocity of the particle in the c.m.s., and V is the velocity of the c.m. relative to the laboratory system. We may immediately write $v' = 1$ in the numerator, and in the denominator, write:

$$v' \cos \theta + V \approx v' (1 + \cos \theta) + \frac{1}{2} (V^2 - v'^2),$$

or, since V is closer to unity than v' :

$$v' \cos \theta + V \approx 1 + \cos \theta + \frac{1}{2} (1 - v'^2) = 1 + \cos \theta + \frac{1}{2u'^2}.$$

The last term on the right can be neglected for all cases except when θ is too close to π . However, it is easy to see that the angles we have found satisfy the inequality $\theta, \pi - \theta \gg 1/u'$; this is equivalent to the inequality:

$$\exp \left\{ \frac{L}{6} - \frac{1}{3} \sqrt{L^2 - \lambda^2} \right\} \ll 1$$

(according to (39) and (45), which is actually satisfied for all $\lambda < \lambda_{\max}$. Thus we can set $v' \cos \theta + V \approx 1 + \cos \theta$, and the formula for transforming angles to the laboratory system takes the form:

$$\chi = \sqrt{1 - V^2} \tan \frac{\theta}{2}. \quad (46)$$

In this connection we note the following curious fact. Independently of any detailed computations, the distribution of outgoing particles, for a collision of two identical particles, is symmetric in the c.m.s., i.e. angles θ occur just as often as $\pi - \theta$. Since $\tan(\pi - \theta)/2 = 1/\tan(\theta/2)$, it follows automatically that, upon averaging over all particles,

$$\overline{\ln \chi} = \ln \sqrt{1 - V^2} = -L. \quad (47)$$

In other words, the geometrical mean of all the angles of separation gives just the value of the velocity of the c.m. and, consequently, the velocities of the incident particles (for a collision of two identical particles).

Substituting the value $\tan(\theta/2) = e^{-\lambda}$ and $\tan(\pi - \theta)/2 = e^{-\lambda}$ in (46) for particles moving in opposite directions in the c.m.s., we obtain:

$$\chi = e^{-L \mp \lambda}.$$

This formula has the special property that when we change from particles going to the right in the c.m.s. to particles travelling to the left, there is merely a change in sign of the quantity λ . We can therefore write

$$\chi = e^{-L - \lambda}, \quad (48)$$

and consider formulae (36) and (48) as giving the angular distribution of all particles in the laboratory system, where λ can take both positive and negative values.

For the transformation of the energy of particles moving to the right, we have

$$u \sim \frac{u'}{\sqrt{1 - V^2}} = e^L u',$$

and for particles moving to the left we get (noting that $\theta \gg 1/u$):

$$u \sim \frac{\theta^2 u'}{\sqrt{1 - V^2}} = u' e^{L - 2\lambda}.$$

Substituting (45), this gives:

$$\mu u \sim M \exp \left\{ \frac{5L}{6} \pm \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right\}.$$

This formula too has the property that it describes particles moving to both the right and the left in the c.m.s., if we write

$$\mu u \sim M \exp \left\{ \frac{5L}{6} + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right\}, \quad (49)$$

and give λ both signs.

Formulae (36) and (49) give the energy distribution of the particles in the laboratory system. The coefficients in these formulas can be made more precise if we use the obvious relations:

$$\int dN = N, \quad \int \mu u dN = E.$$

In the integrations we can, to the accuracy we are considering, expand the exponent in a series in the neighbourhood of the maximum. We then get

$$dN = \frac{N}{\sqrt{2\pi L}} e^{\sqrt{L^2 - \lambda^2}} d\lambda,$$

or, taking account of (6) and (38),

$$dN = \frac{KA}{\sqrt{2\pi L}} e^{-\frac{L}{2} + \sqrt{L^2 - \lambda^2}} d\lambda. \quad (50)$$

It is understood that the coefficient in this formula is actually a slowly varying (non-exponential) function of the ratio λ/L . For the energy we get:

$$\mu u = \frac{5\sqrt{5}}{2\sqrt{3}} \cdot \frac{M}{K} \exp \left\{ \frac{5L}{6} + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right\}. \quad (51)$$

Here the same remarks apply to the coefficient as were made in the last case. From formula (51) it is clear that most of the particles have an energy of order

$$M c^2 \left(\frac{E}{2MAc^2} \right)^{7/12}$$

in the laboratory system.

We note that both the angular and energy distributions of the particles are close to Gaussian if we use the logarithm of these quantities as variables; consequently, they have quite straight tails on both sides of the maximum. The results of a computation based on (51) are shown in Fig. 1.

In conclusion, I should like to thank E. M. Lifshitz, I. Ya. Pomeranchuk, and E. L. Feinberg for discussion of the questions touched upon here, and also L. I. Saruchev for permission to use the drawing of the spectra which he calculated from formula (51).

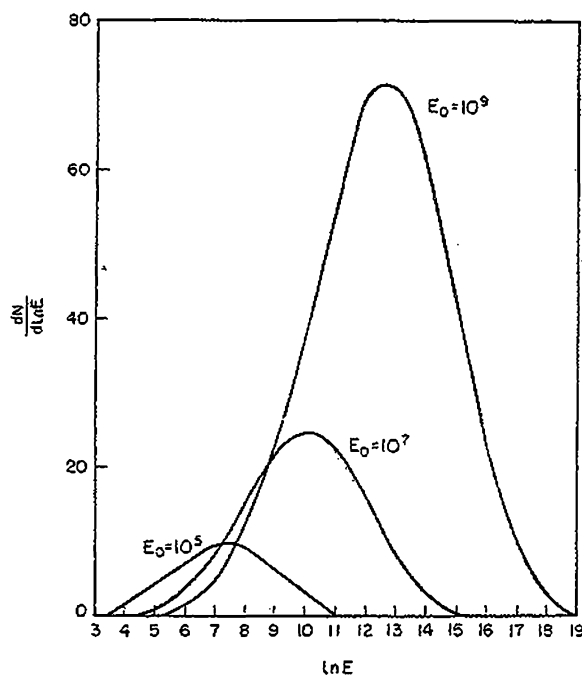


FIG. 1.

Differential energy spectra of secondary particles during nuclear interactions at high energy (for varying energies E_0 of the initial particles). The areas under the curves are proportional to the total number of secondary particles (mesons and nucleons).

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75. THE LIMITS OF APPLICABILITY OF THE THEORY OF BREMSSTRAHLUNG BY ELECTRONS AND OF THE CREATION OF PAIRS AT LARGE ENERGIES

THEORETICAL analysis¹ by the methods used in obtaining the Bethe-Heitler (B-H) equation² for Bremsstrahlung and the formation of pairs, leads to the conclusion that these should be applicable right up to any energies, however great. This conclusion, however, refers to a radiation process taking place at a single isolated atom. If we consider radiation processes occurring in a medium, it may be shown that for sufficiently large energies the Bethe-Heitler theory will not be valid.

In order to discuss in the simplest possible fashion the question, we consider initially the emission of quanta possessing an energy[†] ω much smaller than the energy E of an electron. In this case classical concepts are valid, and lead to the following expression for the energy supplied by radiation in an element of solid angle $d\Omega$ and within the frequency interval $d\omega$:³

$$dI_{n\omega} = \frac{e^2}{4\pi^2} \omega^2 d\omega \left| \int [n \wedge dr] e^{i(\omega t - (k \cdot r \omega))} \right|^2, \quad dr = V(t) dt, \quad k = n \omega. \quad (1)$$

The Bethe-Heitler formulae for low values of ω are obtainable from (1) if it is supposed that V changes abruptly during the collision of an electron with a single atom: that is, $V = V_1$, $V_1 = V_{1z}$ when $t < 0$, and $V = V_2$ when $t > 0$. Multiplying (1) by the probability of the given electron dispersion, it is possible to obtain the Bethe-Heitler results⁴. We now proceed to consider the integral

$$[n \wedge V_1] \int_{-\infty}^0 dt e^{i(\omega - k_z V_1)t} + [n \wedge V_2] \int_0^{\infty} dt e^{i[\omega - (k \cdot V_2)]t},$$

$$(k \cdot V_2) = k V_{2z} \left(1 - \frac{\vartheta^2}{2} \right) + (k \cdot \vartheta V_2), \quad (2)$$

where ϑ is the angle between k and the z -axis.

For low values of ϑ (effective $\vartheta \sim m/E$, where m is the rest energy of the electron), the essential time interval is found to be of the order:

$$t_e \sim E^2/m^2 \omega. \quad (3)$$

Л. Ландау и И. Померанчук, Пределы применимости теории тормозного излучения электронов и образования пар при больших энергиях, Доклады Академии Наук СССР, 92, 535 (1953).

[†] $\hbar = c = 1$

With increasing E , t_e increases very rapidly, and, in consequence of this, the distance of the electron from the nucleus, which considerably exceeds atomic dimensions, plays an important role.

Starting from $E = (m^2 L \omega)^{\frac{1}{2}}$, where L is the shower unit of length,² the separation will be of the order L . Under these conditions it is quite evident that the Bethe-Heitler results cannot be valid, since electrons, positrons and quanta are substantially absorbed over this length. The Bethe-Heitler formulae, however, are infringed at considerably lower energies, because of multiple Coulomb scattering over the separation t_e . Taking account of this distance, we have:

$$(\mathbf{k} \cdot \mathbf{r}(t)) = V(0) k \left(1 - \frac{\vartheta^2}{2}\right) \int \left(1 - \frac{\theta_t^2}{2}\right) dt + k \left(\vartheta \cdot \int \theta_t dt\right), \quad (4)$$

where θ is connected with V in the following manner:

$$V(t) = V(0) + \theta_t, \quad (V(0) \cdot \theta_t) = 0 \quad (5)$$

and θ is due to the multiple scattering. Inserting (4) in (1), we assume for simplicity that $\vartheta = 0$. Because of multiple scattering there occurs in the exponent of (1) a term

$$\frac{1}{2} k \int \theta_t^2 dt. \quad (6)$$

To estimate the effect of this we replace θ_t^2 by $\overline{\theta_t^2}$, obtaining the latter from the theory of multiple scattering⁵. (6) then becomes, as to order of magnitude:

$$\omega E_s^2 t^2 / E^2 L. \quad (7)$$

Replacing t with the quantities in (3) we obtain:

$$E_s^2 E^2 / m^4 \omega L. \quad (8)$$

When $E = E_0(\omega)$, where

$$E_0(\omega) = \frac{m^2}{E_s} \sqrt{\omega L} = \frac{m^2}{E_s} \sqrt{\frac{m^2 \omega}{4 Z^2 n e^6 \ln(191 Z^{-1/3})}} \quad (9)$$

(n being the number of nuclei per cm^3), there appears in the exponent of (1) an additional term of the order of magnitude of unity, and, consequently, the Bethe-Heitler formulae will now become applicable for radiation of frequency ω .

If ω is approximately equal to E , the classical consideration is still valid as to order of magnitude, and we obtain from (9)

$$E_0(E) \equiv E_0 = \frac{m^4 L}{E_s^2} = \frac{m^6}{4 E_s^2 Z^2 n e^6 \ln(191 Z^{-1/3})}. \quad (10)$$

Since $(E_s/m)^2 = 1700$, it follows that $E_0 = m^2 L / 1700$. For lead this gives an energy of 5×10^{12} eV; with elements at the end of the periodic table E_0 19a*

diminishes to 2×10^{12} eV. When E is greater than E_0 , radiation of almost all quanta infringes the Bethe-Heitler formulae. The same is also true with reference to the formation of pairs. For lower energies very large deviations from the Bethe-Heitler formulae would be expected for small values of ω , corresponding to the inequality:

$$\omega \lesssim E^2/E_0. \quad (11)$$

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76. ELECTRON-CASCADE PROCESSES AT ULTRA-HIGH ENERGIES

It has been pointed out¹ that the Bethe-Heitler (B-H) theory of Bremsstrahlung of electrons and positrons as well as their theory of electron-positron pair production by γ -quanta is not applicable at sufficiently high energies because of multiple scattering of electrons and positrons in matter. In ref. 1 only the order of magnitude of the energies at which noticeable deviations from B-H formulae occur was estimated. Now we shall consider radiative processes under conditions where the B-H theory is not justified.

First, let us consider the radiation of low frequency quanta ($\omega \ll E$). The energy radiated into the solid angle element dn and in the frequency interval $d\omega$ is given by formula (1) of ref. 1.

In order to get more accurate expressions we shall integrate (1) of ref. 1 over all directions n of the quanta. The energy dI lost in the interval $d\omega$ is equal to:

$$\begin{aligned} dI &= \frac{e^2}{4\pi^2} \omega^2 d\omega \iiint ([n \wedge dr_1] \cdot [n \wedge dr_2]) e^{i\omega(t_1 - t_2) - i(n \cdot r_1 - r_2)\omega} dn \\ &= \frac{e^2}{4\pi^2} \omega^2 d\omega \iiint e^{i\omega(t_1 - t_2)} \{ (dr_1 \cdot dr_2) - (n \cdot dr_1)(n \cdot dr_2) \} e^{-i(n \cdot r_1 - r_2)\omega} dn \\ &= \frac{e^2}{\pi} \omega^2 d\omega \iint \{ (dr_1 \cdot dr_2) + (dr_1 \cdot \nabla_g)(dr_2 \cdot \nabla_g) \frac{\sin g}{g} e^{i\omega(t_1 - t_2)} \}, \end{aligned} \quad (1)$$

$$g = \omega(r_1 - r_2).$$

Because the effective values for g are large compared with unity in (1) we can neglect the differentiation of $1/g$:

$$dI = \frac{e^2 \omega d\omega}{\pi} \int_{-\infty}^{+\infty} dt_1 dt_2 \frac{e^{i\omega(t_1 - t_2)}}{|r_1 - r_2|} \left[(V_1 \cdot V_2) - \frac{(V_1 \cdot r_1 - r_2)(V_2 \cdot r_1 - r_2)}{(r_1 - r_2)^2} \right] \sin g, \quad (2)$$

where

$$r_1 - r_2 = \int_{t_2}^{t_1} V(\tau) d\tau = \int_0^{t_1 - t_2} V(t_2 + \tau) d\tau \quad (3)$$

$V(t_2 + \tau)$ is expressed in the following form:

$$V(t_2 + \tau) = V_2 \left(1 - \frac{\theta_\tau^2}{2} \right) + \theta_\tau, \quad (V_2 \cdot \theta_\tau) = 0, \quad (4)$$

θ_τ is the angle of multiple scattering at time τ .

(3) and (4) give:

$$r_1 - r_2 = V_2(t_1 - t_2) - \frac{1}{2} V_2 \int_0^{t_1 - t_2} \theta_\tau^2 d\tau + \int_0^{t_1 - t_2} \theta_\tau d\tau.$$

Using the smallness of θ and introducing the variables $T = t_2$, $t_1 - t_2 = t$, we can bring (2) into the form

$$dI = \frac{e^2 \omega d\omega}{\pi} \int_{-\infty}^{+\infty} dT \int_{-\infty}^{+\infty} \frac{dt}{t} e^{i\omega t} \times \overline{\left[\frac{\left(\int_0^t \theta_\tau d\tau \right)^2}{t^2} - \frac{\left(\theta_t \cdot \int_0^t \theta_\tau d\tau \right)}{t} \right]} \sin \omega \left[Vt - \frac{\int_0^t \theta_\tau^2 d\tau}{2} + \frac{\left(\int_0^t \theta_\tau d\tau \right)^2}{2t} \right]. \quad (5)$$

The bar signifies averaging overall angles θ_τ . If θ_τ is neglected under the sine symbol, then from (5), using multiple scattering theory², we get the usual B-H formula in the region $\omega \ll E$.

The energy lost per unit time in the interval $d\omega$ is equal to

$$dI = \frac{e^2 \omega d\omega}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{t} e^{i\omega t} \times \overline{\left[\frac{\left(\int_0^t \theta_\tau d\tau \right)^2}{t^2} - \frac{\left(\theta_t \cdot \int_0^t \theta_\tau d\tau \right)}{t} \right]} \sin \omega \left[Vt - \frac{\int_0^t \theta_\tau^2 d\tau}{2} + \frac{\left(\int_0^t \theta_\tau d\tau \right)^2}{2t} \right]. \quad (6)$$

To perform exactly the averaging in (6) is difficult in view of the fact that the averaged quantity θ_τ appears under the sine symbol. Because of this, for an order of magnitude estimate, we shall replace all terms in (6), separately by their mean values, using²

$$\begin{aligned} \overline{\int_0^t \theta_\tau d\tau} &= \frac{E_s^2 t |t|}{2E^2 L}; \quad \overline{\left(\theta_t \cdot \int_0^t \theta_\tau d\tau \right)} = \int_0^t (\theta_\tau + \varphi, \theta_\tau) d\tau = \int_0^t \theta_\tau^2 d\tau = \frac{E_s^2 t |t|}{2E^2 L}; \\ \overline{\left(\int_0^t \theta_\tau d\tau \right)^2} &= 2 \int_0^t d\tau \int_0^t \overline{(\theta_\tau \cdot \theta_\nu)} d\nu = 2 \int_0^t \overline{\theta_\nu^2} (t - \nu) d\nu = \frac{E_s^2 |t|^3}{3E^2 L}. \end{aligned} \quad (7)$$

Using (7) we have

$$\begin{aligned} dI &\approx -\frac{e^2}{3\pi E^2} \frac{\omega d\omega}{L} E_s^2 \int_0^t dt \cos \omega t \sin \omega V \left(t - \frac{E_s^2 t^2}{12 E^2 L} \right) \\ &\approx \frac{e^2 \omega d\omega}{6\pi E^2 L} E_s^2 \int_0^\infty dt \sin \left[(1-V) \omega t + \frac{E_s^2 t^2 \omega}{12 E^2 L} \right] = \frac{4 d\omega}{3 L} \int_0^\infty dX \sin \left(X + \frac{E^2 X^2}{3 E_0 \omega} \right), \\ E_0 &= \frac{m^4 L}{E_s^2}. \end{aligned} \quad (8)$$

Now let us consider the case, when $\omega \ll E^2/E_0$. The integral, appearing in (8), is equal to $(1/2)\sqrt{\pi/2}\sqrt{3E_0\omega/E^2}$, then dI proves to be of the order:

$$dI \approx \sqrt{\frac{2\pi}{3}} \frac{d\omega}{L} \sqrt{\frac{\omega E_0}{E^2}}. \quad (9)$$

From this the number of quanta emitted per unit time in the frequency interval $d\omega$ is determined by:

$$dN = \sqrt{\frac{2\pi}{3}} \frac{d\omega}{\sqrt{\omega}} \sqrt{\frac{E_0}{E^2 L^2}}. \quad (10)$$

Since the condition $\omega \ll E^2/E_0$ is always going to be satisfied for quanta of sufficiently low energy, it follows from (10) that the infrared catastrophe never takes place in bremsstrahlung, for when $\omega \rightarrow 0$ the spectrum changes from $d\omega/\omega$ into $d\omega/\sqrt{\omega}$. By this very fact the total number of radiated quanta is shown to be finite.

Going over to the case when $\omega \sim E$ it is necessary to use a quantum treatment. An exact calculation is associated with calculational difficulties. Because of this we shall limit ourselves only to estimates as to orders of magnitude.

In the matrix element which determines the radiation the following expression enters as an essential factor:

$$e^{i(p-p'-K \cdot r)}, \quad (11)$$

where p is the momentum of the electron before, p' the momentum after radiation, K is the momentum of the quantum. Taking into account the smallness of the angles between K , p , p' , let us consider first the case where they are all parallel. The difference $p - p' - K$ is equal to $m^2 K/2EE'$. Because of this r_{eff} is of the order of

$$EE'/m^2 K; \quad (12)$$

p , p' are changed because of multiple scattering. The components of p , p' along the direction K at a distance r_{eff} are changed by an amount: $E(E_s^2/E^2)(r_{\text{eff}}/L)$

$E' (E_s^2/E'^2) (r_{\text{eff}}/L)$, respectively. Taking this into account (11) assumes the form

$$\exp \left[i \frac{m^2 r}{2 E E'} K + i \frac{r^2 E_s^2 K}{4 E E' L} \right] \quad (13)$$

r_{eff} equals:

$$\sqrt{\frac{L E E'}{E_s^2 K}} = \frac{E E'}{m^2 K} \sqrt{\frac{E_0 K}{E E'}}, \quad (14)$$

if $E_0 K / E E' \ll 1$, and equals (12) if $E_0 K / E E' \gg 1$. In that manner if $E' < E_0$ and $K \approx E$, then the corresponding process of radiation will be in accordance with B-H. In the opposite case the matrix element would be smaller than in the B-H theory:

$$M = M_{\text{B-H}} \sqrt{\frac{E_0 K}{E E'}}. \quad (15)$$

The probability of the emission of a quantum in a given direction has the form:

$$W = W_{\text{B-H}} \frac{E_0 K}{E E'}. \quad (16)$$

However, the solid angle within which quanta are emitted increases. If the angle between K and p is equal to ϑ then in (11) the factor $e^{i K r \vartheta^2/2}$ appears. According to (14) this factor will not diminish the radiation until ϑ^2 is of the order

$$\vartheta^2 \sim \frac{m^2}{E E'} \sqrt{\frac{E E'}{K E_0}} = \vartheta_{\text{B-H}}^2 \sqrt{\frac{E E'}{K E_0}} \quad (17)$$

(The expression for $\vartheta_{\text{B-H}}^2$ is easily derived from equation (12).)

Therefore the probability of radiation, which is integrated over all directions of quanta, has the form:

$$A dK = dK A_{\text{B-H}} \sqrt{\frac{E_0 K}{E E'}} \sim \frac{dK}{L} \sqrt{\frac{E_0}{E K E'}}. \quad (18)$$

The differential number of quanta has its minimum at $K = E/2$. When $E - K \sim E_0$, A goes over into the B-H expression.

The total number of radiated quanta per unit time is finite and equals

$$\int_0^E A dK = a_e \frac{1}{L} \sqrt{\frac{E_0}{E}}, \quad a_e \sim 1. \quad (19)$$

The energy I , radiated per unit time is:

$$I = \int_0^E K A dK = \frac{b}{L} \sqrt{E E_0} = \frac{E}{L(E)}, \quad b \sim 1. \quad (20)$$

From (20) we conclude that the distance through which an energy of order E is emitted, increases with increasing energy proportionally to \sqrt{E} . Losses in energy per unit length are proportional not to E , as in B-H, but to \sqrt{E} . The penetrability of electrons and positrons increases when $E > E_0$. The probability of pair formation by γ -quanta according to (18) is equal to:

$$dA_\gamma = \frac{a_\gamma}{L} \sqrt{\frac{E_0}{E_+ E_- K}} dE_+; \quad a_\gamma \sim 1; \quad E_+, E_- > E_0. \quad (21)$$

If $E_-, E_+ \lesssim E_0$, then dA_γ becomes the B-H expression.

Integration of (21) gives the total probability of pair production A_γ per unit time:

$$A_\gamma = \frac{\pi a_\gamma}{L} \sqrt{\frac{E_0}{E}}. \quad (22)$$

The mean free path of quanta also increases proportionally to the square-root of their energy.

According to (20) and (22) when $E/E_0, K/E_0 \sim 300-500$, then the mean free path of the "soft component" in materials like lead becomes 10-15 cm and is comparable to the mean free path of high energy nucleons. At ultra-high energies the soft component takes on properties of the hard component.

Since the infrared "catastrophe" is absent the expressions for the radiation corrections of different processes (elastic scattering of electrons by nuclei, Compton effect, electron-electron scattering) should be changed. For these processes the existence of the infrared catastrophe would be important.

REFERENCES

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2. See, e.g. B. ROSSI and K. GREISEN, *Interaction of Cosmic Rays with Matter* (1948).

77. EMISSION OF γ -QUANTA DURING THE COLLISION OF FAST π -MESONS WITH NUCLEONS

For the determination of the bremsstrahlung spectrum of γ -rays occurring during the collision of high-energy π -particles with nucleons or nuclei, it is sufficient to know the wave function ψ of the π -meson outside the nucleon. Proceeding from the assumption that the nucleon is "black" in relation to π -mesons, the external ψ -function is found in the form of the sum of a plane wave and the wave diffracted from the black sphere. By means of this function, the determination is made of the emission cross-section of the γ -quanta originating during the diffraction of π -mesons and during their capture by the nucleons. At high energies of π -mesons, differential cross-sections in the area of small angles reach very high values. The effect of the size of π -mesons on the emission of γ -quanta is revealed. The form factor of the π -meson is introduced in the emission formulae. An experimental study of the bremsstrahlung γ -spectrum of π -mesons could make it possible to determine this form factor.

STRONG interactions between high-energy π -mesons and nucleons result in a great probability of such collisions between them when the π -particle and the nucleon are united into a highly excited short-lived system¹ which disintegrates further into several mesons and, probably, into a certain number of nucleon pairs. Such a progress of the collision of π -mesons with nucleons cannot be described by the methods of perturbation theory, in so far as it is a case of a very strong interaction. Under these conditions, the nucleons (or pairs) represent, in relation to the π -meson, a black body whose radius R determines the cross-section πR^2 . The wave function ψ of the π -meson outside R will, evidently, be a superposition of a plane wave and a wave diffracted from the "black" nucleon. It is found that, under relativistic conditions, it is sufficient to know ψ in the exterior region in order to calculate exactly the bremsstrahlung of γ -quanta of almost any energy if these quanta are flying in a direction which differs little from the direction of the incident π -particle. This situation resembles the one which occurs during the bremsstrahlung of electrons by Coulomb forces, when under extreme relativistic conditions the emission takes place at great distances from the nucleus. The physical cause of such phenomena is the same and lies in the fact that the conservation laws during emission require smaller and smaller momentum transfer from the radiating particles to the foreign body.

The theory which is set forth below may be applied not only to a "black" nucleon (or nucleus), but also to a "grey" one which only partially absorbs mesons. We shall first consider the first case which is a simpler one.

Л. Д. Ландау и И. Я. Померанчук, Излучение γ -квантов при столкновении быстрых π -мезонов с нуклонами, *Журнал Экспериментальной и Теоретической Физики*, 24, 505 (1953).

The wave equation for a particle with spin 0 (π -meson, ref. 2) outside the nucleon (nucleus), with the required exactness, has the following form ($\hbar = c = 1$):

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\psi = \frac{2e}{i} (\mathbf{A} \cdot \nabla) \psi,$$

$$A = \sum_{fj} \sqrt{\frac{2\pi}{f\Omega}} \mathbf{j} (a_{fj} e^{i[(f \cdot r) - \omega t]} + a_{fj}^+ e^{-i[(f \cdot r) - \omega t]}). \quad (1)$$

Let us apply perturbation theory to (1) considering that the right-hand side is small and substituting there ψ for ψ_0 which is equal to the sum of the initial plane wave and the outgoing diffracted wave:

$$\psi_0 = \Phi_0(r) e^{-iEt}.$$

Let us find a solution in the form of a series

$$\psi = \psi_0 + \sum_{p'} \Phi_{p'} e^{-iE't},$$

where

$$(-E'^2 - \nabla^2 + \mu^2) \Phi_{p'} = \frac{2e}{i} \sqrt{\frac{2\pi}{f\Omega}} \sum (\mathbf{j} \cdot \nabla) \Phi_0 e^{-i(f \cdot r)} a_{fj}^+ \quad (2)$$

(the sum is taken over the directions of f)

The component Φ , which corresponds to the emission of the given quanta, is equal to

$$\Phi_{p'} = \int G(r, r') \frac{2e}{i} \sqrt{\frac{2\pi}{f\Omega}} (\mathbf{j} \cdot \nabla) \Phi_0 e^{-i(f \cdot r')} d\tau', \quad (3)$$

where G is the Green function of equation (2),

$$p'^2 = (E - f)^2 - \mu^2. \quad (4)$$

The emission of γ -quanta may take place either during the absorption of the π -meson with a subsequent emergence of many particles, or during the diffraction scattering of π -mesons. Let us first consider the second case, in which at the end of the process there is the γ -quantum and the scattered π -meson, and the nucleon (nucleus) receives a small recoil.

The function G , which is included in (3), should account for the diffraction near the nucleon, and, therefore, it is equal to³:

$$G(r, r') = \frac{p'}{2\pi i} \int \frac{e^{ip'|r'-s|}}{|r'-s|} \frac{e^{ip'|r-s|}}{4\pi|r-s|} d^3s. \quad (5)$$

Integration over s may be done, for example, over the part of the plane perpendicular to vector $r - r'$ and passing through the nucleon which is "outside" the nucleon (Fig. 1). Complementing the s -integration range with the interior part of the nucleon, we shall obtain

$$G(r, r') = \frac{e^{ip'|r-r'|}}{4\pi|r-r'|} + G_1, \quad (6)$$

where G_1 has the form of (5) differing only in the fact that the integration range lies inside the nucleon.

Let us substitute the first term of (6) into (3):

$$\frac{e}{\sqrt{2\pi f \Omega i}} \int \frac{e^{i\mathbf{p}' \cdot (\mathbf{r} - \mathbf{r}')}}{|\mathbf{r} - \mathbf{r}'|} (\mathbf{j} \cdot \nabla \Phi_0) e^{-i(\mathbf{f} \cdot \mathbf{r}')} d\tau'. \quad (7)$$

In order to find the diffraction emission it is necessary to determine the flux of diffracted particles which are accompanied by γ -quanta. For this it is necessary to find Φ at $r \rightarrow \infty$. Then (7) is simplified ($\mathbf{p}' = \mathbf{p}'(r/r)$):

$$\frac{e e^{i\mathbf{p}' \cdot \mathbf{r}}}{\sqrt{2\pi f \Omega i} r} \int e^{-i(\mathbf{p} \cdot \mathbf{r}')} (\mathbf{j} \cdot \nabla) \Phi_0 e^{-i(\mathbf{f} \cdot \mathbf{r}')} d\tau' = \frac{(\mathbf{j} \cdot \mathbf{p}') e e^{i(\mathbf{p}' \cdot \mathbf{r})}}{\sqrt{2\pi f \Omega} r} \int \Phi_0 e^{-i(\mathbf{f} + \mathbf{p}' \cdot \mathbf{r}')} d\tau'. \quad (8)$$

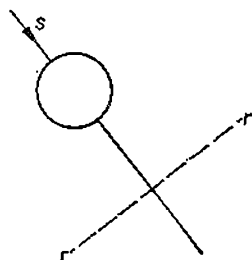


FIG. 1.

Instead of Φ_0 , let us substitute³:

$$\Phi_0 = e^{i(\mathbf{p} \cdot \mathbf{r})} - \frac{p}{2\pi i} \int \frac{e^{i\mathbf{p} \cdot (\mathbf{r} - \mathbf{q})}}{|\mathbf{r} - \mathbf{q}|} d^3 q. \quad (9)$$

The second term represents the diffraction in the initial state of π -meson (outgoing wave). The vector \mathbf{q} , which is perpendicular to \mathbf{p} , lies in the plane passing through the nucleon and satisfies the condition $q^2 \leq R^2$.

The first term in (9) gives zero due to the conservation laws. Instead of (8) we have:

$$\begin{aligned} & - \frac{e e^{i\mathbf{p}' \cdot \mathbf{r}}}{\sqrt{2\pi f \Omega} r} (\mathbf{j} \cdot \mathbf{p}') \frac{p}{2\pi i} \iint \frac{e^{i\mathbf{p} \cdot (\mathbf{r}' - \mathbf{q})}}{|\mathbf{r} - \mathbf{q}|} d^3 q e^{-i(\mathbf{f} + \mathbf{p}' \cdot \mathbf{r}')} d\tau' \\ & \approx \frac{e e^{i\mathbf{p}' \cdot \mathbf{r}}}{\sqrt{2\pi f \Omega} r} (\mathbf{j} \cdot \mathbf{p}') \frac{1}{p - |\mathbf{p}' + \mathbf{f}|} \int e^{-i(\mathbf{f} + \mathbf{p}' \cdot \mathbf{q})} d^3 q \\ & = e \sqrt{\frac{2\pi}{f \Omega}} \frac{e^{i\mathbf{p}' \cdot \mathbf{r}}}{r} \frac{(\mathbf{j} \cdot \mathbf{p}')}{p - |\mathbf{p}' + \mathbf{f}|} R \frac{J_1(|\mathbf{f} \mathbf{\theta} + \mathbf{p}' \mathbf{\theta}'| R)}{|\mathbf{f} \mathbf{\theta} + \mathbf{p}' \mathbf{\theta}'|}. \quad (10) \end{aligned}$$

In calculating (10) we made use of the condition of the smallness of the angles θ between f and p and of θ' between p' and p . θ and θ' are two-dimensional angle vectors:

$$p' = \frac{(p' \cdot p)}{p^2} p + p' \theta'; \quad f = \frac{(f \cdot p)}{p^2} p + f \theta \quad (11)$$

Let us now calculate the term containing G_1 :

$$2e \sqrt{\frac{2\pi}{f\Omega}} \frac{1}{i} \int G_1(r, r') (j \cdot \nabla \Phi_0) e^{-i(f \cdot r')} d\tau'$$

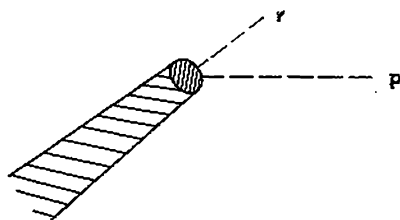


FIG. 2.

G_1 , which is considered as a function of r' , is different from zero only in the area of the geometrical shadow in relation to the direction r , i.e. in the hatched area of Fig. 2. Since we are interested in points r , which lie within the limits of the narrow diffraction cone, these r' lie where the incident wave has not yet reached the nucleus and, therefore, is not yet distorted by diffraction. Therefore, Φ_0 coincides here with the plane wave $e^{i(p \cdot r')}$. Making use of (6) and (5), we have at $r \rightarrow \infty$,

$$\begin{aligned} 2e \sqrt{\frac{2\pi}{f\Omega}} (j \cdot p) \int G_1(r, r') e^{i(p \cdot r')} d\tau' \\ = - \frac{e}{\sqrt{8\pi^3 f\Omega}} \frac{(j \cdot p)}{r} e^{ip'r} p' \iint \frac{e^{ip'|\tau' - s'| + i(p \cdot r')}}{|r' - s'|} d^3 s' d\tau' \\ = e \frac{1}{\sqrt{2\pi f\Omega}} \frac{e^{ip'r}}{r} \frac{(j \cdot p)}{p' - |p - f|} \int e^{i(p \cdot s')} d^3 s' \quad (12) \end{aligned}$$

s' is perpendicular to p' and its magnitude is less than or equal to R' , where R'^2 is the interaction cross-section of the diffracted π -particle with the nucleon (R' need not coincide with R).

Equation (12) contains the Fourier component of the diffracted wave in the final state:

$$e \frac{(j \cdot p)}{r} e^{ip'r} \sqrt{\frac{2\pi}{f\Omega}} \frac{1}{p' - |p - f|} R' \frac{J_1(|p\theta' - f(\theta' - \theta)| R')}{|p\theta' - f(\theta' - \theta)|} \quad (13)$$

Adding (10) and (13), we find $\Phi_{p'}$ ($r \rightarrow \infty$)

$$\frac{e^{ip'r}}{r} e \sqrt{\frac{2\pi}{f\Omega}} \left\{ \frac{R J_1(|p'\theta' + f\theta| R)}{|p'\theta' + f\theta|} \frac{(j \cdot p')}{p - |p' + f|} + \frac{R' J_1(|p\theta' - f(\theta' - \theta)| R')}{|p\theta' - f(\theta' - \theta)|} \frac{(j \cdot p)}{p' - |p - f|} \right\}. \quad (14)$$

The flux of the diffracted particles per unit of solid angle is equal to:

$$e^2 \frac{2\pi}{f\Omega} \{ \}^2 2p'$$

(Here, the factor $2p'$ appears in calculating the flux of particles of spin 0). Dividing it by the flux density of the incident particles $2p$ and multiplying by the number of the quantum states, we obtain the differential cross-section of the emission of a quantum with frequency f , accompanied by π -particles:

$$\sigma_0^d(j) df d\theta d\theta' = \frac{E'}{E} \frac{e^2}{4\pi^2} f \left| \frac{R J_1(|p'\theta' + f\theta| R)}{|p'\theta' + f\theta|} \frac{(j \cdot p')}{p - |p' + f|} + \frac{R' J_1(|p\theta' - f(\theta' - \theta)| R')}{|p\theta' - f(\theta' - \theta)|} \frac{(j \cdot p)}{p' - |p - f|} \right|^2 df d\theta d\theta'. \quad (15)$$

Let us note that (15) can also be obtained by examining the radiative transitions from the initial state taken in the form of a plane and outgoing diffracted wave into the final state described by a plane and incoming⁴ diffracted wave (which has a shadow in front).

In calculating (10) and (13), the most important were distances from the nucleon of the order of

$$r_{\text{eff}} \sim \frac{1}{p - |p' + f|} = \frac{1}{p - p' - f + (p'f\vartheta^2/2p)}, \quad \vartheta = \theta - \theta'.$$

Let us make use of the energy conservation law to transform $p - p' - f$:

$$\begin{aligned} \sqrt{E^2 - \mu^2} - \sqrt{(E - f)^2 - \mu^2} - f &= E - \frac{\mu^2}{2E} - (E - f) + \frac{\mu^2}{2(E - f)} - f \\ &= \frac{\mu^2 f}{2(E - f)E} \approx \frac{\mu^2 f}{2p'p}, \end{aligned} \quad (16)$$

$$r_{\text{eff}} \sim 2p'p/f(\mu^2 + p'^2\vartheta^2).$$

It follows from (16) that, for $E \gg \mu$ and $\vartheta \leq \mu/E$, $r_{\text{eff}} \gg E/\mu^2 \gg R$, which justifies our disregarding the area inside the "nucleon" under these conditions. Let us note that the nucleon is "black" only for $E \gg \mu$.

The expression for the emission probability (15) was obtained on the basis of the wave equation (1), in which the interaction of the π -particle with the electromagnetic field is considered as if the π -particle were a point particle. However, a strong interaction of the π -particle with the nucleonic background should lead to a "smearing out" of the π -particle charge in a certain area of the order of $1/\mu$ in size or smaller. If condition (16) is fulfilled, the entire emission takes place at a great distance from the nucleus. Therefore, the effect of the electrical dimensions of the π -particle on the emission of quanta should lead, under certain conditions, to the appearance in (1) and in all subsequent formulas of the "form factor" of the π -particle F . It is a function from the invariant quantum frequency in the system of rest of the π -particle;

$$F = F\left(\frac{Ef - (\mathbf{p} \cdot \mathbf{f})}{\mu^2}\right). \quad (17)$$

If $E \gg \mu$, $\theta \ll 1$, then

$$F = F\left(\frac{f}{2E} + \frac{Ef\theta^2}{2\mu^2}\right). \quad (18)$$

When the argument of F is small, $F = 1$. At large values of the argument, $F \rightarrow 0$. The following three facts should be stressed when F is introduced.

(1) Emission does not depend on the details which characterise the collision of the π -meson with the nucleus. It is only essential to know the ψ -function at a great distance from the nucleus; therefore, the nucleus cannot seriously affect the properties of the nucleonic vacuum which surrounds the π -particle, and, consequently, the emission shows the structure of an unperturbed π -particle.

(2) In the case of diffraction radiation, it is necessary for the invariant frequencies of the quantum in the rest system of the incident and the diffracted mesons to differ little from each other. This results in the following condition:

$$\frac{Ef - (\mathbf{p} \cdot \mathbf{f})}{\mu^2} - \frac{E'f - (\mathbf{p}' \cdot \mathbf{f})}{\mu^2} = \frac{f^2 - (\mathbf{p} - \mathbf{p}') \cdot \mathbf{f}}{\mu^2} \ll 1.$$

When θ and θ' are small, we have:

$$\left| \frac{\theta^2 E - (\theta - \theta')^2 E' - (\mu^2 f / E E')}{2\mu^2} f \right| \ll 1. \quad (19)$$

(3) Even when (19) is fulfilled, it is apparently, necessary for the π -mesons not to suffer very high accelerations in the process of diffraction; otherwise these accelerations may effect the "form" of the particle. Mathematically, this means the dependence of F on the invariant $p \cdot p' - \mu^2 = E E' - (\mathbf{p} \cdot \mathbf{p}') - \mu^2$, at small values of which F is only the function of the invariant frequency. Thus, the introduction of the form factor for the π -particle which suffers low accelerations imposes the following condition:

$$\frac{f^2}{2E E'} + \frac{E E' \theta'^2}{2\mu^2} \ll 1. \quad (20)$$

It follows from this and from (19) that:

$$f \ll E, \quad \theta' < \mu/2E, \quad \theta \ll \mu^2/Ef\theta'. \quad (20a)$$

The form factor here appears to be dependent on $f\theta^2$

$$F = F(Ef\theta^2/2\mu^2). \quad (20b)$$

When (19) is satisfied, but (20) not, then a more complicated form factor appears (in relation to the emission process under which the π -particle suffers the given acceleration).

If (19) is not fulfilled, then the simple concept of form factor loses its meaning. Instead of it, a more complex value will appear in the formulae for the radiation intensity which is connected with two invariant frequencies of the quantum and which reflects the effect of acceleration, i.e. contains the invariant (20).

Thus, when conditions (19) and (20) are fulfilled, we obtain the following instead of (15):

$$\begin{aligned} \sigma_0^2(j) df d\theta d\theta' = & \frac{E'}{E} \frac{e^2}{4\pi} f \left| F \left(\frac{E\theta^2}{2\mu^2} f \right) \right|^2 \frac{R J_1(|p'\theta' + f\theta| R)}{|p'\theta' + f\theta|} \frac{(j \cdot p')}{p - |p' + f|} \\ & + \frac{R' J_1(|p\theta' - f(\theta' - \theta)| R')}{|p\theta' - f(\theta' - \theta)|} \frac{(j \cdot p)}{p' - |p - f|} \Big|^2 df d\theta d\theta'. \quad (21) \end{aligned}$$

Since $p - f \approx p'$, the argument of the Bessel function contained in the second member of (21) is equal to $|p'\theta' + f\theta| R'$ and at $R' = R$ coincides with the argument of the Bessel function contained in the first member. Therefore, if $R = R'$, a simpler expression for $\sigma_0^2(j)$ will be obtained:

$$\sigma_0^2(j) = \frac{E'}{E} \frac{e^2}{4\pi^2} f \frac{R^2 J^2(|p'\theta' + f\theta| R)}{|p'\theta' + f\theta|^2} \left| \frac{(j \cdot p')}{p - |p' + f|} + \frac{(j \cdot p)}{p' - |p - f|} \right|^2 |F|^2. \quad (22)$$

The most important role is played by small θ and θ' , at which (22) becomes substantially simplified

$$\begin{aligned} \sigma_0^2(j) = & \frac{E'}{E} \frac{e^2}{4\pi^2} + \frac{R^2 J^2(|E'\theta' + f\theta| R)}{|E'\theta' + f\theta|^2} \left| \frac{(j \cdot p')}{(\mu^2/E'^2) + (\theta - \theta')^2} \frac{E}{E'} \right. \\ & \left. - \frac{(j \cdot p)}{(\mu^2/E^2) + \theta^2} \frac{E'}{E} \right|^2 |F|^2. \quad (23) \end{aligned}$$

The condition $E, E' \gg \mu$ is also used here. However, if $R \neq R'$, then, at small θ and θ' , (21) gives:

$$\begin{aligned} \sigma_0^2(j) = & \frac{E'}{E} \frac{e^2}{\pi^2} \frac{|F|^2}{f} \left| \frac{(j \cdot p')}{(\mu^2/E^2) + (\theta - \theta')^2} \frac{E}{E'} R \frac{J_1(|E'\theta' + f\theta| R)}{|E'\theta' + f\theta|} \right. \\ & \left. - \frac{(j \cdot p)}{(\mu^2/E^2) + \theta^2} \frac{E'}{E} R' \frac{J_1(|E'\theta' + f\theta| R')}{|E'\theta' + f\theta|} \right|^2. \quad (24) \end{aligned}$$

Summing in (23) over the polarisation, we obtain:

$$\sigma_0^d = \frac{e^2}{\pi^2} \frac{1}{f} \frac{E'}{E} R^2 \frac{J_1^2(|E'\theta' + f\theta|R)}{|E'\theta' + f\theta|^2} \times$$

$$\left\{ \frac{|\theta \cdot \theta'|^2}{(\mu^2/E'^2) + (\theta - \theta')^2 \theta^2} E_2^2 + \left| \frac{E(\theta \cdot \theta' - \theta)}{\mu^2/E'^2 + (\theta - \theta')^2} \frac{1}{\theta} + \frac{E'\theta}{(\mu^2/E^2) + \theta^2} \right|^2 \right\} |F|^2. \quad (25)$$

The first term of (25) resulted from the polarisation of $j_1 = [p \wedge f]/E f \theta$, and the second from $j_2 = [f \wedge [p \wedge f]]/E f^2 \theta$. Formula (25) may be re-written in the following way:

$$\sigma_0^d = \frac{e^2}{\pi^2} \frac{E'}{E} \frac{1}{f} |F|^2 R^2 \frac{J_1^2(|E'\theta' + f\theta|R)}{|E'\theta' + f\theta|^2} \left\{ \frac{E'^2 \theta^2}{(\mu^2/E^2) + \theta^2} + \right.$$

$$\left. \frac{E^2 (\theta - \theta')^2}{(\mu^2/E'^2) + (\theta - \theta')^2} - 2 \frac{E E' (\theta \cdot \theta - \theta')}{[(\mu^2/E^2) + \theta^2][\mu/E'^2 + (\theta - \theta')^2]} \right\}. \quad (26)$$

In order to obtain the spectral composition of the radiation, it is necessary to integrate (26) over all θ and θ' . In a general form it leads to complicated expressions.

Let us first consider the collision of π -particle with heavy nuclei, when R may be considered large in comparison with $1/\mu$, which results in substantial simplifications. If θ and θ' are small in comparison with μ/E , μ/E' , then the value within the braces in (26) is equal to:

$$\{ \} = \frac{E'^2 E^2}{\mu^4} |f\theta + E'\theta'|^2. \quad (27)$$

Thus, when $\theta \ll \mu/E$, $|\theta - \theta'| \ll \mu/E'$, σ_0^d is equal to:

$$\sigma_0^d = \frac{e^2}{\pi^2} \frac{E'}{E} \frac{R^2}{f} J_1^2(|E'\theta' + f\theta|R) \frac{E'^2 E^2}{\mu^4} |F|^2 \quad (28)$$

If $R \gg 1/\mu$, then the comparison of (26) and (28) shows that in the integral over θ and θ' the main role is played by such θ and θ' for which J_1^2 can be replaced by its mean value

$$J_1^2(|E'\theta' + f\theta|R) \approx \frac{1}{\pi} \frac{1}{|E'\theta' + f\theta|R}.$$

This mean value is obtained when the quickly oscillating factor $\cos^2(|E'\theta' + f\theta|R - 3\pi/4)$ is replaced by $\frac{1}{2}$. In this case

$$\sigma_0^d = \frac{e^2}{\pi^3} \frac{E'}{E} \frac{R}{f} \frac{|F|^2}{|E'\theta' + f\theta|^3} \left\{ \frac{E'^2 \theta^2}{\left[\frac{\mu^2}{E^2} + \theta^2 \right]^2} - 2 \frac{EE'(\theta \cdot \theta - \theta')}{\left(\frac{\mu^2}{E^2} + \theta^2 \right) \left[\frac{\mu^2}{E'^2} + (\theta - \theta')^2 \right]} + \frac{E^2(\theta - \theta')^2}{\left[\frac{\mu^2}{E'^2} + (\theta - \theta')^2 \right]^2} \right\}. \quad (29)$$

Let us introduce variables s and η instead of θ and θ' :

$$\theta = \frac{\mu}{E} s = \frac{1}{\gamma} s; \quad \theta - \theta' = \frac{\mu}{E'} \eta = \frac{1}{\gamma'} \eta, \quad (30)$$

in which case (29) will assume the following form:

$$\sigma_0^d \theta d\theta' = \frac{e^2}{\pi^3} \frac{E'}{E} \frac{R}{\mu} \frac{1}{f} \frac{1}{|s + \eta|^3} \left(\frac{s}{1 + s^2} + \frac{\eta}{1 + \eta^2} \right)^2 \left| F \left(\frac{f}{2E} [s^2 + 1] \right) \right|^2 ds d\eta. \quad (31)$$

In integrating over s and η , the main role is played by s and $\eta \sim 1$. In this case, $|E'\theta + f\theta|R \sim \mu R \gg 1$, which justifies the use of an asymptotic expansion of J_1 . Moreover, according to (30) and (16), the entire radiation is determined by the area outside of the nucleus. Integrating over s and η , we determine the cross section of the emission of the quanta of the given frequency in the case when F may be assumed to be equal to one:

$$\sigma^d(f) = \frac{4e^2}{\pi} \frac{E'}{E} \frac{1}{f} \frac{R}{\mu} \int_0^\infty \frac{dx}{\sinh x} \left[\coth x - \frac{2x}{\sinh^2 x} \right] \approx 2.3 e^2 \frac{E'}{E} \frac{1}{f} \frac{R}{\mu}. \quad (32)$$

Thus, the effective cross-section of the emission of the hard γ -rays caused by the diffraction of π -particles is proportional to the radius of the nucleus R , i.e. it grows slowly with the increase of the atomic number ($\sim A^{1/3}$). According to (32) the emitted spectrum has a simple dependence on the frequency. The angular distribution of the quanta is given by formula (29).

Let us now determine the total energy radiated during the collision, assuming that $F = 1$. As to order of magnitude it is equal to

$$\int df \sigma^d f \sim 2.3 e^2 \frac{R}{\mu} \int_0^E \frac{E'}{E} df = 1.5 e^2 \frac{R}{\mu} E. \quad (33)$$

The total emission cross-section for quanta of frequency larger than f is equal to ($F = 1$):

$$\int_f^E \sigma^d(f) df = 2.3 e^2 \frac{R}{\mu} \left[\ln \frac{E}{ef} + \frac{f}{E} \right], \quad \mu R \gg 1. \quad (34)$$

If $A \sim 1$, then $R \leq 1/\mu$ and in the general case the integration of (26) over the angles is difficult.

In the variables s and η , (26) has the form

$$\sigma_0^d d\theta d\theta' = \frac{e^2}{\pi^2} \frac{E'}{E} \frac{R^2}{f} \frac{1}{|s + \eta|^2} J_1^2(\mu R |s + \eta|) \left(\frac{s}{1 + s^2} + \frac{\eta}{1 + \eta^2} \right)^2 \left| F\left(\frac{f}{2E} s^2\right) \right|^2 ds d\eta. \quad (35)$$

When $f\theta R \ll 1$, then (26) is simplified

$$\sigma_0^d = \frac{e^2}{\pi^2 f} \frac{E'}{E} R^2 \frac{J_1^2(E' \theta' R)}{\theta'^2} \left\{ \frac{\theta^2}{(\gamma^{-2} + \theta^2)^2} - 2 \frac{(\theta \cdot \theta - \theta') E}{(\gamma^{-2} + \theta^2) [\gamma'^{-2} + (\theta - \theta')^2] E'} + \frac{(\theta - \theta')^2 E^2}{E'^2 [\gamma^{-2} + (\theta - \theta')^2]^2} \right\}.$$

This expression coincides for small f ($f \ll E$) with the classical radiation arising from a "sudden" change in the momentum of the π -particles as a result of diffraction. Under these conditions a general connection between the cross sections of the elastic and the inelastic collisions is preserved⁵. Integrating (36) over all directions θ of the photon and assuming that $F = 1$, we shall find the total cross-section of the emission of quanta with frequency f of any directions at the given angle of diffraction θ' , when $f \ll E$:

$$\int \sigma_0^d d\theta = \frac{2}{\pi} \frac{e^2}{f} \frac{R J_1^2(E R \theta')}{\theta'^2} \left[\frac{1 + (\gamma^2 \theta'^2/2)}{(\gamma \theta'/2) [1 + \frac{1}{4} \gamma^2 \theta'^2]^{1/2}} \ln \left(\frac{\gamma \theta'}{2} + \left[1 + \frac{1}{4} \gamma^2 \theta'^2 \right]^{1/2} \right) - 1 \right]. \quad (37)$$

Integrating (36) over all angles of diffraction for a given direction θ of the quantum, we determine the cross-section of the emission of the quanta of frequency f which are accompanied by arbitrary diffracted π -particles:

$$\int \sigma_0^d d\theta' = \frac{2}{\pi} \frac{e^2}{f} R^2 \gamma^2 \int_0^\infty \frac{J_1^2(\mu R b)}{b} db \left\{ \frac{2 + b^2}{1 + s^2} \frac{1}{[(1 + s^2 + b^2)^2 - 4s^2 b^2]^{1/2}} - \frac{1 + s^2 + b^2}{[(1 + s^2 + b^2)^2 - 4s^2 b^2]^{3/2}} - \frac{1}{(1 + s^2)^2} \right\}; \quad s = (E/\mu) \theta. \quad (38)$$

Let us integrate (37) over all angles of diffraction θ' . We shall obtain the total cross section of the emission of photons of frequency f ($f \ll E$)

$$\sigma^d(f) = \frac{4}{f} e^2 R^2 \int_0^\infty \frac{J_1^2(2\mu R b)}{b} db \left[\frac{1 + 2b^2}{b \sqrt{1 + b^2}} \ln(b + \sqrt{1 + b^2}) - 1 \right]. \quad (39)$$

The main role is played here by $b \sim 1$, i.e. diffraction angles of the order of $1/\gamma$. If $2\mu R \gg 1$, then (32) follows from (39).

When $f \sim E$, then formula (35) ($F = 1$), when integrating over the angles θ and θ' , gives the following expression for the total cross section for the emission of the quantum of frequency f :

$$\begin{aligned}\sigma^d(f) &= 4 \frac{E'}{E} \frac{R^2 e^2}{f} \int_0^\infty \frac{J_1^2(2\mu Rq)}{q^2} dq \left[\frac{2q^2 + 1}{q\sqrt{q^2 + 1}} \ln(q + \sqrt{q^2 + 1}) - 1 \right] \\ &= 4 \frac{E'}{E} \frac{e^2}{f} R^2 \mathcal{N}(\mu R).\end{aligned}\quad (40)$$

The total loss of energy connected with diffraction radiation, according to (40), is equal to:

$$\int f \sigma^d(f) df = 2ER^2 e^2 \mathcal{N}(\mu R).\quad (41)$$

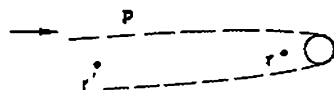


FIG. 3.

The integral cross-section of the emission of quanta having a frequency greater than f is equal to:

$$\int_0^E \sigma^d df = 4e^2 R^2 \left(\ln \frac{E}{ef} + \frac{f}{E} \right) \mathcal{N}(\mu R).\quad (42)$$

If $\mu R \gg 1$, then the expressions (40), (41), and (42) change into (32), (33), and (34).

Let us now turn to the radiation which accompanies the capture of π -particles by the nucleon (nucleus). It may be called "stop" radiation. In order to determine the probability of this process, it is necessary to find the flux of π -particles per nucleon when there is a quantum at infinity. This requires the knowledge of $\Phi_{p'}$ at $r = R$. But the waves come to the surface of the nucleon from the area lying in front of the nucleon (Fig. 3), where G and Φ_0 are equal to

$$G = \frac{e^{ip' \cdot |r-r'|}}{4\pi |r-r'|}, \quad \Phi_0 = e^{i(p \cdot r)}.\quad (43)$$

Therefore,

$$\begin{aligned}\Phi_{p'}(r = R) &= \frac{F}{\sqrt{2\pi f \Omega}} e(j \cdot p) \int \frac{e^{ip' \cdot |r-r'|}}{|r-r'|} e^{i(p-f \cdot r')} d\tau' \\ &\approx - \frac{\sqrt{2\pi} F e(j \cdot p)}{\sqrt{f \Omega} p'} e^{i(p-f \cdot R)} \frac{1}{p' - |p-f|}.\end{aligned}\quad (44)$$

From this we find the flux of particles

$$\frac{4\pi}{f\Omega} e^2 \frac{(\mathbf{j} \cdot \mathbf{p})^2}{p'^2} \frac{\pi R^2}{(p' - |\mathbf{p} - \mathbf{f}|)^2} p' |F|^2.$$

Dividing it by the flux of incident particles $2p$ and summing over f , we find the differential cross-section for a given polarisation f :

$$\sigma_{0c}(j) df d\theta = \frac{e^2}{4\pi} R^2 \frac{(\mathbf{j} \cdot \mathbf{p})^2 |F|^2}{(p' - |\mathbf{p} - \mathbf{f}|)^2 p p'} f df d\theta. \quad (45)$$

Since θ is small, (45) simplified to

$$\sigma_{0c}(j) = \frac{e^2}{\pi} R^2 \frac{(\mathbf{j} \cdot \mathbf{p})^2}{p^2} \frac{|F|^2 E'}{E f} \frac{1}{[(\mu^2/E^2) + \theta^2]^2}. \quad (46)$$

Summing over the polarisation, we have

$$\sigma_{0c} = \frac{e^2}{\pi} R^2 \frac{\theta^2}{(\gamma^2 + \theta^2)^2} \frac{E - f}{E} \frac{1}{f} \left| F \left(\frac{f E \theta^2}{2\mu^2} \right) \right|^2. \quad (47)$$

If $f \ll E$, then (47) corresponds to the classical "stop" radiation originating at a sudden stopping of the charge when its stopping time is small.

The cross-section integrated over $f \geq f_{\min}$, and $\theta \leq \theta_{\max}$ with $F = 1$ is equal to:

$$e^2 R^2 \left[\ln \frac{\theta_{\max}^2 E^2}{e \mu^2} \right] \left[\ln \frac{E}{f_{\min}} - 1 \right]. \quad (48)$$

Unlike the differential radiation, in which the most important role is played by angles θ of the order of $1/\gamma$, considerably larger angles occupy an important place in the "capture" radiation. However, for $\theta > \sqrt{2\mu^2/Ef}$, in accordance with (20a, b), the form factor may begin to limit substantially the intensity of radiation. Therefore, the total probability for the "stop" radiation, taking into consideration the finite size of the π -particle, will be obtained by integrating (47) from $F = 1$ to $\theta \sim \sqrt{2\mu^2/Ef}$

$$\int_{f_{\min} < f < E} \sigma_{0c} df d\theta \approx e^2 R^2 \int_{f_{\min}}^E \frac{df}{f} \ln \frac{E}{f} = \frac{1}{2} e^2 R^2 \ln^2 \frac{E}{f}. \quad (49)$$

If $f \sim E$, then (49) is true only as to order of magnitude. Comparing (49) with (42) or (34), we can see that the "stop" radiation is somewhat greater than the diffraction radiation. The total radiated energy in the case of "stop" radiation under the conditions considered is equal, as to order of magnitude, to:

$$e^2 R^2 \int_0^E \ln \frac{E}{f} df = e^2 R^2 E. \quad (50)$$

It should be mentioned that although the integral cross-sections of γ -quantum radiation are small in comparison to πR^2 , the differential cross-sections of radiation at small angles increase fast and at angles of the order of μ/E reach very great values when E increases.

If $\theta = \mu/E$, then the "stop" radiation has a differential cross-section which, according to (47), is equal to:

$$\frac{e^2}{4\pi} R^2 \left(\frac{E}{\mu} \right)^2 \frac{E - f}{E} \frac{1}{f}. \quad (51)$$

When $E/\mu = \sqrt{4\pi/e^2} \approx 40$, the differential cross-section is found to be equal to:

$$R^2 \frac{E - f}{E} \frac{1}{f}, \quad (52)$$

and corresponds to the nuclear cross-section.

To the "stop" radiation discussed here, which is caused by the capture of a π -meson by the nucleon or the nucleus, we should add the radiation which results from the resultant movement of the nucleon or the nucleus as a whole, as well as the γ -radiation from the secondary mesons and nucleons appearing during the capture of the primary π -meson. γ -radiation from the movement of the nucleon or the nucleus is relatively small because of the large mass of the nucleon or the nucleus as a whole. As for the γ -Radiation from secondary particles, these particles, first of all, are distributed in a comparatively broad solid angle⁶, and, secondly, their energy is considerably smaller than the energy of the primary particle. Therefore, even in the direction coinciding with the direction of one of the secondary particles, the intensity of γ -radiation will be small in comparison with the intensity of radiation in the direction which is close to the direction of the primary π -particle.

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78. THE REMOVAL OF INFINITIES IN QUANTUM ELECTRODYNAMICS

THE solution of the majority of problems in quantum electrodynamics is well known to lead to infinities. Although means exist at the present time of removing these infinities (regularisation), which undoubtedly lead to correct results, there is something artificial about such procedures. Infinities arise in the theory of point interactions which are described by δ -functions (the operators of the interacting fields are taken at one point). The present work aims at developing the idea that the origin of the infinities does not lie in the point nature of the interaction but in the fact that a description of the interactions by means of δ -functions is not permissible.

The general method must consist in the following: when considering an interaction with no matter how small a finite radius a , the coefficient of the corresponding expression should be regarded as a function of the radius of interaction and not as a constant. In the general case the dependence of the coefficient on a must be defined in such a way that, as a decreases, all the physical results tend to finite limits.

This means, in the case of electrodynamics, that the charge e_1 in the expression for the interaction between an electron and the electromagnetic field must be regarded as an as yet unknown function of the radius of interaction.

No use can be made of perturbation theory when carrying out this programme, since, as was first pointed out by Edwards¹, one of the source of infinities is the solution of equations by successive approximations. If we have an equation $f + \lambda I(f) = f_0$, where I is an integral operator and λ is a small quantity, it can have a finite solution even when $I(f)_0 = \infty$. In this case $\lambda I(f)$ does not tend to zero as $\lambda \rightarrow 0$, so that this term plays a part even in determining the zero approximation for f .

Let us aim at obtaining the zero approximations for the fundamental quantities of quantum electrodynamics. In this case we only need to consider those diagrams in which the integrals compensate the small coefficients (powers of e_1^2) standing in front of them. As is well known, all the divergent integrals in electrodynamics are of a logarithmic type†.

It is clear from the foregoing that such a logarithmically divergent integral is in reality of order $1/e_1^2$. Since we are interested in the zero approximation,

Л. Д. Ландау, А. А. Абрикосов и И. М. Халатников, Об устранении бесконечностей в квантовой электродинамике, Доклады Академии Наук СССР, 95, 497 (1954).

† As regards the more strongly divergent so-called "self mass" of a photon, it must be equal to zero by virtue of the law of conservation of charge. This implies that the interaction must be smeared out in such a way that this requirement is satisfied.

it is sufficient for us to consider only those diagrams in which the power of the logarithm is the same as the power of e_1^2 .

Dyson² has already obtained exact equations connecting the Green functions G and $D_{\mu\nu}$ corresponding to electron and photon lines with the vertex part Γ_μ . We shall show that, in the zero approximation, closed integral equations can be written for the Γ_μ .

This is done by using the fact that the addition of a photon line enclosing two vertices does not yield a divergent integral. This means that, in the diagrams representing the successive approximations for the vertex part, we do not need to consider lines representing interference of vertices, which lead to logarithms to lower powers than the orders of perturbation theory. In particular, we need not consider diagrams with intersecting photon lines. More complicated diagrams can be eliminated in the same way. All the diagrams that remain must form a ladder, in which each vertex must be furnished with a

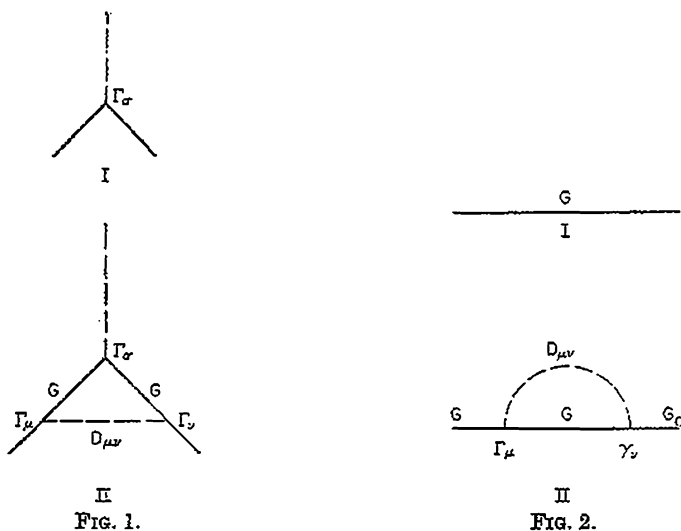


FIG. 1.

FIG. 2.

similar subsidiary ladder. It is understood here that the photon and electron lines are summed over all approximations, i.e. they are represented by the true functions $D_{\mu\nu}$ and G . The summation of such a scheme can easily be carried out, given the assumptions indicated. In fact, the total sum of all the higher approximations for diagram I (see Fig. 1) is the same as diagram II (Fig. 1), in which all the higher approximations are included. We obtain the following equation from this†:

$$\Gamma_\sigma(p, p-l; l) = \gamma_\sigma + \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \Gamma_\sigma(p-k, p-k-l; l) \times G(p-k-l) \Gamma_\nu(p-k-l, p-l; -k) D_{\mu\nu}(k) d^4k. \quad (1)$$

† We use Feynman's notation³; in particular, $\gamma_{1,2,3} = \beta \alpha_{1,2,3}$; $\gamma_0 = \beta$; $d^4k = (2\pi)^{-2} dk_0 dk_1 dk_2 dk_3$.

The equations for G and $D_{\mu\nu}$ are added to this equation. These equations were derived by Dyson by summation of diagrams, no terms being neglected in the present case. This is done by equating (in the case of G) the sum of the higher approximations for diagram I (see Fig. 2) to the total sum of all the approximations for diagram II (Fig. 2). Here, in order to avoid considering higher order diagrams more than once in diagram II, the extra photon lines must only be added to one side of the diagram. This gives

$$G(p) = G^0(p) + \frac{e_1^2}{\pi i} G(p) \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\nu D_{\mu\nu}(k) d^4k G_0(p). \quad (2)$$

Here, G_0 is the electron line to the zero approximation of perturbation theory:

$$G_0 = \frac{1}{\hat{p} - m_1}, \quad (3)$$

where m_1 denotes the "self-mass" of an electron, which it would have "in the absence of charge". This equation can be written as

$$G(p) \left[\hat{p} - m_1 - \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\nu D_{\mu\nu}(k) d^4k \right] = 1. \quad (4)$$

Similarly, we get the equation for $D_{\mu\nu}$:

$$D_{\mu\nu}(k) = D_{\mu\nu}^0(k) - \frac{e_1^2}{\pi i} D_{\mu\sigma}(k) \text{Tr} \left[\int G(p) \Gamma_\sigma(p, p-k; k) \right. \\ \left. \times G(p-k) \gamma_\tau d^4p \right] D_{\tau\nu}^0(k). \quad (5)$$

$D_{\mu\nu}^0$ is usually assumed equal to $\delta_{\mu\nu}/k^2$. However, this is not the most general, or, as we shall see below, the most convenient definition. In view of the gauge invariance, the "longitudinal" (in the four-dimensional sense) photons do not interact with the charged particles, so that the longitudinal part (proportional to $k_\mu k_\nu$) of the tensor $D_{\mu\nu}$ can be chosen arbitrarily. Moreover, for the same reason, it cannot vary under the influence of perturbations. This is in complete accord with the fact that the integral term in (5), corresponding to the Dirac current, is always transverse. Hence, given any choice of the longitudinal part of $D_{\mu\nu}^0$, we can replace $D_{\mu\nu}^0$ in it by $\delta_{\mu\nu}/k^2$ without changing the result. The following equation is finally obtained for the transverse part $D_{\mu\nu}^t$ ($D_{\mu\nu}^t k_\mu = 0$) of the tensor $D_{\mu\nu}$:

$$D_{\mu\sigma}^t \left\{ k^2 \delta_{\sigma\nu} + \frac{e^2}{\pi i} \text{Tr} \left[\int G(p) \Gamma_\sigma(p, p-k; k) G(p-k) \gamma_\nu d^4p \right] \right\} \\ = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \quad (6)$$

We shall be occupied with the solution of equations (1), (4) and (6) in subsequent articles; for the present, we shall indicate some important properties of the integrals featured in these equations. An annoying feature is the integration

over four-dimensional pseudo-Euclidean space. Let us show that, provided we confine ourselves to spatial vectors, the integrations can be carried over to four-dimensional Euclidean space. An important factor here is that all the functions encountered in the equations reduce in the long run to scalar functions of expressions of the type $(k - a)^2$, $(k - b)^2$, etc. where a, b, \dots are constant vectors, and k is the variable vector over which the integration is performed. The definition of these functions is such that, when $(k - a)^2$ changes sign, they have to be analytically continued via the upper half-plane, in which they have no singularities. Since we are starting from a "smeared-out" interaction, Γ_μ , and hence also the integrands, must vanish as $k \rightarrow \infty$.

Let the constant vectors a, b , etc. in the integration have time components equal to zero. The integration over the time component k_0 can now be written as

$$\int_{-\infty}^{+\infty} F(k_0^2) dk_0 = \int_0^{\infty} F(x) \frac{dx}{\sqrt{x}}$$

(of course F also depends on the other components of the vector k). Since $F(x)$ has no singularities in the upper half-plane and vanishes at infinity, we have identically:

$$\int_0^{\infty} \frac{F(x)}{\sqrt{x}} dx = i \int_0^{\infty} F(-x) \frac{dx}{\sqrt{x}}.$$

This is in accordance with the fact that we can replace k_0 by ik_0 in all the four-dimensional integrals in the present case, after which the integration reduces to integration over four-dimensional Euclidean space.

In conclusion, the authors express their sincere thanks to A. D. Galanin, I. Ya. Pomeranchuk and B. L. Ioffe for discussions and useful suggestions.

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79. AN ASYMPTOTIC EXPRESSION FOR THE ELECTRON GREEN FUNCTION IN QUANTUM ELECTRODYNAMICS

In a previous paper¹ we have obtained equations for the fundamental functions of quantum electrodynamics. Since we set ourselves the task of obtaining an exact solution of these equations, we were unable to make use of the usual expressions for the values of G , $D_{\mu\nu}$ and Γ_μ , even with re-normalized coefficients.

In view of the analytic character of the functions $D_{\mu\nu}$ and G , it is sufficient for our purpose to determine them for negative values of p^2 and k^2 (space vectors), after which they may be analytically continued through the upper half-plane to positive (time) values. Using the transformation indicated in a previous work¹, we may convert the integration in the equations for G and $D_{\mu\nu}$ into Euclidean four-dimensional space. It follows that an important part in the determination of G and $D_{\mu\nu}$ is played by Γ_μ from pure space vectors. By carrying out the same transformation in the equation for Γ_μ , we see that such "special" quantities G , $D_{\mu\nu}$ and Γ_μ constitute a closed system of quantities which do not require consideration of Γ_μ in other regions of momentum values for their determination. In the present paper our limited objective is the determination, in addition to G and $D_{\mu\nu}$, of "spatial" values of Γ_μ and of those values which may be obtained from their analytical continuation.

In solving the equations it is of fundamental importance to know the behaviour of all the functions for large values of the momenta, $p \gg m$, and $k \gg m$. We shall seek the functions G and $D_{\mu\nu}$ in the form:

$$G(p) = \frac{\beta(p^2)}{\hat{p}}; \quad (1)$$

$$D_{\mu\nu}(k) = \frac{1}{k^2} \left[d_2(k^2) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + d_1(k^2) \frac{k_\mu k_\nu}{k^2} \right], \quad (2)$$

where the functions $\beta(p^2)$, $d_2(k^2)$, $d_1(k^2)$ are slowly varying functions of p^2 and k^2 . With respect to Γ_μ , we shall show that the equation for this satisfies an expression of the form:

$$\Gamma_\mu(p, q; l) = \gamma_\mu \alpha(f^2) \quad (3)$$

Л. Д. Ландау, А. А. Абрикосов и И. М. Халатников, Асимптотическое выражение для гриновской функции электрона в квантовой электродинамике, *Доклады Академии Наук СССР*, 95, 773 (1954).

where $\alpha(f^2)$ is a slowly changing function of its argument; f^2 represents any of the quantities p^2 , q^2 and l^2 , when these are of the same order of magnitude. In the case when one of these quantities is small compared with the two others, we choose the largest as f^2 . Since we are evaluating the "spacial" Γ_μ , for which $(-p^2)^{\frac{1}{2}}$, $(-q^2)^{\frac{1}{2}}$ and $(-l^2)^{\frac{1}{2}}$ represent the sides of an ordinary triangle (that is, they satisfy the inequality that $(-q^2)^{\frac{1}{2}} + (-l^2)^{\frac{1}{2}}$ is greater than $(-p^2)^{\frac{1}{2}}$, etc.) we are able analytically to continue the solution obtained only for those non-purely spatial Γ for which either p^2 , q^2 and l^2 are of the same order of magnitude, or, when one of these is small, the two remaining ones are sufficiently close to one another.

We now insert expressions (1), (2) and (3) into equation (1) of our previous paper¹. It is easy to see that the integral on the right-hand side, with the exception of the slowly changing functions, diverges logarithmically when $|k^2| \gg |p^2|$, and when $|k^2| \gg |q^2|$. Correspondingly, this region of the integration plays a decisive role in the integration. Within this region it is permissible to neglect p , q and l in comparison with k , so that all the expressions standing within the integral are functions of k only. After averaging over the direction of the vector k , γ_μ may be taken outside the integral sign, confirming equation (3), and a scalar expression remains within the integral. In order to convert the integral over d^4k into a simple one over dk^2 , we use the change of k_0 to ik_0 indicated in the previous paper¹. When d^4k is changed into $i(2\pi)^{-2} d\Omega$, where $d\Omega$ is an element of volume in the four-dimensional Euclidean space, and may be written in the form $2\pi^2 R^3 dR$, where R is the radius, which in the present case is equal to $(-k^2)^{\frac{1}{2}}$. We thus obtain:

$$d^4k \rightarrow \frac{i}{4} (-k^2) d(-k^2). \quad (4)$$

Introducing a new variable: $\xi = \ln(-p^2/m^2)$, we obtain finally:

$$\alpha(\xi) = 1 + \frac{e_1^2}{4\pi} \int_{\xi}^{\infty} \alpha^3(z) \beta^2(z) d_1(z) dz. \quad (5)$$

We may note that the quantity d_1 does not appear in this equation.

We now pass to equation (4) of the previous paper¹, for G . It is easy to see that the integral in this expression diverges for large values of $k \gg p$. Although this divergence is at first glance of the first order with respect to k , averaging over directions leads, as is well known, in view of the vector character of the quantity k , to a logarithmic divergence once again. This means that, in expanding the integrand in inverse powers of k , we ought always to take into account the terms of the next order of smallness. Such terms in G are found directly from equation (1), where we may clearly, in view of the slight dependence of β on p^2 , restrict the expansion in a series of denominators; that is, we may write

$$G(p-k) \approx -\frac{\beta(k^2)}{\hat{k}} - \beta(k^2) \frac{1}{\hat{k}} \hat{p} \frac{1}{\hat{k}}. \quad (6)$$

It is, however, also necessary to take account of a correction of the order of p/k in $\Gamma_\mu(p, p-k; k)$ also. For the determination of this correction we must again return to the equation for Γ_μ . Since we are concerned only with small corrections, arising in passing from $p=0$ to the final value of p , we shall consider the change in the integral on the right-hand side as the changes occurring in each of the terms within the integral. Evaluation of the expression obtained in this way shows that when $k \gg l$, the function within the integral is inversely proportional to k^5 : that is, the integral converges for large values of k . The integral assumes a logarithmic form, however, in the region where $l \gg k \gg p$, which should thus be found also in the main region of the integration. For greater and for smaller values of k , the integral gives relatively small corrections, which may be neglected within the accuracy under contemplation.

Within the indicated region of integration, it is evidently sufficient to take account only of the correction p/k , since these are much greater than p/l . It may be shown that it is sufficient to limit ourselves in the expression for $\Gamma_\mu(p, p-l; l)$ to corrections of the form:

$$[\Delta_1(p^2, l^2) \hat{l} \gamma_c + \Delta_2(p^2, l^2) \gamma_c \hat{l}] \frac{\hat{p}}{l^2}; \quad (7)$$

and it is not difficult to show that corrections of other forms do not arise from G in the form given in (6), which when inserted into the integral again give an expression of type (7); and therefore, when comparing the left-hand and right-hand sides of the equation are seen to be equal to zero.

By inserting (6) and (7) into the equation for Γ_c (equation (1) of ref. 1), and using the transformation (4), we obtain the following equations for Δ_1 and Δ_2 :

$$\left. \begin{aligned} \Delta_1(\xi, \eta) &= \frac{e_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \{ \Delta_1(\xi, z) [\bar{d}_1(z) - d_t(z)] \\ &\quad + [\Delta_2(\xi, z) + \alpha(z)] [\bar{d}_1(z) + d_t(z)] \} dz; \\ \Delta_2(\xi, \eta) &= \frac{e_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \{ \Delta_1(\xi, z) [\bar{d}_1(z) + d_t(z)] \\ &\quad + [\Delta_2(\xi, z) + \alpha(z)] [\bar{d}_1(z) - d_t(z)] \} dz, \end{aligned} \right\} \quad (8)$$

where $\xi = \ln(-p^2/m^2)$, and $\eta = \ln(-l^2/m^2)$.

Inserting expressions (6) and (7) in the integral of equation for G , in which the main part of the integration concerns the region $p \ll k < \infty$, we obtain, after similar transformations:

$$\frac{1}{\beta(\xi)} = 1 + \frac{e_1^2}{4\pi} \int_{\xi}^{\infty} [\alpha(z) + \Delta_1(\xi, z) + \Delta_2(\xi, z)] \beta(z) d_1(z) dz. \quad (9)$$

Adding together the two equations in (8), we obtain for the sum of Δ_1 and Δ_2 :

$$\Delta_1(\xi, \eta) + \Delta_2(\xi, \eta) = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) d_1(z) [\Delta_1(\xi, z) + \Delta_2(\xi, z) + \alpha(z)] dz. \quad (10)$$

We should note that in this equation, and in equation (9), the quantity d_t has again been eliminated. Thus, the equations for the determination of Γ_μ and G may be solved independently of the evaluation of the photon function $D_{\mu\nu}$. The linear part of d_1 is, as has already been indicated, an arbitrary quantity, which we preserve in the equations as an index of their gauge-invariance. It is clear that the dependence of G and Γ_μ on d_1 in no way contradicts the gauge-invariance of the equations, since these quantities do not remain unchanged during a gauge transformation.

We now introduce the notation:

$$\frac{\Delta_1(\xi, \eta) + \Delta_2(\xi, \eta)}{\alpha^2(\eta) \beta(\eta)} = q(\xi, \eta); \quad (11)$$

when $q(\xi, \eta)$ satisfies the equation

$$q(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d_1(z) q(\xi, z) dz + \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) d_1(z) dz. \quad (12)$$

Equation (9) may now be re-written in the form:

$$\frac{1}{\beta(\xi)} = 1 + q(\xi, \infty). \quad (13)$$

Equations (5), (12) and (13) provide the possibility of an unambiguously determination of α and β . After differentiation of equations (5) and (12) (the latter with respect to η), we obtain:

$$\frac{d\alpha}{d\xi} = -\frac{e_1^2}{4\pi} \alpha^3(\xi) \beta^2(\xi) d_1(\xi); \quad (14)$$

$$\frac{\partial q(\xi, \eta)}{\partial \eta} = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta^2(\eta) d_1(\eta) q(\xi, \eta) + \frac{e_1^2}{4\pi} \alpha(\eta) \beta(\eta) d_1(\eta). \quad (15)$$

To this it is necessary to add the boundary conditions from the integral equations:

$$\alpha \rightarrow 1 \quad \text{when} \quad \xi \rightarrow \infty; \quad q = 0 \quad \text{when} \quad \xi = \eta.$$

Introducing a new function $r(\xi, \eta) = \alpha(\eta) q(\xi, \eta)$, we have:

$$\frac{\partial r(\xi, \eta)}{\partial \eta} = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta(\eta) d_1(\eta).$$

Hence

$$r(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta(z) d_1(z) dz, \quad \frac{\partial r(\xi, \eta)}{\partial \xi} = -\frac{e_1^2}{4\pi} \alpha^2(\xi) \beta(\xi) d_1(\xi).$$

When $\eta \rightarrow \infty$, we have from (13) that $r(\xi, \infty) = [1/\beta(\xi)] - 1$, from which:

$$\frac{d\beta}{d\xi} = \frac{e_1^2}{4\pi} \alpha^2(\xi) \beta^3(\xi) d_1(\xi).$$

It follows from this equation and from (14) that $\alpha\beta = \text{constant}$, or, taking account of the gauge conditions:

$$\alpha(\xi) \beta(\xi) = 1. \quad (16)$$

It is easy to see that this condition follows automatically from the Ward theorem²:

$$\Gamma_{\mu}(p, p; 0) = \frac{\partial G^{-1}(p)}{\partial p_{\mu}},$$

if account is taken of the fact that the slowly changing function β gives, on differentiation, a term of higher order.

After the insertion of (16) in equation (14) we obtain finally, taking account of the boundary condition:

$$\alpha(\xi) = \exp \left[\frac{e_1^2}{4\pi} \int_{\xi}^{\infty} d_1(z) dz \right], \quad \beta(\xi) = \exp \left[-\frac{e_1^2}{4\pi} \int_{\xi}^{\infty} d_1(z) dz \right]. \quad (17)$$

Equation (17) expresses essentially the gauge transformation of the quantities G and Γ_{μ} (for slowly changing values of d_1). The most convenient choice is evidently that for which $d_1 = 0$ (in contrast with the usual supposition that $d_1 = 1$). With this there is usually no divergence of Γ_{μ} and G , and these quantities (for Γ_{μ} within the limitations indicated above) are equally simply to their zero values. We should note that this normalisation consists essentially of the Lorentz condition that $\partial A_i / \partial x_i = 0$.

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80. AN ASYMPTOTIC EXPRESSION FOR THE PHOTON GREEN FUNCTION IN QUANTUM ELECTRODYNAMICS

WE obtained in earlier articles^{1, 2} the zero approximation general integral equations of quantum electrodynamics and asymptotic expressions for the electron Green function G and the vertex part Γ_μ . We shall now use the results obtained for finding the photon Green function $D_{\mu\nu}$.

The integral in equation (6) of ref. 1 for $D_{\mu\nu}$ diverges quadratically in this case for large p . However, since the integral must be zero at $k = 0$ because of the law of conservation of charge (transversality of the Dirac current), we only have to consider the difference between the integral and its value at $k = 0$. Noting that the terms containing p^3 in the denominator yield zero, on averaging we again arrive at an integral of the logarithmic type. The region $p \ll k$ can easily be seen to provide an insignificant contribution to the integral, so that we can confine ourselves to the region $p \gg k$.

We naturally have to take into account terms of the second order in k/p in $\Gamma_\mu(p, p - k; k)$ when evaluating the logarithmic integral, just as we did in our previous article. These terms are found by using the equation for Γ_σ (equation (1) of ref. 1), and changing the variable of integration from k to $k' = p - k$ in the integral that results. This now takes the form

$$\int D_{\mu\nu}(p - k') \Gamma_\mu(p, k'; p - k') G(k') \Gamma_\sigma(k', k' - l; l) G(k' - l) \times \Gamma_\nu(k' - l, p - l; k' - p) d^4k.$$

It is assumed here that $p \gg l$. The expansion of $G(k' - l)$ in powers of l/k' evidently actually gives rise to terms of the second order in l , leading to an integral which is logarithmic in the region $p \gg k' \gg l$. No first-order terms in l can appear, since they contain k'^3 in the denominator and vanish on integration over the angle. Terms of the type described can appear both from $G(k' - l)$ and from $\Gamma_\sigma(k', k' - l; l)$. The dependence on l in the final Γ , yields terms of order l^2/p^2 , and not of order l^2/k'^2 , and these can be neglected.

We start by finding the additional terms produced by $G(k' - l)$. The required expression is found to be

$$\frac{e_1^2}{24\pi} \left\{ 2d_t(\xi) \frac{\hat{l}\gamma_\sigma\hat{l} - l^2\gamma_\sigma}{p^3} + [d_1(\xi) - d_t(\xi)] \frac{l^2\hat{p}\gamma_\sigma\hat{p} - p\hat{l}\gamma_\sigma\hat{l}\hat{p}}{p^4} \right\} \times \alpha^2(\xi) \int_{\eta}^{\xi} \alpha(z) \beta^2(z) dz. \quad (1)$$

Here, $\xi = \ln(-p^2/m^2)$, $\eta = \ln(-l^2/m^2)$.

Л. Д. Ландау, А. А. Абрикосов и И. М. Халатников, Асимптотическое выражение для греновской функции фотона в квантовой электродинамике, Доклады Академии Наук СССР, 95 1177 (1954).

It turns out that it is sufficient to seek the correction to $\Gamma_\sigma(p, p-l; l)$ in the form

$$s(\xi, \eta) \alpha^3(\xi) \left\{ 2d_0(\xi) \frac{\hat{l} \gamma_\sigma \hat{l} - l^2 \gamma_\sigma}{p^2} + [d_1(\xi) - d_0(\xi)] \frac{l^2 \hat{p} \gamma_\sigma \hat{p} - \hat{p} \hat{l} \gamma_\sigma \hat{l} \hat{p}}{p^4} \right\}. \quad (2)$$

On substituting this expression in the integral and equating terms on the left and right-hand sides of equation (1) of ref. 1 for Γ_σ , we get the following equation for $s(\xi, \eta)$:

$$s(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\eta}^{\xi} \alpha^2(z) \beta^2(z) d_1(z) s(z, \eta) dz + \frac{e_1^2}{24\pi} \int_{\eta}^{\xi} \alpha(z) \beta^2(z) dz. \quad (3)$$

On substituting expression (2) for the correction to Γ_σ in equation (6) of ref. 1 for $D_{\mu\nu}$, we get

$$d_1(\eta) = 1 - 8d_0(\eta) \left[\frac{e_1^2}{4\pi} \int_{\eta}^L \alpha^2(z) \beta^2(z) d_1(z) s(z, \eta) dz + \frac{e_1^2}{24\pi} \int_{\eta}^L \alpha(z) \beta^2(z) dz \right]. \quad (4)$$

Here, L denotes the upper limit of the values of η that we consider, this being determined by the radius of "smearing out" of the interaction. In accordance with our accepted scheme of "smeared-out interactions", the magnitude of the interaction must fall rapidly when the momentum exceeds a critical value $P \sim 1/a$, where a is the range of interaction. The condition $p \gg P$ transforms in the logarithmic scale simply to $\xi > L$, where $L = \ln(-P^2/m^2)$, since an appreciable variation of p corresponds to a fairly insignificant change in $\ln(p^2/m^2)$. The fact of smearing out therefore implies that all the integrals connected with the interaction must have the upper limit L . Let us emphasise that, in accordance with our fundamental scheme, the constant e_1 is in fact a function of the "radius of smearing out" a , i.e. of L . There is no need for us to make use of the smearing out when finding the functions G and Γ_μ in (2), since the divergence was determined there by the function $d_1(\xi)$, which is arbitrary and not connected with the nature of the interaction. However, if we introduce a smearing out, we must automatically replace infinity by L as the upper limit in the integrals appearing in the exponents α and β .

Notice that now $\beta(L) = 1$. This is in accordance with the obvious fact that, for momenta $p \gg P$, the interaction disappears and an electron must behave as though free and not interacting with the field.

Comparing (4) with (3), we get

$$\frac{1}{d_1(\eta)} = 1 + 8s(L, \eta). \quad (5)$$

Differentiating (3) with respect to ξ and using the relations $\alpha = 1/\beta$ and $d\beta(\xi)/d\xi = (e_1^2/4\pi) d_1(\xi) \beta(\xi)^2$, we get

$$\frac{\partial s(\xi, \eta)}{\partial \xi} = s(\xi, \eta) \frac{1}{\beta(\xi)} \frac{d\beta(\xi)}{d\xi} + \frac{e_1^2}{24\pi} \beta(\xi), \quad (6)$$

whence

$$\frac{\partial}{\partial \xi} \left(\frac{s(\xi, \eta)}{\beta(\xi)} \right) = \frac{e_1^2}{4\pi},$$

and, taking into account that $s = 0$ for $\xi = \eta$, by equation (3),

$$s(\xi, \eta) = \frac{e_1^2}{24\pi} (\xi - \eta) \beta(\xi). \quad (7)$$

On substituting $\xi = L$ in this and using the fact that $\beta(L) = 1$, we finally obtain from (5):

$$d_t(\eta) = \frac{1}{1 + \frac{e_1^2}{3\pi} L - \frac{e_1^2}{3\pi} \eta}. \quad (8)$$

Notice that this expression, unlike those for G and Γ_μ , does not contain the arbitrary function d_1 , as is in fact required by gauge invariance.

This formula is applicable for $|k^2| \gtrsim m^2$. When $k^2 \sim m^2$, it yields

$$d_t^0 = \frac{1}{1 + \frac{e_1^2}{3\pi} L}. \quad (9)$$

Since, when k is small compared with m , the integration is carried out over $p \gg m$, and not over $p \gg k$, (9) also holds for $k \ll m$, and in particular, as $k \rightarrow 0$.

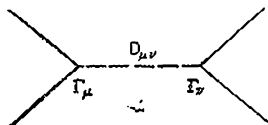


FIG. 1.

We can now find the connection between the constant e_1 and the true charge of an electron. We do this by considering, e.g. the scattering of an electron by an electron. It is illustrated diagrammatically in Fig. 1. The free ends correspond to $p^2 = m^2$. In order to be able to apply the formulae obtained for Γ_μ in ref. 2, it is necessary that the momentum of a photon on a photon line be not very large compared with m . In this case we can neglect η in the denominator of (8), i.e. replace d_t by d_t^0 . According to Dyson³, each free end corresponds

to an additional multiplication by $\sqrt{\lim_{p^2=m^2} (\hat{p} - m) G(p)}$, which in our case is equal to $\sqrt{\beta(\xi=0)}$.

Noting that Fig. 1 contains 4 electron free ends, two vertices and one photon line, it may easily be seen that the net result is to obtain the ordinary formula for the scattering amplitude, with the product $e_1^2 \alpha^2(0) \beta^2(0) d_t^0$ in place of the square of the charge. On substituting the relevant expressions, we finally get

$$e^2 = \frac{e_1^2}{1 + \frac{e_1^2}{3\pi} L}.$$

The same formula is naturally obtained if we consider any other effect. As was to be expected, it satisfies the requirement of gauge invariance. On expressing e_1^2 in terms of e^2 , we get

$$e_1^2 = \frac{e^2}{1 - \frac{e^2}{3\pi} L}. \quad (10)$$

It is clear from this equation that, given e^2 , e_1^2 increases as L increases and becomes of order unity for L close to $3\pi/e^2$. Since our fundamental equations were derived on the assumption that $e_1^2 \ll 1$, our expressions (including (10)) cease to be applicable for $L > 3\pi/e^2$. This means that our entire theory is only applicable for $\xi < 3\pi/e^2$, and it cannot be used to consider smaller dimensions and larger energies.

In actual fact the theory becomes invalid much sooner, due to gravitation. The gravitational interaction becomes of the order of the electromagnetic one when $p^2 \sim e^2/K$ (K is the gravitational constant). This corresponds to $p \sim 2 \times 10^{-6} \text{ g} \sim 10^{27} \text{ eV}$ and $L \sim 100$.

The expression for d_t can now be written as

$$d_t(k^2) = \frac{e^2}{e_1^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(-\frac{k^2}{m^2}\right)}. \quad (11)$$

(for $k > m$). Discounting a "re-normalisation" factor, d_t proves to be independent of the radius of "smearing out", as was to be expected.

An important point as regards the derivation of (11) is that we have not been considering any charged particles other than electrons. This is no doubt justifiable in regard to particles (such as π -mesons or nucleons) which interact strongly with non-electromagnetic fields, since these have a finite "radius" thanks to specific interactions, and their interaction with an electromagnetic field decreases more rapidly with increase of the wave vector than is the case in (11).

On the other hand, it is possible that particles exist in nature that are not capable of strong interactions (μ -mesons?). In this case the denominator of

the formula for d_t will contain, instead of the second term, a sum of similar expressions for different sorts of particles. Since the product of $e^2/3\pi$ with the logarithm of the mass ratio is always small, this amounts in practice to multiplying the second term in the denominator by an integral factor, equal to the number of varieties of this kind of particles.

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81. THE ELECTRON MASS IN QUANTUM ELECTRODYNAMICS

ONE of the most interesting problems of quantum electrodynamics is that of the mass of an electron, and in particular, of the roles of the electromagnetic and "self" mass. The behaviour of the Green function for an electron $G(p)$ when $p \sim m$ is of fundamental importance for the solution of this problem. We obtained in ref. 2 an asymptotic expression for the Green function for an electron, valid for $p \gg m$. We cannot solve the problem of the mass of an electron simply by taking the value of this expression for $G(p)$ with $p \sim m$, and means of correcting the expression have to be found.

Suppose that the function $G(p)$ has the form

$$G(p) = \frac{\beta(p^2)}{\hat{p} - m(p^2)}, \quad (1)$$

where $\beta(p^2)$ is the function introduced in ref. 2, and $m(p^2)$ is a new slowly varying function of p^2 . When p is large the ratio of the second term in the denominator to the first becomes quite insignificant, and always far less than the inaccuracy in the function $\beta(p^2)$ that we have defined. In spite of this, such a term can validly be considered, since in contrast with the first it is an even function of the vector p , so that the error in determining β cannot affect its value. When p is large, $G(p)$ can be written as

$$G(p) = \frac{\beta(p^2)}{\hat{p}} + \frac{\beta(p^2) m(p^2)}{p^2}; \quad (2)$$

the first term here is an odd function, and the second an even function of p .

We write the equation for the function G :

$$G^{-1}(p) = \hat{p} - m_1 + \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\nu D_{\mu\nu}(k) d^4k \quad (3)$$

and separate out on both sides even functions of p , which must be equated to one another. To find the even part of the integral on the right-hand side, we change the sign of p and at the same time change the sign of the variable of integration k . The even part of the integrand will now consist of the product of the even part of Γ (relative to a change of sign of all the variables appearing in it), with the even part of G and the product of the odd part of Γ_μ with the odd part of G . In the first product, it is sufficient to take the expression $\alpha(k^2) \gamma^\mu$

Л. Д. Ландау, А. А. Абрикосов и И. М. Халатник, Масса электрона в квантовой электродинамике, *Доклады Академии Наук СССР* 96, 261 (1954).

for Γ_μ , and the second term of (2) for G . In the second product we take G in its earlier form $\beta(p^2)/\hat{p}$, whilst the odd part of Γ_μ has to be determined from equation (1) of ref. 1 for Γ_μ .

When equating the odd parts of the two sides of the equation, we change the signs of p and l and again simultaneously change the sign of the variable of integration k in the integral on the right-hand side. After this, the odd part of the integral expression will contain either the even part of one of the G or the odd part of one of the Γ_μ . The other variants lead to higher order terms. We first consider the effect of the even correction to G . Let $p \ll l$. It is easily seen that the principal term of the expression is now produced by the odd correction to $G(p - k)$. The corresponding integral is logarithmic in the region $p \ll k \ll l$ and is rapidly convergent beyond this region. Direct evaluation gives us for the integral:

$$-\frac{e_1^2}{4\pi} \frac{l_\sigma}{l^2} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \alpha(z) m(z) [3d_t(z) + d_1(z)] dz,$$

where, as before, $\xi = \ln(-p^2/m^2)$, $\eta = \ln(-l^2/m^2)$.

We shall now seek the odd part of $\Gamma_\sigma(p, p - l; l)$ as $(l_\sigma/l^2) \alpha^2(\eta) \beta(\eta) t(\xi, \eta)$ and substitute the corresponding correction in $\Gamma_\mu(p, p - k; k)$. It can be shown that the corrections to the remaining Γ yield expressions of a higher order of smallness. After this, we get the following integral equation for $t(\xi, \eta)$:

$$\begin{aligned} t(\xi, \eta) = & \frac{e_1^2}{4\pi} \int_{\xi}^L \alpha^2(z) \beta^2(z) d_1(z) t(\xi, z) dz \\ & - \frac{e_1^2}{4\pi} \int_{\xi}^L \alpha(z) \beta(z) m(z) [3d_t(z) + d_1(z)] dz. \end{aligned} \quad (4)$$

On substituting in the integrand in (3) the expressions for the corrections to G and Γ_μ , we finally get for the even part of (3):

$$\begin{aligned} -\frac{m(\xi)}{\beta(\xi)} = & -m_1 + \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d_1(z) t(\xi, z) dz \\ & - \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) m(z) [3d_t(z) + d_1(z)] dz, \end{aligned} \quad (5)$$

$$m(\xi) = \beta(\xi) [m_1 - t(\xi, L)]. \quad (6)$$

On differentiating (4) with respect to η and using the equations $\alpha\beta = 1$ and $d\beta/d\xi = (e_1^2/4\pi) \beta d_1$, we get

$$\frac{\partial t(\xi, \eta)}{\partial \eta} = \frac{1}{\beta(\eta)} \frac{d\beta(\eta)}{d\eta} t(\xi, \eta) - \frac{e_1^2}{4\pi} m(\eta) [3d_t(\eta) + d_1(\eta)], \quad (7)$$

whence

$$\frac{\partial}{\partial \eta} \left(\frac{t(\xi, \eta)}{\beta(\eta)} \right) = - \frac{e_1^2}{4\pi} \frac{m(\eta)}{\beta(\eta)} [3d_t(\eta) + d_l(\eta)];$$

noting that, by (4), $t(\xi, \eta) = 0$ for $\xi = \eta$,

$$\frac{t(\xi, \eta)}{\beta(\eta)} = - \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \frac{m(z)}{\beta(z)} [3d_t(z) + d_l(z)] dz,$$

whence differentiation with respect to ξ gives

$$\frac{1}{\beta(\eta)} \frac{\partial t(\xi, \eta)}{\partial \xi} = - \frac{e_1^2}{4\pi} \frac{m(\xi)}{\beta(\xi)} [3d_t(\xi) + d_l(\xi)].$$

On substituting $\eta = L$ in this, we obtain (noting that, from (6), $\beta(L) = 1$)

$$\frac{d}{d\xi} \frac{m(\xi)}{\beta(\xi)} = - \frac{e_1^2}{4\pi} \frac{m(\xi)}{\beta(\xi)} [3d_t(\xi) + d_l(\xi)],$$

or, on using the relationship $d\beta/d\xi = (e_1^2/4\pi) \beta d_l$,

$$\frac{dm(\xi)}{d\xi} = - \frac{3e_1^2}{4\pi} d_l(\xi) m(\xi).$$

On substituting for d_t the expression obtained in ref. 3, we finally get

$$m(\xi) = m(0) \left(1 - \frac{e^2}{3\pi} \xi \right)^{9/4}, \quad (8)$$

where the constant $m(0)$ is the value of $m(\xi)$ at $\xi = 0$, i.e. the observed mass of an electron.

This formula only holds provided there are no particles in nature other than electrons that only have a worthwhile interaction with electromagnetic fields. If all particles of this kind have spin $\frac{1}{2}$, the above formula becomes

$$m(\xi) = m \left[1 - \frac{\nu e^2}{3\pi} \xi \right]^{9/4},$$

where ν is the number of types of particles. Equation (8) becomes, in the usual notation,

$$m(p^2) = m \left[1 - \frac{e^2}{3\pi} \ln \left(- \frac{p^2}{m^2} \right) \right]^{9/4}. \quad (9)$$

Equation (8) determines $m(p^2)$ when $|p^2| \gg m^2$. The case $p^2 \ll m^2$ is no different from $p^2 \sim m^2$, i.e. it corresponds with our accuracy to $m(p^2) = m$. When $p^2 = m^2$ the denominator of (1) vanishes, so that the case of p^2 very close to m^2 ($e^2 \ln [m^2/(p^2 - m^2)] \sim 1$) requires special consideration.

As was to be expected, (9) proves to be gauge invariant, and the "smearing out" radius does not appear explicitly in it. It follows from (9) that the mass

of the particle falls as k^2 increases. By (6), m_1 is equal to $m(L)$, i.e. corresponds to the value k^2 referred to the "smearing out" radius,

$$m_1 = m \left(\frac{e^2}{e_1^2} \right)^{3/4}.$$

At the limit of applicability of the theory $e_1 \sim 1$ and $m_1 \sim m e^{3/2}$, i.e. a very small quantity. As indicated in ref. 3, this region is not reached because of the need to take gravitational effects into account. It can evidently be supposed that we shall have $m(\infty) = 0$, i.e. the mass of an electron is of wholly electromagnetic origin.

Thus analysis shows (see also refs 1-3) that it is possible to find in quantum electrodynamics expressions for the fundamental functions, which directly satisfy the equations, in such a way that regularisation simply amounts to a change of notation:

$$\begin{aligned} \Gamma_\mu &= \exp \left[\frac{e_1^2}{4\pi} \int_0^L d_1(\xi) d\xi \right] \cdot \Gamma'_\mu, \\ G &= \exp \left[- \frac{e_1^2}{4\pi} \int_0^L d_1(\xi) d\xi \right] \cdot G', \\ D &= \frac{e^2}{e_1^2} D', \end{aligned}$$

where Γ'_μ , G' and D' are functions that transform when p^2 is small to functions for free particles. At the same time it is possible to obtain asymptotic expressions for momenta $e^2 \ln(p^2/m^2) \sim 1$, which cannot be obtained by using ordinary perturbation theory.

The applicability of the formulae obtained is limited by two factors: firstly there is the role of gravitation, and secondly, as already mentioned in ref. 3, when p^2 increases the effective charge increases and becomes of the order of unity when p^2 is sufficiently large, thereby destroying the conditions for validity of the theory. The latter factor is of great fundamental significance, since in all probability it refers to other interactions, and not merely to electrodynamics. It seems that, when p^2 is large, we always arrive in the region of "strong coupling", even though we have weak coupling in the region of moderate energies.

As applied to processes at ordinary energies, the formulae obtained naturally yield the results of the first non-vanishing approximation of perturbation theory. The scattering of light by light is an exception to this. In ordinary perturbation theory such scattering is described by a diagram in the form of a square of four electron lines with four photon tails. It is easily seen that joining of two squares by two photon lines (so that four free tails again remain) yields an effect differing by e^4 , but containing the second power of the logarithm,

i.e. of the same order of magnitude from the point of view of our theory. Hence the solution of the problem must be given in this case by a summation of diagrams with an infinite set of squares, joined by photon lines.

As regards effects at large energies, their evaluation on the basis of the proposed theory requires an additional evaluation of the quantities $\Gamma_e(p, q; k)$ for cases when one of p^2 , q^2 , or k^2 is large by comparison with the other two. Furthermore, it can be seen that, in a number of cases (e.g. the Compton effect at very large energies), the outside photon lines yield an effect of the same order of magnitude from the point of view of our theory, which requires the formation of new integral equations in accordance with the same principles.

In conclusion, we express our thanks to I. Ya. Pomeranchuk, A. D. Galanin and B. L. Ioffe for valuable discussions.

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82. ON THE ANOMALOUS ABSORPTION OF SOUND NEAR A SECOND ORDER PHASE TRANSITION POINT

THE general theory of phase transitions enables us to consider the phenomenon of sound absorption in the vicinity of a λ -point. We shall show that in the vicinity of a λ -point, an anomalously large absorption of sound should be observed.

In the vicinity of a second-order transition point, the thermodynamic potential Φ can be represented in the form

$$\Phi(p, T, \eta) = \Phi_0(p, T) + A(p, T) \eta^2 + C(p, T) \eta^4 + \dots \quad (1)$$

Here $C > 0$; $A < 0$ in the low temperature (asymmetrical) phase; $A > 0$ in the high temperature (symmetrical) phase; the transition point is determined by the condition $A(p, T) = 0$. For a given pressure in the vicinity of the transition point, the function $A(p, T)$ is expanded in a series in the difference $T - T_\lambda$ (T_λ is the transition temperature)

$$A(p, T) = a(p) (T - T_\lambda). \quad (2)$$

The parameter η characterised the degree of asymmetry. In the symmetrical phase $\eta \equiv 0$ and in the asymmetrical phase the parameter η is non-vanishing. The dependence of η on temperature in the vicinity of the λ -point is determined from the condition of minimality of the potential Φ . We shall find the non-vanishing value of η_0^2 from the condition $\partial \Phi / \partial \eta = 0$:

$$\eta_0^2 = -\frac{A}{2C} = \frac{a}{2C} (T_\lambda - T), \quad (3)$$

corresponding to some equilibrium state of the asymmetrical phase. Suppose that a system found in the asymmetrical state is split off from the equilibrium state. The rate of approach of the parameter η to the equilibrium value η_0 is determined by the transport equation

$$\frac{d\eta}{dt} = \gamma \frac{\partial \Phi}{\partial \eta}; \quad (4)$$

γ is a transport coefficient which we shall assume does not possess any singularities in the vicinity of T_λ . Expanding the derivative $\partial \Phi / \partial \eta$ in a series in the

Л. Д. Ландау и И. М. Халатник, Об аномальном поглощении звука вблизи точек фазового перехода второго рода, Доклады Академии Наук СССР, 96, 469 (1954).

difference $\eta - \eta_0$ by means of equation (4), we find:

$$\frac{d\eta}{dt} = 2\gamma C \eta_0^2 (\eta - \eta_0) = -\frac{1}{\tau} (\eta - \eta_0). \quad (5)$$

Hence, it follows that the relaxation time τ , which characterises the establishment of equilibrium in the asymmetrical phase, is equal to

$$\tau = -\frac{1}{2C\gamma\eta_0^2}. \quad (6)$$

In the vicinity of the λ -point we have

$$\tau = \frac{1}{\gamma a(T_\lambda - T)}. \quad (7)$$

Consequently, the relaxation time in the vicinity of the transition point increases rapidly according to the rate of approach to the λ -point. Establishment of equilibrium in the vicinity of the λ -point thus takes place extremely slowly, and this should lead to a noticeable attenuation of sound. Quite obviously, this anomalous attenuation of sound will be observed only in the asymmetrical phase, i.e. below T_λ (for a given pressure). In the symmetrical phase, however, the parameter η is identically equal to zero in all states, equilibrium as well as non-equilibrium, and consequently there will be no anomalies of sound absorption.

Suppose that in the system we are considering, a sound wave of frequency ω is propagated, i.e. adiabatic periodic compression and rarefactions take place in the system. According to M. Leontovich and L. Mandelstam†, the square of the velocity of sound in the low temperature phase will be

$$c^2 = \frac{1}{1 - i\omega\tau} [c_{eq}^2 - i\omega\tau c_\eta^2]. \quad (8)$$

Here, c_{eq} is the velocity of sound for a process so slow that the system is at all times in equilibrium. This velocity in our case, is obviously equal to the equilibrium velocity of sound c_η in the low temperature phase‡.

$$c_{eq} = c_{II}. \quad (9)$$

The velocity c_η is the velocity of sound for a process so rapid that the parameter η , as a result of propagation of the sound wave, remains constant. In the vicinity of the λ -point, the velocity c_η is equal to the equilibrium velocity of sound in the high temperature phase

$$c_\eta = c_I. \quad (10)$$

Thus, we have finally

$$c^2 = \frac{1}{1 - i\omega\tau} [c_{II}^2 - i\omega\tau c_I^2]. \quad (11)$$

† See, for example, ref. 1.

‡ We denote the value of a quantity in the low temperature phase by a Roman numeral II, and one in the high temperature phase by a Roman numeral I.

The value of the coefficient of sound absorption α is equal to the imaginary part of the wave vector

$$k = \frac{\omega}{c} = \omega \sqrt{\frac{1 - i \omega \tau}{c_{\text{II}}^2 - c_{\text{I}}^2 i \omega \tau}}. \quad (12)$$

Assuming that the difference in the values of the velocities c_{II} and c_{I} is small in comparison with c_{I} , we find†:

$$\alpha = \text{Im } k = \frac{\omega^2 \tau}{1 + \omega^2 \tau^2} \frac{1}{2 c_{\text{I}}^3} (c_{\text{I}}^2 - c_{\text{II}}^2). \quad (13)$$

The relaxation time in the vicinity of the λ -point is determined by formula (7). It increases rapidly on approach to the λ -point. According to equation (13), on approaching the λ -point (for a given frequency of sound, ω) the coefficient of sound absorption also increases. However, in the immediate vicinity of the λ -point ($\omega \tau \approx 1$), the value of α attains a maximum, and then commences to decrease. Such is the general picture of the phenomenon of anomalous absorption of sound in the vicinity of the λ -point in the low temperature phase. In the high temperature phase, however, there should be no anomalous absorption of sound whatsoever.

The magnitude of the change in the velocity of sound at the λ -point, $c_{\text{II}} - c_{\text{I}}$, which enters into the formula for α , can be expressed by the usual well-known value of the change in specific heat at the λ -point.

Bearing in mind that later on we shall apply the results to the case of the λ -point in helium II, we shall obtain the unknown relationship for the transition into liquid. We know that the volume V and temperature T at the transition point are continuous (as independent variables, however, we choose the pressure p and the entropy S) i.e. their rates of change ΔV and ΔT are equal to zero:

$$\Delta V = 0, \quad \Delta T = 0. \quad (14)$$

If we differentiate these equations with respect to entropy along the transition curve, we have

$$\Delta \frac{\partial V}{\partial S} + \frac{dp}{dS} \Delta \frac{\partial V}{\partial p} = 0, \quad (15)$$

$$\Delta \frac{\partial T}{\partial S} + \frac{dp}{dS} \Delta \frac{\partial T}{\partial p} = 0. \quad (16)$$

Taking into account the equality of the derivatives, $\partial V/\partial S = \partial T/\partial p$ we eliminate the quantity $\Delta \partial V/\partial S$ from both equations:

$$\Delta \frac{\partial V}{\partial p} = \Delta \frac{\partial T}{\partial S} \left(\frac{dS}{dp} \right)^2. \quad (17)$$

† P. E. Stepanov² made use of a formula similar to (13), for analysing the data concerning the absorption of elastic vibrations in β -brass in the vicinity of the Curie point. The author, however, did not take into account the existence, in the vicinity of the phase transition point, of the temperature dependence of the value of the relaxation time.

Hence we obtain the required relationship connecting the rate of change of velocity of sound and the specific heat at the λ -point:

$$\Delta \frac{1}{c^2} = -\frac{T}{V} \left(\Delta \frac{1}{c_p} \right) \left(\frac{dS}{dp} \right)^2 \quad (18)$$

The derivative dS/dp along the curve of phase equilibrium can be expressed by the coefficient of thermal expansion $\partial V/\partial T$ and the derivative $\partial T_\lambda/\partial p$:

$$\frac{dS}{dp} = \left(\frac{dS}{dp} \right)_T + \left(\frac{dS}{dT} \right)_p \frac{dT_\lambda}{dp} = - \left(\frac{dV}{dT} \right)_p + c_p \frac{d \ln T_\lambda}{dp}. \quad (19)$$

The results obtained above relating to the absorption of sound in the vicinity of the λ -point are applicable in all cases of second order phase transitions†. We shall use them for analysing data on the absorption of first (normal) sound in the vicinity of the λ -point in helium II. Chase³ observed the anomalous increase in the coefficient of sound absorption in helium II on approach to the λ -point. Above the λ -point, however, no anomalously large absorption of sound was observed. By using the data of ref. 4 for c_p and $\partial V/\partial T$ and those of ref. 5 for the dependence of T_λ on the pressure, we find, by means of equation (18), the value of the change of velocity at the λ -point:

$$c_{II} - c_I \approx 12.5 \text{ m/sec} \quad (20)$$

Further, comparing the experimental values of the absorption coefficient at different temperatures (the frequencies used in ref. 3 were equal to 2 and 12.1 Mc/s) we find the value of the relaxation time τ . The experimentally-obtained value of the relaxation time follows very well from the temperature dependence given by formula (7):

$$\tau = \frac{4 \times 10^{-13}}{T_\lambda - T} \text{ sec.} \quad (21)$$

Within the temperature range in which the measurements were carried out, the value of $\omega \tau$ did not exceed 10^{-2} . Consequently, the term $\omega^2 \tau^2$ in the denominator of expression (13) need not be taken into account. The term mentioned should have an effect only at $T_\lambda - T \approx 3 \times 10^{-5} \text{ K}$ (for the frequency of sound considered). Above this value of T , a reduction in the value of the sound absorption coefficient α should be expected.

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† In particular, they should be applicable for transitions from a normal into a super-conducting state, where the anomalous absorption of sound in the vicinity of the transition point has not yet been experimentally observed by anyone.

83. A STUDY OF FLOW SINGULARITIES USING THE EULER-TRICOMI EQUATION

We consider the reflexion of a weak discontinuity (discontinuity of the first derivatives of the velocity with respect to the co-ordinates) from a sonic line. We take their point of intersection as the origin of the x, y co-ordinates, and the x -axis along the direction of the velocity at this point; it now corresponds to the origin in the η, θ plane of the hodograph†. Let the incident discontinuity correspond to the characteristic Oa (see Fig. 1) in the plane of the hodograph. Continuity of the x, y co-ordinates on the discontinuity implies continuity of the first derivatives Φ_η, Φ_θ . On the contrary, the second derivatives of $\Phi(\eta, \theta)$ are expressible in terms of the first derivatives of the velocity with respect to the co-ordinates and must therefore have a discontinuity. On the other hand, the functions Φ themselves do not need to have any singularities on the characteristic Oa in the regions 1 and 2 on either side of it. Such a solution of the Euler-Tricomi equation $\Phi_{\eta\eta} - \eta \Phi_{\theta\theta} = 0$ has the form

$$\Phi = -A\eta\theta - B\theta^{1/6}\zeta^2 F\left(\frac{13}{12}, \frac{19}{12}, 3; \zeta\right) \quad (1)$$

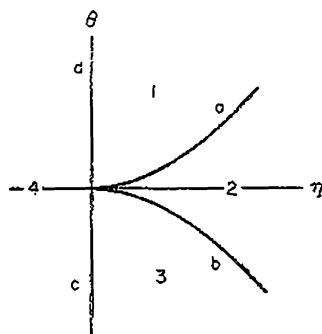


FIG. 1.

in the domain 1, and the same form but with a different constant C instead of B in domain 2. The first term here is the particular solution of lowest power in θ and η , which leads to no singularities of the flow in the physical plane (cf. ref. 1, section 111). The second term does lead to a jump of the second deriv-

Л. Д. Ландау и Е. М. Лифшиц, Исследование особенностей течения при помощи уравнения Эйлера — Трикоми, Доклады Академии Наук СССР, 96, 725 (1954).

† The notation is the same as in ref. 1, Sections 110 and 111.

atives of Φ when the first are continuous (this is the second term in ref. 1 (110.6) with $k = 11/12$). The notation $\zeta = 1 - (4\eta^3/9\theta^2)$ is introduced in (1).

The second characteristic Ob corresponds to the discontinuity reflected from the sonic line. The form of the function Φ close to it is established by the analytic continuation of functions (1) in accordance with formulae (110, 11-13) of ref. 1 (in which we have to put $k = 11/12 + \varepsilon$, then let ε tend to zero). The following expression, in which terms up to and including order ζ^2 are retained, is obtained after some calculations for the function Φ in domain 4 close to the characteristic Ob:

$$\begin{aligned}\Phi &= -A\theta\eta + \frac{B}{\pi}(-\theta)^{11/6} \left\{ \zeta^2 \ln|\zeta| - \frac{2^9 \times 3^4}{335} + \frac{288}{7}\zeta \right. \\ &\quad \left. + \left[2\gamma - 2\ln 2 + \frac{2781}{770} + \psi\left(\frac{17}{6}\right) + \psi\left(\frac{13}{6}\right) \right] \right\} \\ &= -A\theta\eta + \frac{B}{\pi}(-\theta)^{11/6} \{ \zeta^2 \ln|\zeta| - 108 + 41.1\zeta + 4.86\zeta^2 \} \quad (2)\end{aligned}$$

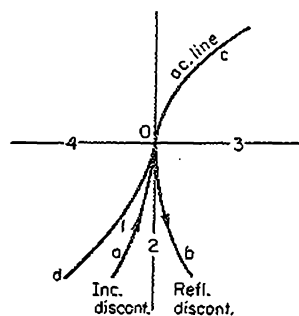


FIG. 2.

($\psi(z) = \Gamma'(z)/\Gamma(z)$, γ is Euler's constant). A similar transformation of the function Φ in domain 1 from the neighbourhood of Oa to the neighbourhood of Ob yields the same expression (2) with $\frac{1}{2}C$ instead of B . The condition for continuity of the x, y co-ordinates on the reflected discontinuity therefore leads to the relationship $C = 2B$.

The condition for the Jacobian $\Delta = \partial(x, y)/\partial(\theta, \eta)$ to be positive is automatically fulfilled close to Oa (where $\Delta \cong A^2$), whilst the evaluation with the aid of (2) gives, close to Ob,

$$\Delta \cong -\frac{16}{\pi} \left(\frac{3}{2} \right)^{1/6} AB\eta^{1/4} \ln|\zeta|.$$

Since $\ln|\zeta| \rightarrow -\infty$ on approaching Ob, the condition $\Delta > 0$ leads to $AB > 0$.

On the incident discontinuity (characteristic $\theta = + (2/3)\eta^{2/3}$) we have $x = \Phi_\theta = -A\theta$, $y = \Phi_\eta = -A\eta$. Since the gas velocity is along the positive direction of the x -axis, the discontinuity, in order to be "arriving" relative to the point

of intersection, must lie in the half-plane $x < 0$. It follows from this that the constant A , and hence B , C also, are positive. The equations of the line of discontinuity in the physical plane will be

$$-y = \left(\frac{3}{2}\right)^{2/3} A^{1/3} (-x)^{2/3} = 1.31 A^{1/3} (-x)^{2/3}. \quad (3)$$

The equation of the reflected discontinuity is obtained by differentiating function (2) with respect to θ and η , and reads

$$-y = 1.31 A^{1/3} x^{2/3} + 115 B A^{-5/6} x^{5/6} \quad (4)$$

where the correction term, absent in (3), is retained.

The shape of the transition line is determined with the aid of the analytic continuation of the function (1) in the region close to the $\eta = 0$ axis. Close to the upper half-plane, we get

$$\Phi = -A\eta\theta - B|\theta|^{11/6} \frac{2\Gamma(\frac{1}{6})}{\Gamma(\frac{17}{12})\Gamma(\frac{23}{12})} = -A\eta\theta - 6.25B|\theta|^{11/6}, \quad (5)$$

and close to the lower semi-axis, the same expression with $\sqrt{3}B$ instead of B (only the terms of lowest powers in η are retained here). On differentiating with respect to θ and η , then putting $\eta = 0$, we get the following equations for the two branches of the transition line:

$$y = -11.4BA^{-5/6}(-x)^{5/6}, \quad y = 11.4\sqrt{3}BA^{-5/6}x^{5/6}. \quad (6)$$

The relative disposition of all the lines is shown in Fig. 2, where the same notation is used for the lines and domains as in Fig. 1.

Discontinuity of the derivatives of the velocity on the incident discontinuity can be characterised by a jump of the derivative $(\partial\eta/\partial x)_y = -\Phi_{\theta\theta}/A$. Evaluation with the aid of (1) gives

$$\left[\left(\frac{\partial\eta}{\partial x}\right)_y\right]_2^1 = 8\left(\frac{3}{2}\right)^{1/6} \frac{B}{A^2} \eta^{-1/4} = 8.56 BA^{-7/4} |y|^{-1/4}. \quad (7)$$

On a reflected weak discontinuity, the derivatives of the velocity do not in general have a jump, though the velocity distribution has a special type of transcendental singularity. Having found the co-ordinates x , y from function (2) (retaining only the first term in the brackets), we can write the dependence of η on x for a given y in the following parametric form:

$$\eta = \frac{|y|}{A} + \frac{x - x_0}{2\sqrt{A|y|}} - \frac{1}{6A} |y| \zeta, \\ x - x_0 = \frac{1}{3\sqrt{A}} |y|^{3/2} \zeta - \frac{8}{\pi} \left(\frac{2}{3}\right)^{5/6} \frac{B|y|^{7/4}}{A^{7/4}} \zeta \ln |\zeta|, \quad (8)$$

where ζ plays the role of parameter, and $x_0 = x_0(y)$ is the equation of the line of discontinuity in the physical plane.

The Euler-Tricomi equation should be applicable to the problem of the possible termination of a shock wave from below relative to the flow, when it

intersects with a sonic line (the point O on Fig. 3a, with respect to which the shock wave is "arriving")†; the intensity of the shock wave would be small close to such a point, i.e. the flow would be near-sonic. We have considered different variants of the flow picture, differentiated in particular by the number of characteristics terminating at the point O (it needs to be remembered here that there need not be any singularities on the "arriving" characteristics, since these could only be produced by extraneous factors, having no relation to the termination of the shock wave). However, on none of these variants did it prove possible to construct a solution of the Euler-Tricomi equation which would satisfy all the necessary conditions. It would also seem that no solutions exist that would correspond to the termination of the shock wave along with the sonic line at their point of intersection (Fig. 3b). We suppose therefore

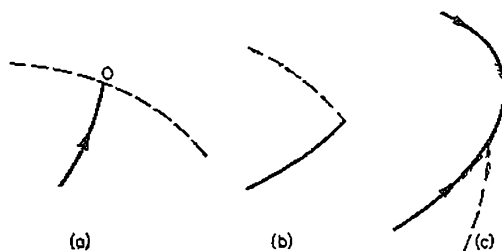


FIG. 3. The full-drawn lines are the shock waves, the dotted line is the sonic wave.

(though we have no strict proof of this assertion) that a shock wave cannot "terminate" and must consequently either carry on to infinity or bend round, as illustrated in Fig. 3c, in such a way as to be "issuing" relative to both its ends. This last case must always hold for a shock wave originating in a local supersonic zone.

Some remarks may also be made regarding the shape of the local supersonic zone formed for a flow with $M < 1$. Up to the point where a shock wave appears, this zone must wholly border on the surface of the body in the flow. Otherwise the sonic line, passing behind the body, would be perpendicular to the current line at at least one point, with its concavity turned towards the supersonic zone; whereas the sonic line can only be concave towards the subsonic region at such a point (see e.g. ref. 1, section III). On the other hand, if the supersonic region passed behind the body (which must always occur for values of M sufficiently close to 1), this boundary must be made up both of the sonic line and of that part of the shock wave intersecting it (though the intensity of the shock wave is by no means zero at the points of intersection).

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† As regards the "start" of the shock wave, it can only occur at a point of supersonic flow, and the investigation of its properties does not present any particular difficulty (cf. e.g., ref. 1, section 107).

84. ON THE QUANTUM THEORY OF FIELDS

IN the last seven or eight years, owing to the notable work of Schwinger, Feynman, Dyson and others, quantum electrodynamics has made very great progress. Perturbation theory has been reconstructed in a relativistically invariant way, and it has been shown that this can be used as a foundation on which to develop an unambiguous procedure leading to finite expressions for the effects for which infinities were previously obtained. Moreover, by means of the same procedure it has been possible to calculate, in closed form, the corrections to any order. The striking agreement between the results thus obtained and experiment has fully confirmed the correctness of the methods which have been developed.

Thus an electrodynamics, based on the concept of a point interaction described by the product of operators at one and the same point in space, could be freed from infinities, and in consequence its field of applicability was extended in a remarkable manner.

The situation in the theory of π -meson interaction is very different. For some types of interaction (in particular, pseudovector coupling), the removal of the infinities in the same way as for electrodynamics was found to be quite impossible. In the case of pure pseudoscalar coupling, where the plan could be formally carried out, a comparison with observation has shown that the value of the coupling constant renders impossible any application of perturbation theory. On the other hand, the existing theory offers no possibility of calculating quantities except in the form of series in perturbation theory. These are most probably asymptotic, and so, for large values of the coupling constant, they give no information concerning the quantities they represent.

However, even in electrodynamics, the method at present existing for the removal of infinities has retained, to a considerable extent, the nature of a recipe, and this is a serious obstacle to the further development of the theory. We shall therefore attempt, first of all, to explain the existing theory without using quantities which are actually infinite, and we shall thus be able to determine, at the same time, the limits of applicability of this theory, which is usually, but unjustifiably, assumed to be unrestricted.

Since the consideration of a point interaction leads at once to infinities, it appears reasonable to regard it as the limit of some "smoothed-out" interaction with a finite radius, as this radius decreases to zero. In doing so, we have no reason to suppose that the constant e_1 , which appears as a coefficient in the interaction and is the "intrinsic" charge of the electron in quantum electro-

L. Landau, On the quantum theory of fields, *Niels Bohr and the Development of Physics* (ed. W. Pauli), Pergamon Press (London, 1955), p. 52.

dynamics, is independent of the radius of interaction. Furthermore, the dependence of this constant on the radius of interaction should be so defined that the final result of the theory (the expressions for physical effects) is independent of the radius of interaction, since this was introduced as an auxiliary quantity and is therefore devoid of physical significance.

An approach of this kind means, essentially, the rejection of any unjustified consideration of point interactions by means of a δ -function.

All the divergences in electrodynamics are logarithmic, as is well known. The only exception is the quadratic divergence in the intrinsic mass of the photon. This latter fact, however, is not a serious objection, for it is easy to see that the appearance of a mass of the photon, under the influence of its interaction with charged particles, contradicts the law of conservation of charge. If the "smoothing-out" is effected in such a way that the law of conservation of charge is not thereby violated the corresponding expressions should reduce to zero identically.

The logarithmic divergence of the integrals in perturbation theory, which are taken over the momenta of virtual particles, always occurs in the range $p \ll k \ll \Lambda$, where k is the variable of integration, $\Lambda \sim 1/a$ is the upper limit of integration, a being the order of magnitude of the radius of the "smoothed-out" interaction, and p is the order of magnitude of the four-dimensional momenta considered, if it is large compared with the mass m of the electron. If $|p^2| \ll m^2$, the lower limit of the logarithmic range of integration is m . If p^2 for the electron is close to m^2 , an additional logarithmic range occurs, connected with what is called the "infrared catastrophe," which we shall touch on below.

It is well known also that the degree of the logarithmic divergence nowhere exceeds the order of perturbation theory that is being applied, that is, the quantity $\ln(\Lambda^2/p^2)$ enters any expression in a degree not greater than that of e_1^2 , the square of the charge†. We emphasise that, in such an approach to the problem, the charge e_1 is the unobservable charge of the electron, which depends on the value of Λ chosen, and in no way coincides with the physical charge e .

Thus the convergence of a series in perturbation theory is directly connected with the value of the parameter $e_1^2 \ln(\Lambda^2/p^2)$. The condition that perturbation theory is applicable over the entire range is thus

$$e_1^2 \ln\left(\frac{\Lambda^2}{m^2}\right) \ll 1. \quad (1)$$

As is well known, the results hereby obtained can be re-normalised, that is, if the physical charge of the electron is defined by its interaction with quanta of zero frequency and its mass, as the physical mass of the electron, the undetermined constant Λ disappears from the formulae for the physical effects. The quantities e and m are then expressed in terms of the "intrinsic" e_1 and m_1 in the form of series in powers of e_1^2 and $\ln(\Lambda^2/m^2)$.

† For simplicity, we shall always write in $\ln(\Lambda^2/p^2)$, with the understanding that, if $|p^2| \ll m^2$, it should be replaced by $\ln(\Lambda^2/m^2)$.

This re-normalisability of the theory is in reality only approximate, since terms of the order of p^2/Λ^2 are neglected. Since, in applying perturbation theory, Λ is restricted by the inequality (1), this error cannot possibly be made arbitrarily small. However, it is easy to see that its order of magnitude is $\exp(-\text{constant}/e^2)$, that is, it is extremely small, because e^2 is small. We note that the exponential character of the error is an additional argument in favour of the asymptotic nature of the series.

If the condition (1) is not fulfilled, the use of perturbation theory becomes impossible. The problem therefore arises of finding the fundamental quantities without the direct use of perturbation theory. In the work of L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov¹⁻⁴ such a calculation was performed on the assumption that

$$e_1^2 \ll 1, \quad (2)$$

which is a much weaker condition than (1). For the calculation it is necessary to take into account that, when the condition (2) is fulfilled, we can neglect all terms which, for a given power of the logarithm, contain higher powers of e_1^2 than the principal terms. As we have already said, the lowest power of e_1^2 is the same as the power of the logarithm. Hence, in order to obtain the first approximation, it is sufficient to confine ourselves to a combination of terms of the type $[e_1^2 \ln(\Lambda^2/p^2)]^n$.

As base functions it is natural to take, in the first place, the Green function of the electron $G(p)$ and that of the photon $D_{\mu\nu}(k)$, that is, the exact expressions for the electron and photon lines, taking into account the corrections of all orders. These functions are expressed in terms of the corresponding operators ψ and A for the electron and the vector potential. In a co-ordinate representation,

$$\left. \begin{aligned} D_{\mu\nu}(x-x') &= \frac{i}{4\pi} \langle T(A_\mu(x) A_\nu(x')) \rangle \\ G(x-x') &= -i \langle T(\psi(x) \bar{\psi}(x')) \rangle \end{aligned} \right\} \quad (3)$$

where the average is taken over the physical vacuum, and the symbol T denotes that the product is taken in the order of the time sequence of x and x' and, in the case of G , with the appropriate sign.

To determine these functions it is necessary to use also the vertex part $\Gamma_\mu(p, q; k)$ (where $q = p - k$). Here it can be shown that, to determine the functions $G(p)$ and $D(k)$ for the space vectors p and k , it is sufficient to use those Γ_μ in which all three vectors are also spatial^{1, 2}. Since G and D for time vectors can be found from G and D for space vectors by analytic continuation, it suffices to consider only "spatial" Γ_μ . In such Γ_μ , either the quantities p^2 , q^2 and k^2 are of the same order of magnitude or, if one is comparatively small, the other two are close to equality. The case where one of the three quantities is large compared with the other two, which is quite possible for non-spatial vectors, is here excluded. For such Γ_μ , the series in perturbation theory contains only logarithms of the type $\ln(\Lambda^2/f^2)$, where f^2 is the greatest of p^2 , q^2 and k^2 , and then only in powers not greater than that of e_1^2 . We note

that this does not in general hold for non-spatial Γ . In particular, in the case where k^2 is large compared with p^2 and q^2 , $\ln(k^2/p^2)$ and $\ln(k^2/q^2)$ occur in the formulae, and their common power may be double⁵ that of e_1^2 (this question has been analysed in greater detail by V. V. Sudakov⁶).

The Green function for the photon may, from considerations of relativistic invariance, be always written in the form

$$D_{\mu\nu}(k) = D_t(k^2) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + D_l(k^2) \frac{k_\mu k_\nu}{k^2}. \quad (4)$$

In using perturbation theory, it is usually supposed that in the zero-order approximation $D_t^0 = D_l^0 = 1/k^2$. As a result of the perturbations, D_t changes, whilst D_l remains equal to its zero-order value, because of the transverseness of the Dirac current (conservation of charge). Such a choice of D_l , however, is not at all convenient. It is important that, in consequence of gauge invariance, the choice of the function $D_t(k^2)$ is in general arbitrary. This means that the expressions for the physical effects are independent of D_t . The quantities which we are considering, apart from D_t , do not exhibit gauge invariance. The gauge invariance of D_t follows from the fact that the transverse components $A_\mu[A_\mu - (k_\mu k_\nu A_\nu/k^2)]$ do not vary under gauge transformation. On the other hand, the operator ψ does vary under gauge transformation, and the quantities G and Γ_μ change correspondingly.

We shall not discuss these changes here. The general theory of gauge transformations developed by I. M. Khalatnikov and the author, allows us to calculate G and Γ_μ in terms of their values for $D_t = 0$, in the case where D_t is arbitrary. In what follows, we shall therefore use the value $D_t = 0$, which considerably simplifies the formulae and is usually the most natural procedure (it corresponds directly to the Lorentz condition $\partial A_\mu/\partial x_\mu = 0$).

If we put $D_t = 0$, it can be shown that, on applying perturbation theory to the quantity Γ_μ (which is spatial,) the terms of the type $[e_1^2 \ln(\Lambda^2/p^2)]^n$, which we are considering, reduce identically to zero. The same is true of the radiative corrections to G , if we suppose $|p^2| \gg m^2$, i.e. we neglect the corrections to the mass.

To determine D_t , we can now use Dyson's equation⁷

$$D_t \left(k^2 + \frac{e_1^2}{3\pi i} \text{Tr} \left[\int G(p) \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\mu d^4 p \right] \right) = 1,$$

where we can substitute $\Gamma_\mu = \gamma_\mu$ and $G = 1/(\gamma p - m)$. We then obtain

$$k^2 D_t(k^2) = \frac{1}{1 + \frac{e_1^2}{3\pi} \ln \left(\frac{\Lambda^2}{-k^2} \right)} (k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2). \quad (5)$$

In the case $k^2 \ll m^2$ we have $\ln(\Lambda^2/m^2)$ instead of $\ln(\Lambda^2 - k^2)$ in this formula. For $k^2 = \Lambda^2$ (5) gives $D_i = 1/k^2$, which corresponds to the fact, for values of $k^2 \gg \Lambda^2$, interaction is absent, and the particle behaves as if free.

Formula (5) is, in the following familiar sense, conditional in nature. In its derivation it was supposed that the only particle which takes part in vacuum polarisation is the electron, which most probably is not true. It is very probable that the interaction, with the electromagnetic field, of the particles capable of strong non-electromagnetic interaction (nucleons, π -mesons) decreases rapidly with wave length beyond the "radius" of the particle, so that they do not make an appreciable contribution to vacuum polarisation. However, it is possible that there are particles (μ -mesons?) incapable of strong interaction, which make an important contribution to vacuum polarisation. At present, nothing can be asserted regarding the number of kinds of such particles or their properties.

If vacuum polarisation is effected only by particles with spin $\frac{1}{2}$ and charge equal to that of the electron, then, in the formula (5), a coefficient appears equal to the number ν of kinds of particle:

$$\left. \begin{aligned} k^2 D_i(k^2) &= \frac{1}{1 + \frac{\nu e_1^2}{3\pi} \ln\left(\frac{\Lambda^2}{-k^2}\right)}, \quad (|k^2| \gg m^2), \\ k^2 D_i(k^2) &= \frac{1}{1 + \frac{\nu e_1^2}{3\pi} \ln\left(\frac{\Lambda^2}{m^2}\right)}, \quad (|k^2| \lesssim m^2). \end{aligned} \right\} \quad (6)$$

Particles with charge Ze evidently make a contribution of Z^2 to ν . Particles with spin 0 make a contribution of $\frac{1}{2}Z^2$ to ν , as is shown by a comparison with perturbation theory⁸ (this question has been analysed in greater detail by L. P. Gor'kov and I. M. Khalatnikov). In the case of particles with spin 1, the divergence which arises is not logarithmic but quadratic⁹. We shall examine this situation more closely below.

An analysis of physical effects shows that the physical charge e is related to e_1 by

$$e^2 = e_1^2 \lim_{k^2 \rightarrow 0} k^2 D_i(k^2). \quad (7)$$

We therefore obtain from formula (6)

$$e^2 = \frac{e_1^2}{1 + \frac{\nu e_1^2}{3\pi} \ln\left(\frac{\Lambda^2}{m^2}\right)} \quad (8)$$

or

$$e_1^2 = \frac{e^2}{1 - \frac{\nu e^2}{3\pi} \ln\left(\frac{\Lambda^2}{m^2}\right)}. \quad (9)$$

Expressing the function D_t in terms of e^2 we have

$$k^2 D_t(k^2) = \frac{e^2}{e_1^2} \frac{1}{1 - \frac{\nu e^2}{3\pi} \ln\left(\frac{-k^2}{m^2}\right)}. \quad (10)$$

Formula (10) is evidently re-normalisable, that is, we can replace the unobservable charge e_1 by the physical charge e in every formula, by multiplying the function D_t by the renormalising factor $Z_3 = e_1^2/e^2$, whereupon all quantities will no longer contain the cut-off limit Λ or the charge e_1 related to it.

The charge e_1 defined by formula (9) is always greater than the charge e . This is natural, since vacuum polarisation should lead to a decrease of the "original" charge. This result is related to the properties of the function D_t , which increases monotonically with k^2 . This is one example of a general property of Green functions, which was found in the works of Källen¹⁰ and Lehmann¹¹.

Formula (10) satisfies the condition derived in the work of Gell-Mann and Low¹². This condition can be derived particularly simply by starting from the concepts explained here.

The above theory shows that, in calculating the base functions corresponding to some values $(k^2)_0$, it is sufficient to consider a range of values of k^2 of the order of $(k^2)_0$ and, in particular, large compared with this quantity. The range of values of $k^2 \ll (k^2)_0$ has only a negligible correction effect (actually of the order of $[k^2/(k^2)_0]^2$), since the corresponding range of integration is small, and the integral converges well for small k . We can thus "move down" from the limit Λ to the values of k in which we are interested, and take no notice, in the calculation, of much smaller values of k .

It follows from this that if we consider values of $k^2 \gg m^2$, the mass cannot appear in the formula for D_t . Hence, from dimensional considerations, the dimensionless quantity $k^2 D_t$ can depend only on the two dimensionless quantities e_1^2 and k^2/Λ^2 . On the other hand, the quantity $e_1^2 D_t$ determines the physical effects, and therefore, for a given value of the physical charge e , it cannot depend on the cut-off limit Λ . This is possible only if

$$e_1^2 k^2 D_t = f \left[\frac{-k^2}{\Lambda^2} \psi(e_1^2) \right],$$

where f and ψ are arbitrary functions, and the relation

$$\frac{m^2}{\Lambda^2} \psi(e_1^2) = \chi(e^2)$$

holds, χ being again some arbitrary function. Introducing the function φ which is the inverse of f , these relations can be conveniently written in the form

$$\varphi(e_1^2 k^2 D_t) = \frac{-k^2}{\Lambda^2} \psi(e_1^2) = \frac{-k^2}{m^2} \chi(e^2). \quad (11)$$

A comparison with formula (6) shows that, in the approximation we are considering,

$$\varphi(x) = \psi(x) = \chi(x) = e^{-3\pi/\nu x}.$$

These formulae are to be regarded as the asymptotic values of the functions for small x . The results in ref. 13 allow us to calculate the next approximation also. Here we obtain

$$\varphi(x) = e^{-3\pi/\nu x} x^{-9/4\nu},$$

$$\chi(x) = \psi(x) e^{-5/3}$$

There is no sense in refining the difference between $\psi(x)$ and $\varphi(x)$, since the nature of the "smoothing-out" is seen therein.

If we considered higher approximations, we should obtain

$$\varphi(x) = e^{-3\pi/\nu x} x^{-9/4\nu} F(x), \quad (12)$$

where $F(x)$ is a series of powers of x , and a similar expression for $\chi(x)$. We notice that the presence of an exponential term, which is not decomposed into a series, is some argument in favour of the asymptotic character of the series $F(x)$.

Formula (6) has a range applicability which is considerably wider than that of perturbation theory. However, even (6) does not make it possible to conclude the process of passing to the limit. As follows from (9), for sufficiently large Λ there is always a point where e_1 becomes of the order of unity, and the application of the approximation concerned becomes impossible.

Thus "weak-coupling" electrodynamics is a theory which is, fundamentally, logically incomplete. It might be thought that this makes it necessary to supplement it by "strong coupling" at high energies. We shall show, however, that there are serious reasons for supposing that, if we regard the physical charge e as a function of e_1 and Λ , then, for a given Λ , no increase in e_1 can lead to an increase of e above some limit, which tends to zero as Λ increases†.

To show this, we consider values of e_1 which are not too small compared with unity, and, at the same time, values of k which are small compared with Λ , such that $(\nu e_1^2/3\pi) \ln(\Lambda^2/k^2) \gg 1$. It follows then from formula (6) that $D_t = 3\pi/\nu e_1^2 k^2 \ln(\Lambda^2/k^2)$. We introduce, instead of the vector potential, the quantity $\mathfrak{A}_\mu = e_1 \mathcal{A}_\mu$. Then the interaction term in the Lagrangian will not contain the charge e_1 , while the term corresponding to the Lagrangian of the free field will contain e_1^2 in the denominator. The function \mathfrak{D}_t corresponding to the vector \mathfrak{A} takes the form

$$\mathfrak{D}_t = \frac{3\pi}{\nu k^2 \ln\left(\frac{\Lambda^2}{k^2}\right)}. \quad (13)$$

† The considerations given in what follows are due to I. J. Pomeranchuk and the author.

This expression does not contain the charge e_1 . Hence it may be thought that it is obtained essentially by neglecting the free field term in the Lagrangian. It is difficult to imagine that the legitimacy of neglecting this term would diminish as e_1 increases further.

The applicability of formula (13) for large e_1 means that the charges $e_1 \gg 1$ polarise the vacuum to such an extent that the "effective charge" $\sqrt{[e_1^2 k^2 D_t(k^2)]}$ becomes of the order of unity even if the ratio $\Lambda^2/k^2 \sim 1$. In other words, if within some radius a there is concentrated an arbitrarily large charge, then, owing to vacuum polarisation, the total charge inside a radius $2a$ will be of the order of unity.

Let us consider the physical charge e as a function of e_1 and Λ . Since we suppose that, as e_1 tends to infinity, \mathfrak{D}_t tends to the limit (13), we have for the physical charge (see equations (7) and (8)):

$$e^2 \rightarrow \frac{3\pi}{\nu \ln\left(\frac{\Lambda^2}{m^2}\right)}. \quad (14)$$

Since this expression tends to zero as $\Lambda \rightarrow \infty$, we reach the conclusion that within the limits of formal electrodynamics, a point interaction is equivalent, for any intensity whatever ($e_1^2 \rightarrow \infty$) to no interaction at all.

It is curious that a completely paradoxical situation has resulted. For 25 years it was supposed that the use of the δ -function leads inevitably to infinite interactions. However, formula (8) shows convincingly that the δ -function (e_1 independent of the cut-off radius) leads in fact to zero interaction, and even an unlimited increase of e_1 does not seem to save the situation.

Conversely, the theory considered, for a given physical charge e , seems to have a "ceiling," in that it cannot in principle be used to discuss an energy greater than Λ_k , the value of Λ for which $e_1 \rightarrow \infty$ (as we have already said, this practically coincides with the value corresponding to $e_1 \sim 1$), or consequently a distance less than $1/\Lambda_k$.

Of course, no unambiguous physical conclusions can be drawn from the result obtained, that the point interaction is zero in the case of electrodynamics. The energies Λ for which $e^2 \sim 3\pi/\nu \ln(\Lambda^2/m^2)$ are in every case very large. At these energies, the effects of gravitational interaction may exceed the electromagnetic effects, so that a discussion of electrodynamics as a closed system becomes physically incorrect. The idea is very attractive that this "crisis" in electrodynamics occurs for just those energies where the gravitational interaction is comparable with the electromagnetic. Since the effective charge in the critical range is of the order of unity, this means that

$$\kappa \Lambda^2 \sim 1,$$

where κ is the gravitational constant. This gives a value of the order of 10^{28} eV for the critical Λ . Using formula (14), we then obtain $\nu \approx 12$. From this standpoint, the value of the physical charge e of the electron would be automatically determined by the theory. At present, of course, it is impossible to say whether

these ideas have any real significance. It is quite possible, in particular, that $\nu < 12$ and that the gravitational effects appear considerably before the effective charge becomes of the order of unity†.

If particles with spin 1 take part in vacuum polarisation, the situation is essentially different. In this case, as we have said, the divergence is not logarithmic but quadratic. This leads to the result that the difficulties mentioned above arise, not for extremely large values of Λ , but for those such that $e^2 \Lambda^2 / M^2 \sim 1$, where M is the mass of a particle. The probable way out of these difficulties, given above, then becomes clearly impossible. We shall discuss below the situation which would arise in this case. It is evidently still more pronounced for particles of higher spin¹⁵. Here we merely note that there is no evidence of the existence of such particles.

Let us now pass to the mass of the electron. In order to find the relation between the physical and "intrinsic" masses, we write the Green function of the electron in the form

$$G(p) = \frac{1}{\gamma p - m(p^2)}. \quad (15)$$

For large values of p^2 , the ratio of the second term in the denominator to the first becomes totally negligible, and is always much less than the error in formula (15). Nevertheless, it is justifiable to consider this term, since, unlike the first, it is an even function of the momentum.

We now write Dyson's formula for G :⁷

$$G^{-1}(p) = \gamma p - m_1 - \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\nu D_{\mu\nu}(k) d^4k, \quad (16)$$

and substitute the expression (15) for G (for Γ_μ we write γ_μ as before; the legitimacy of this is closely examined in ref. 4). Then, after some calculations, we obtain

$$m(p^2) = m_1 + \frac{3e_1^2}{4\pi} \int_{p^2}^{\Lambda^2} m(k^2) D_t(k^2) d(k^2). \quad (17)$$

This integral equation is solved by elementary methods, and after substituting the expression (6) for $D_t(k^2)$ we obtain

$$m(p^2) = m_1 \left[1 + \frac{\nu e_1^2}{3\pi} \ln \left(\frac{\Lambda^2}{-p^2} \right) \right]^{9/4\nu} \quad \text{for } |p^2| \gg m^2. \quad (18)$$

The physical mass is obtained from this by putting $p^2 \sim m^2$, so that the relation between m_1 and m is given by the formula

$$m_1 = m \left(\frac{e^2}{e_1^2} \right)^{9/4\nu}. \quad (19)$$

According to this formula, m_1 decreases when Λ increases. This makes it reasonable to suppose that if we were able to extend the theory to $\Lambda \rightarrow \infty$,

† We do not consider here the possibility that other non-electromagnetic interactions begin to play an important part at high energies.

m_1 would reduce to zero; this would mean that the mass of the electron is electromagnetic in origin. Thus, these concepts lead to a peculiar return to the long abandoned idea of a purely electromagnetic mass of the electron.

A special case is formed by the electron Green function G and the vertex parts Γ_μ , when the squares of the momenta of the electrons approach m^2 , a circumstance which is related to the so-called "infrared catastrophe." This topic was discussed by A. A. Abrikosov¹⁶, who showed that the corresponding value of the function G is

$$m^2 G = - \frac{\hat{p} + m}{\left(1 - \frac{p^2}{m^2}\right)^{1 + (3e^2/2\pi)}}. \quad (20)$$

Contrary to general opinion, this function has at $p^2 = m^2$ not a simple pole, but a branch point. We point out that, for $D_i \neq 0$, the power of the denominator depends on $\lim_{k^2 \rightarrow 0} k^2 D_i(k^2)$.

The expression for $\Gamma_\mu(p, q; k)$ (where $q = p - k$) is different from γ_μ only in the case where both p^2 and q^2 are close to m^2 . We shall not write it out here.

As has already been said, the theory given above does not apply to $\Gamma_\mu(p, q; k)$ when the smallest of p^2 , q^2 and k^2 is very small compared with the largest, and the two largest are not very close to each other. Particularly important effects are found when the inequalities $k^2 \gg p^2$ and $k^2 \gg q^2$ hold. In this case, terms appear in the formula which depend on the product $e^2 \ln(k^2/p^2) \ln(k^2/q^2)$, and increase considerably more rapidly with k than the terms considered above. The analysis carried out by V. V. Sudakov⁶ has shown that the effects which occur here are also connected with the infrared catastrophe.

If we consider physical effects at very high energies, it is necessary to take into account the fact that the diagrams of higher orders, which are not given here (i.e. those which do not reduce to a variation of G , D and Γ in the simplest diagram), although they do not lead to divergences contain the logarithms of the ratios of the energies of various particles. Hence, in calculating the corresponding effects, we must, following the method developed above, sum the diagrams which, for a given order of perturbation theory, contain the highest powers of the logarithms. We note that here also the degree of the logarithm is usually twice that of the diagram.

In the consideration of the Compton effect, given by A. A. Abrikosov¹⁶, it is found that the chief difference compared with the elementary theory concerns the infrared catastrophe. The Compton diagram with electron termini, for which $p^2 = m^2$, reduces to zero, as would be rigorously true. Here it can be shown that if we consider the emission of additional photons, the total effective cross-section of such a Compton effect is given, to the approximation concerned†, simply by the Klein-Nishina formula. It should, however, be

† We notice that since, as we have said already, terms with squares of logarithms appear in this case, it is here assumed that the product of e^2 and the logarithm is small compared with unity, in contrast to what was assumed in the preceding part of the article.

mentioned that the amplitude of coherent scattering (scattering at $\theta = 0$) is multiplied by a factor $\exp[(4\pi/e^2) \ln^2(\omega/m)]$ increasing with the energy of the photon.

Let us now pass from quantum electrodynamics to another fundamental problem of quantum field theory, the theory of meson interactions. The situation here differs radically from that in quantum electrodynamics. Whereas quantum electrodynamics permits us to calculate even small corrections, meson theories have essentially given no quantitative results at all which are correct.

The chief cause of this failure of meson theories is the strength of meson interactions. The interaction of an electron with an electromagnetic field is weak. The corresponding dimensionless constant, involving the charge on the electron, is very small ($e^2/3\pi \sim 1/1000$). On the other hand, all the experimental data on meson interactions show that they cannot be regarded as weak from any point of view.

Numerous attempts at quantitative calculations of meson interactions have been chiefly concerned with two types of interaction, pseudoscalar and pseudovector. Both these interactions relate to mesons which, as is shown by experiment, have zero spin and odd parity with respect to the proton (for the π^0 -meson the odd parity is absolute); these interactions are written down similarly to those of particles with an electromagnetic field, where the meson field (pseudoscalar coupling) or its gradient (pseudovector coupling) plays the part of the vector potential. The pseudovector coupling cannot be renormalised (we shall return to this question in more detail below). Attempts to apply perturbation theory to pseudoscalar coupling have led only to the result that the values of the corresponding constant, calculated from the appropriate experiments, differ from one another by factors of ten or even a hundred. We notice also that the corresponding dimensionless constant g^2 , which is analogous to the charge, is of the order of 10–15 in the majority of effects, and this completely rules out the possibility of applying perturbation theory.

Despite the fact that the consideration of weak meson interactions has thus proved physically inadequate, we shall nevertheless analyse the situation which arises here, on the grounds that such an analysis is essential in order to understand the problem itself.

We begin with weak pseudoscalar coupling. Here too only logarithmically divergent integrals occur (apart from the rest mass of the meson). We can, therefore employ the approach explained at the beginning of this article. However, the simplification which occurs in electrodynamics for $D_t = 0$ as regards the vertex part Γ , which was found to be simply equal, in the approximation considered, to its "zero-order" value, is here absent. In the approximation considered, however, it is found to be possible to sum all diagrams giving effects of the required order (just as, in electrodynamics, we discuss "spatial" Γ). It can be shown¹ that, in this case, Γ is the sum of its zero-order value γ_5 and Fig. 1, where all lines correspond to the exact Green functions G and D , and all vertices to exact values of Γ . All additional corrections to this

diagram contain the powers of g^2 which exceed the power of the logarithm by at least unity.

As well as Dyson's equations, we thus obtain a whole system of equations from which the functions G , D and Γ can be determined. The solution of this system is fairly complicated, particularly so since it is necessary, in calculating G and D , to take account in $\Gamma(p, p-k; k)$ of small corrections of order p/k for $k \gg p$, and of corrections of order k^2/p^2 for $k \ll p$.

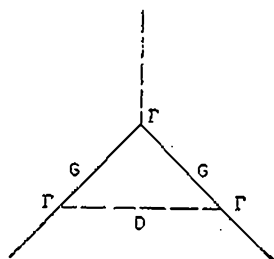


FIG. 1.

This question has been considered in detail in the work of A. A. Abrikosov, A. D. Galanin and I. M. Khalatnikov¹⁷, where it was shown that for charge-symmetric theory

$$\begin{aligned}
 G(p) &= \frac{1}{\gamma p - M} \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-p^2} \right) \right]^{-3/10} \\
 D(k) &= \frac{1}{k^2} \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-k^2} \right) \right]^{-4/5} \\
 \Gamma(p, p-k; k) &= \gamma_5 \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-f^2} \right) \right]^{1/5}
 \end{aligned} \tag{21}$$

where f^2 is the greatest of p^2 , $(p-k)^2$ and k^2 , Λ is the upper cut-off limit, and g_1 is the value of the unobservable constant of the pseudoscalar interaction. All these formulae are written for $p^2 \gg M^2$, $k^2 \gg M^2$, $f^2 \gg M^2$. If these inequalities do not hold, we must write $\ln(\Lambda^2/M^2)$ in each formula.

This theory can be re-normalised. The value of the physical constant g which characterises the interaction at small energies is obtained, by comparison with physical effects, in the form

$$g^2 = g_1^2 (\gamma_5 \Gamma)^2 [(\gamma p - M) G]^2 (k^2 D), \tag{22}$$

where all the quantities are taken for small energies. This gives

$$g^2 = \frac{g_1^2}{1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{M^2} \right)} \tag{23}$$

or

$$g_1^2 = \frac{g^2}{1 - \frac{5g^2}{4\pi} \ln\left(\frac{\Lambda^2}{M^2}\right)}. \quad (24)$$

Substitution of (24) in (21) gives

$$G(p) = \frac{1}{\gamma p - M} \left[\frac{g_1^2}{g^2} \left(1 - \frac{5g^2}{4\pi} \ln\left(\frac{-p^2}{M^2}\right) \right) \right]^{-3/10},$$

$$D(k) = \frac{1}{k^2} \left[\frac{g_1^2}{g^2} \left(1 - \frac{5g^2}{4\pi} \ln\left(\frac{-k^2}{M^2}\right) \right) \right]^{-4/5},$$

$$\Gamma(p, p - k; k) = \gamma_3 \left[\frac{g_1^2}{g^2} \left(1 - \frac{5g^2}{4\pi} \ln\left(\frac{-f^2}{M^2}\right) \right) \right]^{1/5}.$$

These formulae, disregarding the inessential constant factors, depend only on g , and not on g_1 or Λ .

The formulae (23) and (24) which re-normalise g coincide exactly, apart from the coefficient, with formulae (8) and (9) which re-normalise the charge. In particular, it follows from them that g_1 increases with Λ , and this corresponds to a strengthening of the interaction as the energy increases. However small the value of g , we always reach a region, for sufficiently large Λ , where $g_1 \sim 1$, that is, the interaction cannot be regarded as weak.

While discussing the pseudoscalar theory of meson interaction, let us examine also the problem of the scattering of one meson by another. It is generally asserted that this phenomenon, not being re-normalisable, cannot be considered within the limits of the theory. To remove the non-renormalisability, an attempt is made to introduce additional terms into the Lagrangian¹⁸, but then a new undetermined constant appears in the end.

Let us see how the problem of the scattering of one meson by another appears from the standpoint of the present article. In applying perturbation theory, the amplitude of scattering of one meson by another at small energies apart from a numerical factor, is taken as $g_1^4 \ln(\Lambda^2/M^2)$. This result is obviously not re-normalisable, that is, for a given value of g_1 it depends also on the cut-off limit Λ . This non-renormalisability, however, must not be regarded as a defect of the theory, but has a definite physical significance. The introduction of the cut-off means that virtual particles with energies greater than Λ are excluded from consideration. The re-normalisability, for instance, of the scattering of mesons by nucleons should really be regarded only as signifying that only virtual particles with energies of the order of those of the colliding particles take part in this phenomenon. In the scattering of mesons by mesons, however, virtual particles of larger energies also take part.

It is clear that, to obtain a concrete result, it is necessary that the part played by particles of various initial energies should diminish from some energy onwards. If a theory of pseudoscalar meson coupling can exist, then, as we have seen, the weak coupling must pass into a strong one at some energy.

It is natural to suppose that the boundary where the weak coupling becomes strong is the most important energy in this case.

Thus we arrive at the following method of solving the problem. It is necessary to obtain an expression for the scattering of one meson by another, taking into account all corrections of relative order $[g_1^2 \ln(\Lambda^2/M^2)]^n$, and to substitute in it $g_1^2 \sim 1$ and $\ln(\Lambda^2/M^2) = 4\pi/5g^2$. However, it is found that, to take into account corrections of the order required, it is necessary to sum a set of "parquet" diagrams consisting of an infinite number of nucleon squares joined by meson lines, and it is laborious to find the corresponding formulae.

Let us now examine the problem of pseudovector coupling. This coupling differs noticeably from pseudoscalar coupling in the following respect. In pseudoscalar coupling (as in electrodynamics), the coefficient in the interaction operator is a direct measure of the coupling strength. It is clear from a comparison with pseudoscalar coupling that in pseudovector theory a similar part is played not by the coefficient f in the interaction operator, but by $f k/M$. In other words, pseudovector coupling with a small constant f , unlike pseudoscalar coupling for a given g_1 , automatically gives strong coupling at high energies.

This state of affairs has the result which is usually called the non-renormalisability of pseudovector coupling. All the divergences here are not logarithmic, but quadratic, and the absence of the logarithmic situation completely changes the character of the theory. The characteristic feature of the variants considered above was the very slow change of the coupling strength with energy. Between perturbation theory and the strong coupling lay a region of "renormalisable" theory, which was large for small g . In the case of pseudovector coupling, the region between perturbation theory and the strong coupling is completely absent. We note that the position is quite similar to that in the electrodynamics of particles with spin 1 (and probably with higher spins).

Thus we reach the conclusion that, in all the variants of meson theory, we inevitably find strong coupling at high energies. Since meson couplings are in reality not weak even at energies of the order of Mc^2 , this means that the coupling becomes strong even for $E \gg Mc^2$. Here we are speaking, not of a coupling of a given intensity, but of the increase of the effective coupling with energy. The construction of such a theory of strong coupling at high energies appears to be the main problem in this field. The theory of meson showers evolved by Fermi¹⁹ and further developed by the author²⁰ shows that the theory of strong coupling must lead ultimately to a hydrodynamic picture. The consideration of diagrams with a small number of lines in the case of strong coupling is clearly insufficient, and diagrams with a large number of lines must be an important factor; the future theory must relate such diagrams to the equations of relativistic hydrodynamics.

There is, however, another possibility. We have seen in the case of electrodynamics that a point interaction can lead to the absence of any interaction, even if its intensity increase without limit. The possibility cannot be excluded that this is a general property of point interactions. In this case, the construction

of meson theories is possible only by abandoning the point interaction, that is, by renouncing essentially all the methods at present existing. The great difficulties which arise in a physical "smoothing-out" of particles, as opposed to a purely formal "smoothing-out" such as was discussed in the present article, are well known. In this case, therefore, the theory of meson interactions would draw a blank.

We emphasise that the physical "smoothing-out" related to the introduction of some "fundamental length" of the order of 10^{-13} cm must inevitably have some effect on electrodynamics, although at these energies no logical difficulties arise. Unfortunately, the chief electrodynamic effects at high energies, namely bremsstrahlung and pair production by photons, occur (in the system where the electron is at rest), at energies of the order of its rest mass, so that the study of these phenomena at high energies can give no information in this direction. An elementary calculation shows that a fundamental length of the order of 10^{-13} cm must lead to important changes in the Compton effect or in the annihilation of positrons at energies of the order of 10^{11} eV. The study of these phenomena might be of the greatest importance to theoretical physics.

It is great pleasure to me to contribute this article to the present volume dedicated to Niels Bohr, the great physicist whose pioneer work has determined the immense progress, of the modern quantum theory.

Note added in proof. Since this paper was written I. J. Pomeranchuk brought forward new arguments in favour of the absence of physical interactions for point particles. He succeeded in proving this statement rigorously in electrodynamics (as compared to the not quite rigorous considerations given in the text). He gave also strong arguments in favour of a similar result for pseudoscalar meson coupling. These important results give additional strength to the point of view that meson theories cannot be constructed without deep changes in the basic principles of modern theoretical physics but of course cannot completely exclude the other possibility.

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85. ON THE ROTATION OF LIQUID HELIUM

According to the basic concepts of the theory of liquid helium II,¹ the "superfluid" portion of the bulk liquid can execute only potential motion and, in particular, cannot participate in the rotation of the liquid as a whole (for example, in a cylindrical vessel, rotating about its axis). It is well known, that this theoretical conclusion was verified by experiments carried out by E. L. Andronikashvili, thereby enabling the ratio of ρ_s/ρ_n to be measured directly as a function of temperature.

In these experiments, the liquid was set into a rotatory, oscillatory motion with a large period (and, correspondingly, with a small linear velocity). Later, experiments were also carried out with fast, uniform rotation of liquid helium in a cylindrical vessel^{2, 3}; in Andronikashvili's experiments, the linear velocity of the liquid attained 40 cm/sec, and in Osborne's experiments it reached 70 cm/sec. As a result of this, however, it was found that the depth of the meniscus formed corresponds to a rotation of the entire mass of the liquid, and not only to a rotation of its "normal part".

The aim of the present notes is to indicate the possible explanation of these experimental results, which arises in a natural manner from a consideration of thermodynamical stability.

As a result of the rotation (with a given angular velocity Ω) of the normal liquid, thermodynamical equilibrium is attained when the liquid is rotating as a whole. When liquid helium II is rotating, however, the carrying along of the whole of its mass should require the transition of liquid helium II into the non-superfluid phase (helium I), which should be associated with a large expenditure of energy, and therefore does not correspond to thermodynamical equilibrium. But on the other hand, rotation of only the normal part of the mass is not energetically most favourable.

It is natural to suppose that the most favourable state corresponding to thermodynamic equilibrium will be a "layered" structure of the rotating liquid, as follows. As a result of the rotation of the vessel, the normal part of the bulk helium rotates as a whole, but relative to the superfluid motion, the cylindrical volume of the liquid is subdivided into a number of coaxial cylindrical layers, between which are confined surfaces of tangential discontinuities in the superfluid velocity. In each of these layers, a superfluid rotatory motion takes place with circular streamlines and with a velocity distributed according to the law

$$v_s^{(i)} = \frac{b_i}{r}, \quad (1)$$

where r is the distance to the axis of the liquid and b_i is a constant which is different for different layers (which we label with the suffix i). This distribution satisfies the equation of continuity of an incompressible liquid ($\text{div } \mathbf{v}_s = 0$), and at the same time also the condition of potentiality ($\text{curl } \mathbf{v}_s = 0$), necessary for superfluid motion. At the boundaries of the layers, the velocity undergoes a discontinuous change in its magnitude. The presence of just one such discontinuity is essential in any case for the existence of potential rotation; the distribution in equation (1) cannot be extended to the entire bulk of the liquid, since on the axis of the cylinder the velocity should become infinite.

Rotation of the superfluid part of the bulk liquid increases its effective moment of inertia and as a result it is energetically beneficial. The formation of a surface discontinuity is acting, however, in the reverse direction, a jump in velocity indicates the presence of a "surface rotation" of velocity, i.e. it represents the breakdown of the flow potentiality and thus, of some "local" destruction of superfluidity. A finite expenditure of energy is required, which can be described, from the macroscopic point of view, by assigning to the surface of discontinuity a certain surface tension α . It is natural to assume that with small values of the velocity jump, α will tend towards a constant limit: this should imply the necessity of a finite expenditure of energy for every "local" breakdown of superfluidity.

Definite dimensions of the layers correspond to the thermodynamic equilibrium for every given value of Ω . According to the extent of increase of Ω , the layer thickness, naturally, decreases, and at sufficiently high velocities of rotation the layers become very thin. From the outside, this motion will completely imitate the rotation of the mass of the liquid as a whole, which is also observed, apparently, experimentally. The presence of a true layered structure should, however, develop in a not completely smooth, "jagged" surface of the meniscus.

We emphasise that the motion described as a uniform rotation corresponding to thermodynamic equilibrium is thus thermodynamically reversible, i.e. it is not accompanied by a dissipation of energy. In this respect, the destruction of superfluidity taking place here differs from the destruction which takes place as a result of the so-called critical velocities.

We shall now perform a quantitative calculation as a result of the above-made assumptions about the constancy of α . The free energy (at a given temperature) of a rotating cylinder of liquid is equal to:

$$F = \frac{\rho_n \pi R^4 \Omega^2}{4} + \pi \rho_s \sum_i b_i^2 \ln \frac{r_i}{r_{i+1}} + 2\pi \alpha \sum_i r_{i+1} + \frac{I \Omega^2}{2} \quad (2)$$

(where R is the radius of the cylinder; I is the moment of inertia of the vessel; the suffix $i = 0, 1, 2, \dots$ labels successive layers in the direction outside-inside, so that $i = 0$ corresponds to the layer lying next the wall of the vessel; r_i are the radii of the boundaries of separation between layers, so that $r_0 = R$). The first two terms represent the kinetic energy of normal and superfluid motion respectively, and the third term represents the surface energy of the discontinuities. In thermodynamic equilibrium there should be a minimum

(for a given temperature and Ω) value of $F' = F - M\Omega$, where M is the moment of momentum of the rotating system (see, for example ref. 4). In the given case

$$M = \frac{\pi R^4}{2} \varrho_n \Omega + \varrho_s \pi \sum_i b_i (r_i^2 - r_{i+1}^2) + I \Omega, \quad (3)$$

$$F' = -\frac{\varrho_n \pi R^4 \Omega^2}{4} - \frac{I \Omega^2}{2} + \pi \varrho_s \sum_i \left[b_i^2 \ln \frac{r_i}{r_{i+1}} - \Omega b_i (r_i^2 - r_{i+1}^2) \right] + 2\pi \alpha \sum_i r_{i+1} \quad (4)$$

The first two terms in the expression for F' are independent of the presence of layers, and we shall neglect them below. Minimising F' with respect to b_i , we obtain

$$b_i = \frac{\Omega}{2} \frac{r_i^2 - r_{i+1}^2}{\ln \frac{r_i}{r_{i+1}}} \quad (5)$$

(for small layer thicknesses, b tends towards Ωr^2 , which corresponds to rotation of the liquid as a whole). The corresponding value of F' is

$$F' = 2\pi \alpha R \sum_i \left\{ x_{i+1} - \lambda \frac{(x_i^2 - x_{i+1}^2)^2}{\ln \frac{x_i}{x_{i+1}}} \right\}, \quad (6)$$

where the dimensionless quantities $x_i = r_i/R$ and the dimensionless parameter

$$\lambda = \frac{\varrho_s \Omega^2 R^3}{8\alpha}$$

are introduced.

Let us consider the limiting case $\lambda \ll 1$ (slow rotation). In this case, it is found that $1 \gg x_1 \gg x_2 \gg \dots$. In connection with this, each successive term of the summation in equation (6) turns out to be small relative to the preceding one, and in order to determine each value of x_{i+1} , it is sufficient to minimise one term of the summation for given values of the preceding x_i . If we introduce the relation $\xi_{i+1} = x_{i+1}/x_i \ll 1$, we can write an individual term of the summation in equation (6) in the form $x_i [\xi_{i+1} + \lambda x_i^3 / \ln \xi_{i+1}]$, and form its minimality condition with respect to ξ_{i+1} , we obtain

$$\sqrt{\xi_{i+1}} \ln \frac{1}{\sqrt{\xi_{i+1}}} = \frac{\sqrt{\lambda} x_i^3}{2}. \quad (7)$$

Thus, even for an arbitrarily small velocity of rotation, an infinite number of layers occurs, the radii x_i of which are asymptotically concentrated with respect to direction towards the axis of the cylinder (actually, it is clear that the small layer thicknesses are limited by atomic distances, and therefore the

number of layers is large, but not infinite). The maximum radius is given by the relationship $-\sqrt{x_1} \ln \sqrt{x_1} = \sqrt{\lambda}/2$.

In the reverse limiting case $\lambda \gg 1$ (fast rotation), the entire volume of liquid is split up into thin layers, so that at distances r , which are not too small (in relation to R), from the axis, the thicknesses h of the layers slowly change from layer to layer. In the limit $h \rightarrow 0$, the energy F' should correspond to the liquid rotating as a whole:

$$F' = -\frac{\pi \rho_s \Omega^2}{4} \sum (r_i^4 - r_{i+1}^4) = -2\pi \alpha R \lambda \sum (x_i^4 - x_{i+1}^4).$$

Subtracting this value from equation (6) and rearranging with respect to powers of $\Delta = h/R$, we obtain for the excess energy of the layered structure, per unit interval of x (i.e. per $1/\Delta$ layers), the following expression:

$$2\pi \alpha R \left\{ \frac{x}{\Delta} + \frac{4\lambda}{3} x \Delta^2 \right\}.$$

Minimising with respect to Δ , we obtain

$$\Delta = \left(\frac{3}{8\lambda} \right)^{2/3}, \quad h = \left(\frac{3\alpha}{\rho_s \Omega^2} \right)^{1/3}, \quad (8)$$

as well as determining the unknown thickness of the layers.

Not knowing α , we can attempt to obtain an approximate estimate for the numerical value of h , substituting for it a quantity of the correct dimensions (the dimensions of $h^3 \Omega^2$) from the following macroscopic quantities: transition temperature T_λ , density ρ , velocity of sound u . By this method we obtain:

$$h^3 \Omega^2 \sim \left(\frac{k T_\lambda u^4}{\rho} \right)^{2/3}.$$

Taking for u the characteristic value of ~ 20 m/sec, we obtain $h^3 \Omega^2 \sim 0.3$. Thus, for angular velocities of 8–16 r.p.s (as in ref. 2) this estimate gives a value for h of about 0.05 cm.

The problem of a detailed picture of the raggedness of the surface of the meniscus is associated with the shape of the layers close to their emergence to the free surface of the liquid and requires special considerations.

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86. ON POINT INTERACTIONS IN QUANTUM ELECTRO DYNAMICS

At the basis of quantum electrodynamics lies the idea of point interactions between quanta and electrons. It is well known that, with such interactions direct calculations lead to infinities. In recent years a relativistically invariant perturbation theory¹ has been established and it has been shown that, based on this theory, it is possible to work out a well-defined procedure leading to finite expressions for those effects for which, previously, infinities were obtained. Another approach to a solution of the equations of quantum electrodynamics was developed in ref. 2. It proceeds from a consideration of point interactions as the limit of a certain "smeared out" interaction with a finite range, when this range diminishes to zero. Furthermore, the constant e_1 standing in the form of a multiplier in the interaction is a function of the "smearing out" radius $1/\Lambda$. With increasing Λ (approach to centre of the particle) e_1 increases. This is connected with the apparent characteristic of polarisation of a vacuum to reduce every charge "introduced" in the vacuum (see also ref. 3).

Whilst $e_1^2 \ll 1$, the connection between e_1 and the re-normalised charge e is brought about by means of the D -function found in ref. 2. The Fourier component of this function is equal to

$$D(k^2) = \frac{1}{k^2} \frac{1}{1 + \frac{\nu e^2}{\nu\pi} \ln \frac{\Lambda^2}{k^2}}. \quad (1)$$

Here ν is the effective number of different weakly interacting particles in nature (see ref. 2 for more details in this connection). Formula (1) is valid when $k^2 \gg m^2$ (m is the mass of an electron). But if $k^2 \ll m^2$, then in (1) k^2 must be replaced by m^2 .

The interconnection between e_1^2 and e^2 , is well known to be obtained from the expression:

$$e^2 = e_1^2 \lim_{k^2 \rightarrow 0} k^2 D(k^2) = \frac{e_1^2}{1 + \frac{\nu e_1^2}{3\pi} \ln \frac{\Lambda^2}{m^2}}. \quad (2)$$

Expressions (1) and (2) can be simplified as

$$D(k^2) = \frac{3\pi}{v e_1^2} \frac{1}{\ln \frac{\Lambda^2}{k^2}} \frac{1}{k^2}, \quad (3)$$

$$e^2 = \frac{3\pi}{v} \frac{1}{\ln \frac{\Lambda^2}{m^2}}, \quad (4)$$

when the condition is fulfilled that

$$\frac{v e_1^2}{3\pi} \ln \frac{\Lambda^2}{k^2} \gg 1. \quad (5)$$

With values of e_1^2 not very small in comparison with unity this condition is already satisfied for values of k^2 , which are not vanishingly small in comparison with Λ^2 .

The inverse proportionality of the D -function to e^2 implies the possibility of neglecting the action of a free magnetic field in the Lagrangian of a system of electrons and quanta†.

Actually, $e_1 \Lambda$ enters into the interaction of a field with electrons and, in the free field, Λ itself. If the action of the free field is neglected, the mean value for the physical vacuum of

$$\langle T A(X) A(Y) \rangle \quad (6)$$

must be inversely proportional to e_1^2 . On the other hand, (6) is only distinguished by an unimportant multiplier from the D -function, which thus proves to be inversely proportional to e_1^2 . Precisely this result is also secured in (3).

If the Lagrangian of the free field ceases to play an effective role even with $e_1^2 \ll 1$, then it seems natural to think that with a further increase of e_1^2 the role of the free field will be still smaller. Therefore the D -function must be inversely proportional to e_1^2 also with $e_1^2 > 1$. Thence it follows that $e_1^2 D$ is independent of e_1^2 and therefore always has the form:

$$e_1^2 D = \frac{3\pi}{v \ln \frac{\Lambda^2}{k^2}} \frac{1}{k^2} \quad (7)$$

(if merely k^2 is not very close to Λ^2). Equations (2) and (7) lead to the following connection between e_1^2 and e^2 :

$$e^2 = \frac{3\pi}{v \ln \frac{\Lambda^2}{m^2}} \rightarrow 0 \quad \Lambda^2 \rightarrow \infty. \quad (8)$$

† In addition, it is understood that k^2 is not very close to Λ^2 .

The vanishing of e can be explained in the following manner. Let any large charge $e_1(A)$ be situated in a sphere of radius $1/A$. Then, at a distance $1/k$ only a few times greater than $1/A$ there will be a charge $\sqrt{e_1^2 k^2 D(k^2)} \approx \sqrt{[3\pi/\nu \ln(A^2/k^2)]}$ equal in order of magnitude to unity. This means that the "introduced" charge e_1 having the dimension $1/A$ is surrounded by a dense cloud, with dimensions also of order $1/A$, formed by polarised charges, as the result of which the total charge inside a radius $2/A$ will be of the order of unity. (The situation here is reminiscent of that which is brought about close to a Coulomb centre with a charge greater than $\sqrt{137}$.)

The rapid fall of e_1^2 to unity in fact denotes the absence of a wide region of strong interaction in electrodynamics discussed in ref. 2. Only the existence of this region could have made $e^2 \neq 0$.

We reach the fundamental conclusion that, from formal quantum electrodynamics, there apparently follows the equality to zero of the charge on the electron. The reservation "apparently" refers to a certain lack of vigour in the argument given here.

The result we have obtained indicates the logical openness of quantum electrodynamics. It must be emphasised, furthermore, that the insufficiency of the theory indicated here is directly caused, not by the infinities (which have been considered for the last 25 years) but by the conversion of the physical interaction to zero.

The physical consequences of internal inconsistencies of quantum electrodynamics can be different ones.

The energies A_0 at which

$$e^2 \approx \frac{3\pi}{\nu \ln \frac{A_0^2}{m^2}} \quad (8a)$$

are very large in any case. For these energies the effects of gravitational interaction can exceed the electromagnetic effects. It is a tempting idea that "a crisis" of electrodynamics arises precisely with those energies, for which gravitational is equal to electromagnetic interaction. Since the effective charge in the critical zone is of the order of unity, this corresponds to

$$\kappa A_0^2 \sim 1 \quad (9)$$

where κ is the gravitational constant, (8a) and (9) give for ν the value $\nu \approx 12$. From such a point of view the magnitude of the physical charge e would have been automatically established from the theory. If $\nu < 12$ the gravitational effects appear significantly earlier, than the effective charge of the order of unity. On the contrary, with $\nu > 12$ gravitational effects will not "save" electrodynamics because they set in too late.

The inference from this that $e = 0$ was reached by us by considering only purely electromagnetic interactions. The possibility is not excluded that considerations of non-electromagnetic interactions of particles (such as the

$\mu - e$ disintegration† etc.) can vastly change the nature of electrodynamics at high energies, which is accordingly indicated in the re-normalising of the charge.

The method described above for investigating the problem of re-normalising the charge is not transferred directly to meson theory⁴. If, however, the vanishing of e is a reflection of the common properties of every point interaction, then present-day meson theory will prove to be fully untenable. Furthermore, it is necessary to bear in mind that because of the large value of g^2 the role of gravitation in the meson region is completely negligible. For this very reason, in meson theories, there is no region of applicability of perturbation theory. In electrodynamics there is known to be such a region in view of the smallness of e^2 and its properties cannot be seriously changed with any change of electrodynamics in the high energy region.

The revision of existing theories for the case in which it follows from the theory that g , along with e , is equal to zero, would have demanded the introduction of completely new physical concepts. Such concepts would have had to make it possible to indicate in the theory the nature of extended elementary particles. This means that a new universal length would have been introduced into physics which would have automatically given Λ an upper limit. The existence of the new length would have had to appear not only in the meson region, but also in purely electromagnetic processes. The possibility is, therefore, not excluded that all electromagnetic processes, accompanied by large momentum transfers, proceed quite otherwise than present day theory predicts. These deviations can occur in conditions in which the momentum transfer in the centre of mass system is of the order of a few hundred MeV, if the new length is $\cong 10^{-13}$ cm. Thus an experiment of the type of the Compton effect with energies of about $(2-5) \times 10^{10}$ eV could have emerged as a solution for the present day field theory. A similar role can be played by phenomena like the annihilation of positrons of very high energy (10^{10} – 10^{11} eV), the wide-angle scattering of electrons by electrons in the same energy range and other effects under the same conditions. Important information for the theory can be given by experiments made with smaller energies, but, on the other hand, with great accuracy.

It is necessary, however, to emphasise that ordinary bremsstrahlung or pair production by γ -quanta, even in the region of super-high energies, cannot shed light on the region of applicability of electrodynamics. The fact is that, as is well known⁵, bremsstrahlung occurs in the rest system of the electron, by its interaction with soft quanta ($\omega \sim m$) completely independently of what is the initial energy of the particle in the laboratory system. Ideas of just such a type also concern the phenomenon of the pair production.

† The possible role of this interaction was brought to our attention by Ya. B. Zel'dovich.

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87. THE GAUGE TRANSFORMATION OF THE GREEN FUNCTION FOR CHARGED PARTICLES

A gauge transformation is carried out on the Green function and the vertex operator for charged particles interacting with an electromagnetic field.

GAUGE invariance arises in the field theory of charged particles interacting with an electromagnetic field. Given a gauge transformation of the potential of the electromagnetic field

$$A_\mu \rightarrow A_\mu + \partial \varphi / \partial x_\mu \quad (1)$$

(φ is an arbitrary operator function), the ψ -function of the particle is transformed as follows:

$$\psi \rightarrow \psi e^{ie\varphi} \quad (2)$$

(e is the charge, \hbar and c are taken to be unity).

We shall attempt to determine here how the Green function for the particles will change under such a gauge transformation. The Green function for particles is well known to be

$$G(x x') = \langle (\psi(x) \bar{\psi}(x'))_+ \rangle. \quad (3)$$

The brackets denote a vacuum expectation value.

The Green function $G(x x')$ will change under the gauge transformation (1). From equation (2), it can be written in the form:

$$G(x x') = G_0(x x') \langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle, \quad (4)$$

where $G_0(x x')$ stands for the Green function for the particular case when the longitudinal (in the four-dimensional sense) part of the photon's Green function is equal to zero¹. The Fourier components of the Green function for photons

$$D_{\mu\nu}(x x') = i \langle (A_\mu(x) A_\nu(x'))_+ \rangle \quad (5)$$

can be written in the general case in the form¹

$$D_{\mu\nu}(k) = 4\pi d_i \frac{1}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + 4\pi d_l \frac{k_\mu k_\nu}{k^4} \quad (6)$$

Л. Д. Ландау и И. М. Халатников, Градиентные преобразования функций Грина заряженных частиц, *Журнал Экспериментальной и Теоретической Физики*, 29, 89 (1955).

L. Landau and I. M. Khalatnikov, The gauge transformation of the Green's function for charged particles, *Soviet Phys.-JETP*, 2, 69 (1956).

The terms containing d_t and d_l represent respectively the transverse and longitudinal parts of the function $D_{\mu\nu}$. Now let us compute the vacuum expectation value of the chronological product. The longitudinal part of the Green function turns out to be the original function. It does not depend upon interactions with the field. Thereby the problem reduces to calculating the vacuum expectation value of the expression $e^{ie\varphi(x)} e^{-ie\varphi(x')}$, where the operators φ represent a free field. Expanding the free field φ into plane waves, we have

$$\varphi = \sum_k \varphi_k = \sum \lambda(k^2) (a_k e^{ikx} + a_k^+ e^{-ikx}) \quad (7)$$

a_k and a_k^+ represent respectively creation and annihilation operators for longitudinal photons; $\lambda(k^2)$, the amplitude which characterises the contribution of the photons, by the four-dimensional wave vector k . We also include in $\lambda(k^2)$ a normalisation factor. In view of the smallness of this factor (it contains the reciprocal volume) we shall, in what follows, leave out terms of higher power than λ^2 . Expressions of the type $e^{ie\varphi}$ can be expanded by means of equation (7) into the infinite product

$$e^{ie\varphi(x)} = \prod_k e^{ie\varphi_k}, \quad e^{-ie\varphi(x')} = \prod_l e^{ie\varphi_l'}. \quad (7a)$$

The operators a_k and a_k^+ corresponding to different values of the wave vectors commute with each other. Let us now take advantage of this circumstance and take into account the fact that the vacuum expectation values of terms of first order in the operator a_k (or φ_k) are zero. The vacuum expectation values of the products $\varphi_k \varphi_l'$, in accordance with what we said above, can differ from zero only if $k = l$. With the help of (7a) we obtain for the vacuum expectation value of the expression $e^{ie\varphi(x)} e^{-ie\varphi(x')}$,

$$\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle = \left\langle \left[\prod_k \left(1 + ie\varphi_k - \frac{e^2}{2} \varphi_k^2 \right) \prod_l \left(1 - ie\varphi_l' - \frac{e^2}{2} \varphi_l'^2 \right) \right]_+ \right\rangle \quad (8)$$

$$\begin{aligned} &= \prod_k \left(1 - \frac{e^2}{2} \langle \varphi_k^2 \rangle - \frac{e^2}{2} \langle \varphi_k'^2 \rangle + e^2 \langle \varphi_k \varphi_k' \rangle \right) \\ &= \exp \left\{ -\frac{1}{2} e^2 \sum_k \langle \varphi_k^2 \rangle + \langle \varphi_k'^2 \rangle - 2 \langle \varphi_k \varphi_k' \rangle \right\}. \end{aligned}$$

As we have already noted, the vacuum expectation value of the product of the two operators a_k and a_l^+ is always zero except when $k = l$. Thereby the Green function for the φ field becomes

$$\Delta_F(x x') = i \langle (\varphi(x) \varphi(x'))_+ \rangle = i \langle \varphi_k \varphi_k' \rangle_+. \quad (9)$$

Considering equation (9), we re-write equation (8) in the form

$$\begin{aligned} &\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle \\ &= \exp \{ i e^2 (\Delta_F(0) - \Delta_F(x x')) \}. \end{aligned} \quad (10)$$

Now let us apply the results we have obtained and write the final formula for the Green function $G(xx')$ for charged particles; from (4) and (10) we have

$$G(x x') = G_0(x x') \exp \{i e^2 (\Delta_F(0) - \Delta_F(x x'))\}. \quad (11)$$

Formula (11) links the Green function for charged particles $G(xx')$ with its value $G_0(xx')$ computed under the assumption that the longitudinal part of the photon D -function is equal to zero. The longitudinal part of the Green function for photons is connected with the Δ -function of the φ field, as evidenced by the expression:

$$D_{F\mu\nu}(x x') = \frac{\partial^2}{\partial x_\mu \partial x'_\nu} \Delta_F(x x'). \quad (12)$$

In accordance with equation (6), the expression for the Fourier-components becomes

$$\Delta_F(k) = 4\pi \frac{d_l(k)}{k^4}. \quad (13)$$

It is not generally possible to write in Fourier components form the Green function as it appears in formula (11). It is possible, however, to obtain a formula for the change in the Fourier components of the Green function for particles, for an infinitesimal gauge transformation of the potential. Performing a variation on equation (11) we obtain

$$\delta G(x x') = i e^2 G(x x') (\delta \Delta_F(0) - \delta \Delta_F(x x')). \quad (14)$$

We re-write this expression in Fourier components; with the help of equation (13), we find

$$\delta G(p) = \frac{i e^2}{\pi} \int \frac{\delta d_l(k)}{k^4} \{G(p) - G(p - k)\} d^4 k. \quad (15)$$

Let us apply the resulting formula to the case of spin $\frac{1}{2}$ particles. The unperturbed Green function of the free particle is given by

$$G_0(p) = \frac{1}{(\hat{p} - m)}, \quad \hat{p} = \gamma_\mu p_\mu.$$

Let $d_l(k)$ be a slowly varying function of the argument (k^2), so that the condition $(e^2/\pi) d_l(k) \ll 1$ is satisfied. Let us substitute into the right side of (15) the unperturbed value of the function $G(p)$. For slow variations of the function $d_l(k)$ in (15), significant contributions to the integral come from the region $k^2 \gg p^2$. Because of this, the term $G(p - k)$ may be neglected with respect to $G(p)$ on the right side of (15). Equation (15) is then satisfied by the Green function

$$G(p) = \frac{\beta(p^2)}{(\hat{p} - m)} \quad (16)$$

where $\beta(p^2)$ is a slowly varying function of its argument.

For the case when $p^2 \gg m^2$, from equation (15) and from the expression

$$d^4 k = \frac{i}{4} (-k^2) d(-k^2),$$

we obtain

$$\delta \beta(p^2) = -\frac{e^2}{4\pi} \beta(p^2) \int_{-p^2}^{\infty} \frac{\delta d_l}{-k^2} d(-k^2). \quad (17)$$

Let us denote $\beta(p^2)$ when $d_l = 0$ by the expression $\beta_t(p^2)$. Calculations show that in the case of spin $\frac{1}{2}$, we have $\beta_t(p^2) = 1$.¹

From equation (17) we obtain for a finite gauge transformation the well-known formula

$$\beta(p^2) = \exp \left\{ -\frac{e^2}{4\pi} \int_{-p^2}^{\infty} d_l \frac{d(-k^2)}{-k^2} \right\}. \quad (18)$$

The problem for spin zero can be solved in an analogous fashion. In this case the Green function is written in the form

$$G(p) = \frac{\beta(p^2)}{p^2 - m^2}. \quad (19)$$

Calculations show that for slowly varying d_l , the quantity $\beta(p^2)$ similarly turns out to be a slowly varying function. For a finite gauge transformation we obtain, analogously to equation (18),

$$\beta(p^2) = \beta_t(p^2) \exp \left\{ -\frac{e^2}{4\pi} \int_{-p^2}^{\infty} d_l \frac{d(-k^2)}{-k^2} \right\} \quad (20)$$

where $\beta_t(p^2)$ denotes $\beta(p^2)$ when $d_l = 0$.

Let us now turn to the gauge transformation of the vertex operator $\Gamma_\mu(x x'; \xi)$. We shall start from the vacuum expectation value of the chronological product

$$\langle \langle \psi(x) A_\mu(\xi) \bar{\psi}(x') \rangle \rangle_+ \quad (21)$$

The vertex operator $\Gamma_\mu(x x'; \xi)$ is linked to the function $B_\mu(x x'; \xi)$ by the integral

$$B_\mu(x x'; \xi) = e^2 \int G(x x'') \Gamma_\mu(x'' x'''; \xi') G(x''' x') \times D_{\nu\mu}(\xi' \xi) d^4 x'' d^4 x''' d^4 \xi'. \quad (22)$$

Under gauge transformation, the function $B_\mu(x x'; \xi)$ changes as follows:

$$B_\mu(x x'; \xi) \rightarrow B_{0\mu}(x x'; \xi) \langle \langle e^{ie\varphi(x)} \cdot e^{-ie\varphi(x')} \rangle \rangle_+ \\ \div G_0(x x') \left\langle \left\langle e^{ie\varphi(x)} \frac{\partial \varphi(\xi)}{\partial \xi_\mu} e^{-ie\varphi(x')} \right\rangle \right\rangle_+. \quad (23)$$

Here $B_{0\mu}(xx'; \xi)$ denotes the function B_μ when the longitudinal part of the photon D -function is equal to zero. The vacuum expectation value of the product which appears in the term containing $B_{0\mu}$ in (23) is calculated from (10). As for the factor which appears with $G_0(xx')$, simple calculations involving equations (7) and (8) yield for it

$$\left\langle \left(e^{ie\varphi(x)} \frac{\partial \varphi(\xi)}{\partial \xi_\mu} e^{-ie\varphi(x')} \right)_+ \right\rangle = \exp \{ ie^2 [\Delta_F(0) - \Delta_F(x x')] \} \\ \times e \frac{\partial}{\partial \xi_\mu} (\Delta_F(x \xi) - \Delta_F(\xi x')). \quad (24)$$

Substituting (24) in (23) and taking (11) into account, we finally obtain

$$B_\mu(x x'; \xi) = B_{0\mu}(x x'; \xi) \exp \{ i e^2 (\Delta_F(0) - \Delta_F(x x')) \} + B_{1\mu}(x x'; \xi), \quad (25)$$

$$B_{1\mu}(x x'; \xi) = e G(x x') \frac{\partial}{\partial \xi_\mu} (\Delta_F(x \xi) - \Delta_F(\xi x')). \quad (26)$$

Let us clarify the connexion between each of the terms in (25) and the Green functions for particles and photons. First we show that the second term in (25) coincides exactly with that part of (22) which corresponds to the longitudinal part of the photon function $D_{\mu\nu}^l$:

$$B_{1\mu}(x x'; \xi) = e \int G(x x'') \Gamma_\nu(x'' x'''; \xi') G(x''' x') D_{\nu\mu}^l(\xi' \xi) d^4 x'' d^4 x''' d^4 \xi'. \quad (27)$$

This expression can be obtained rigorously by going to Fourier components. Using equation (26), the Fourier components of $B_{1\mu}(xx'; \xi)$ become

$$B_{1\mu}(p, p - k; k) = e(G(p) - G(p - k)) k_\mu \Delta_F(k). \quad (28)$$

The Fourier component of the right side of (27) is evidently given by (see equation (6)):

$$e G(p) \Gamma_\nu(p, p - k; k) G(p - k) \frac{k_\nu k_\mu}{k^4} 4\pi d_t(k). \quad (29)$$

We shall now make use of the famous generalized theorem of Ward²

$$k_\nu \Gamma_\nu(p, p - k; k) = -(G^{-1}(p) - G^{-1}(p - k)). \quad (30)$$

Expression (29) then becomes:

$$e(G(p) - G(p - k)) \frac{k_\mu}{k^4} 4\pi d_t(k). \quad (31)$$

Comparison of (31) and (28), in view of (13), verifies the correctness of (27). If equation (27) is proved, then it follows from (22) and (25) that

$$B_{0\mu}(x x'; \xi) \exp \{ i e^2 [\Delta_F(0) - \Delta_F(x x')] \} = e \int G(x x'') \Gamma_\nu(x'' x'''; \xi') G(x''' x') \\ \times D_{\nu\mu}^t(\xi' \xi) d^4 x'' d^4 x''' d^4 \xi' \quad (32)$$

($D_{\mu\nu}^t$ is the transverse part of the D -function).

Re-writing this expression for an infinitesimal gauge transformation and going to Fourier components, we find

$$\begin{aligned}
 & i e^2 \int \{G(p) \Gamma_\mu(p, p-k; k) G(p-k) - G(p-r) \Gamma_\mu(p-r, p-r-k; k) \\
 & \quad \times G(p-k-r)\} \frac{\delta d_l(r)}{r^4} d^4 r = G(p) \delta \Gamma_\mu(p, p-k; k) G(p-k) \\
 & + i e^2 \int \{[G(p) - G(p-r)] \Gamma_\mu(p, p-k; k) G(p-k) \\
 & \quad + G(p) \Gamma_\mu(p, p-k; k) [G(p-k) - G(p-k-r)]\} \frac{\delta d_l(r)}{r^4} d^4 r.
 \end{aligned} \tag{33}$$

Solving for $\delta \Gamma_\mu$ we finally obtain

$$\begin{aligned}
 G(p) \delta \Gamma_\mu(p, p-k; k) G(p-k) = & -i e^2 \int \{G(p) \Gamma_\mu(p, p-k; k) (G(p-k) \\
 & - G(p-k-r)) + G(p-r) [\Gamma_\mu(p-r, p-r-k; k) G(p-k-r) \\
 & - \Gamma_\mu(p, p-k; k) G(p-k)]\} \frac{\delta d_l(r)}{r^4} d^4 r.
 \end{aligned} \tag{34}$$

For the case of spin $\frac{1}{2}$ particles, the change in the vertex operator $\Gamma_\mu(p, p-k; k)$ under infinitesimal gauge transformation ($p^2 \gg m^2$) can be found by a method similar to the one that was used for the Green function. If $d_l(r)$ is a slowly varying function in equation (34), then all the terms on the right side except the first can be neglected (upon integration in the significant region of large r^2). After this it is easily found (assuming that $(p-k)^2$, p^2 and k^2 are all of the same order of magnitude) that

$$\delta \Gamma_\mu(p, p-k; k) = -i e^2 \Gamma_\mu(p, p-k; k) \int_{-p^2}^{\infty} \frac{\delta d_l(r)}{-r^2} d(-r^2). \tag{35}$$

This result is found to be in conformity with Ward's theorem.

In conclusion we wish to extend our thanks to A. A. Abrikosov for taking part in discussions with us.

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88. A HYDRODYNAMIC THEORY OF MULTIPLE FORMATION OF PARTICLES

1. INTRODUCTION

Experiment shows that in collisions of very fast particles a large number of new particles are formed in multi-prong stars. The energy of the particles which produce such stars is of the order of 10^{12} eV or more. A characteristic feature is that such collisions occur not only between a nucleon and a nucleus but also between two nucleons. For example, the formation of two mesons in neutron-proton collisions has been observed at comparatively low energies, of the order of 10^9 eV, in cosmotron experiments¹.

Fermi^{2,3} originated the ingenious idea of considering the collision process at very high energies by the use of thermodynamic methods. The main points of his theory are as follows.

(1) It is assumed that, when two nucleons of very high energy collide, energy is released in a very small volume V in their centre of mass system. Since the nuclear interaction is very strong and the volume is small, the distribution of energy will be determined by statistical laws. The collision of high-energy particles may therefore be treated without recourse to any specific theories of nuclear interaction.

(2) The volume V in which energy is released is determined by the dimensions of the meson cloud around the nucleons, whose radius is $\hbar/\mu c$, μ being the mass of the pion. But since the nucleons are moving at very high speeds, the meson cloud surrounding them will undergo a Lorentz contraction in the direction of motion. Thus the volume V will be, in order of magnitude,

$$V = \frac{4\pi}{3} \left(\frac{\hbar}{\mu c} \right)^3 \frac{2M c^2}{E'}, \quad (1.1)$$

where M is the mass of a nucleon and E' the nucleon energy in the centre of mass system.

(3) Fermi assumes that particles are formed, in accordance with the laws of statistical equilibrium, in the volume V at the instant of collision. The particles formed do not interact further with one another, but leave the volume in a "frozen" state.

С. З. Белецкий и Л. Д. Ландау, Гидродинамическая теория множественного образования частиц, *Успехи Физических Наук*, 56, 309 (1955).

S. Z. Belenkiy and L. D. Landau, Hydrodynamic theory of multiple production of particles, *Nuovo Cimento, Supplement*, 3, 15 (1956).

(4) Fermi considered both head-on collisions, which according to his calculations lead to an isotropic angular distribution, and other (peripheral) collisions. In the latter case both the conservation of energy and that of angular momentum were taken into account in the statistical theory. For peripheral collisions an anisotropic angular distribution in the centre of mass system is obtained.

Fermi's basic idea regarding the application of statistical methods to the study of collision processes is certainly very fruitful, but the various hypotheses involved and the quantitative derivations are not very convincing^{4,5}.

The assumption that the number of particles in a multi-prong star is determined by the number of particles formed in the volume V at the instant of collision is unjustified. At that instant, on account of the high density of particles and the strong interaction between them, the concept of the number of particles has no meaning. Even if we assume that particles are formed at the initial instant, the assumption of strong interactions cannot be reconciled with the assumption that this interaction ceases immediately the particles leave the volume in question.

In reality the system expands, and the number of particles becomes definite only when the interaction between them becomes small, and the particles move away freely. This was pointed out by Pomeranchuk⁴. Moreover, Fermi³ calculated incorrectly the peripheral collisions and the resulting angle and energy distributions of the particles. Fermi's derivation is difficult to reconcile with the theory of relativity. According to Fermi, in a significantly non-head-on collision the interaction extends to the whole volume of the meson cloud, i.e. to a distance $\hbar/\mu c$, in a collision time of the order of $(\hbar/\mu c) (Mc^2/E')^{1/2}$. This means that the perturbation must be propagated with a velocity considerably exceeding that of light.

The defects of Fermi's theory arise mainly because the expansion of the compound system is not correctly taken into account. Landau⁶ has shown that the expansion of the system can be considered on the basis of relativistic hydrodynamics. The use of hydrodynamics in this case is as consistent as that of thermodynamics, since their ranges of applicability coincide.[†]

Qualitatively, the collision process may be described as follows⁶.

(1) When two nucleons collide, a compound system is formed, and energy is released in a small volume V subject to a Lorentz contraction in the transverse direction.

At the instant of collision, a large number of "particles" are formed; the "mean free path" in the resulting system is small compared with its dimensions, and statistical equilibrium is set up.

(2) The second stage of the collision consists in the expansion of the system. Here the hydrodynamic approach must be used, and the expansion may be regarded as the motion of an ideal fluid (zero viscosity and zero thermal con-

[†] The conditions of applicability of thermodynamics and hydrodynamics are comprised in the requirement $l/L \ll 1$, where l is the "mean free path" and L the least dimension of the system.

ductivity)†. During the process of expansion the "mean free path" remains small in comparison with the dimensions of the system, and this justifies the use of hydrodynamics.

Since the velocities in the system are comparable with that of light, we must use not ordinary but relativistic hydrodynamics. Particles are formed and absorbed in the system throughout the first and second stages of the collision. The high density of energy in the system is of importance here. In this case the number of particles is not an integral of the system, on account of the strong interaction between the individual particles.

(3) As the system expands, the interaction becomes weaker and the mean free path becomes longer. The number of particles appears as a physical characteristic when the interaction is sufficiently weak. When the mean free path becomes comparable with the linear dimensions of the system, the latter breaks up into individual particles. This may be called the "break-up" stage. It occurs with a temperature of the system of the order $T \approx \mu c^2$, where μ is the mass of the pion. (All temperatures are in energy units.)

2. THERMODYNAMIC RELATIONS IN THE BREAK-UP OF THE SYSTEM

In considering the break-up of the system we can neglect the interaction between particles, and thus a number of simple relations are obtained.

Let us consider some part of the system at a break-up temperature T_k . We use the following expression for the relativistic pion density in this part of the system:

$$n_\pi = \frac{g_\pi}{2\pi^2} \left(\frac{T_k}{\hbar c} \right)^3 \cdot F(z_\pi), \quad (2.1)$$

where $z_\pi = \mu c^2/T_k$. Here g_π is the number of possible states of a particle in our case (π^+ , π^- and π^0 mesons), $g_\pi = 3$;

$$F(z_\pi) = z_\pi^3 \int_0^\infty \frac{x^2 dx}{\exp [z_\pi \sqrt{(1+x^2)}] - 1}. \quad (2.2)$$

The function $F(z_\pi)$ can be put in the form

$$F(z_\pi) = z_\pi^3 \sum_{m=0}^{\infty} \frac{K^2 [z_\pi(1+m)]}{1+m}, \quad (2.3)$$

† This may be made clear by the following qualitative arguments. If viscosity and thermal conductivity are to be negligible, the Reynolds number $L V/l v$ must be much greater than unity. Here L is the least dimension of the system, V the "macroscopic" velocity, v the "molecular" velocity and l the mean free path. Since V and v are of the order of c , the condition $R \gg 1$ corresponds to $l/L \ll 1$.

where $K_2(z)$ is the modified Bessel function of the second kind^{7,8}. The series (2.3) is rapidly convergent. The asymptotic expressions for $K_2(z)$ for large and small z are

$$\begin{aligned} K_2(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[1 + \frac{15}{8} \frac{1}{z}\right] \quad \text{for } z \gg 1, \\ K_2(z) &= \frac{2}{z^2} \quad \text{for } z \ll 1. \end{aligned} \quad (2.4)$$

Using (2.4), we easily see that at high temperatures

$$n_\pi = 0.365 \left(\frac{T}{\hbar c}\right)^3,$$

and at low temperatures

$$n_\pi = g_\pi \left(\frac{T_\mu}{2\pi \hbar^3}\right)^{3/2} \exp\left(\frac{-\mu c^2}{T}\right).$$

For the pion energy density ε_π we have

$$\varepsilon_\pi = T_k \left(\frac{g_\pi}{2\pi^2}\right) \left(\frac{T_k}{\hbar c}\right)^4 \Phi(z_\pi), \quad (2.5)$$

where

$$\Phi(z_\pi) = z_\pi^2 \sum_{m=0}^{\infty} \frac{3K_2[z_\pi(1+m)] + z_\pi(1+m)K_1[z_\pi(1+m)]}{(1+m)^2}, \quad (2.6)$$

$K_1(z)$ being the hyperbolic Bessel function of the first kind.

The asymptotic expressions for $K_1(z)$ for large and small z are

$$\left. \begin{aligned} K_1(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[1 + \frac{3}{8} \frac{1}{z}\right] \quad \text{for } z \gg 1, \\ K_1(z) &= \frac{1}{z} \quad \text{for } z \ll 1. \end{aligned} \right\} \quad (2.7)$$

It is easy to see that for $z_\pi \rightarrow 0$ (very high temperatures) $\Phi(0) = 6.49$, and for $z \gg 1$ we have $\varepsilon_\pi/n_\pi = \mu c^2 + 3T/2$. Finally, the expression for the pion entropy density (in absolute units) is

$$S_\pi = \left(\frac{T}{\hbar c}\right)^3 \left(\frac{g_\pi}{2\pi^2}\right) G(z_\pi), \quad (2.8)$$

where

$$G(z_\pi) = z_\pi^2 \sum_{m=0}^{\infty} \frac{4K_2[z_\pi(1+m)] + z_\pi(1+m)K_1[z_\pi(1+m)]}{(1+m)^2}. \quad (2.9)$$

The values of the functions $F(z)$, $\Phi(z)$ and $G(z)$ are given in Table 1.

TABLE 1

z	$F(z)$	$\Phi(z)$	$G(z)$	$F^*(z)$	$\Phi^*(z)$	$G^*(z)$
0	2.40	6.49	8.65	1.80	5.68	7.57
0.5	2.17	6.30	8.31	1.72	5.58	7.37
0.7	2.02	6.12	8.02	1.65	5.47	7.19
0.9	1.86	5.90	7.67	1.56	5.33	6.95
1	1.78	5.78	7.48	1.52	5.24	6.81
1.2	1.62	5.51	7.07	1.41	5.05	6.51
1.5	1.39	5.06	6.42	1.25	4.72	6.00
2	1.05	4.27	5.31	0.982	4.07	5.07
3	0.561	2.78	3.33	0.546	2.72	3.27
6	0.0599	0.471	0.531	0.0599	0.471	0.531
7	0.0268	0.237	0.263	0.0268	0.237	0.263
8	0.0117	0.115	0.127	0.0117	0.115	0.127

In this table the values $F^*(z)$, $\Phi^*(z)$, $G^*(z)$ relate to a Fermi gas, and $F(z)$, $\Phi(z)$, $G(z)$ to a Bose gas. For a Fermi gas we have instead of (2.2)

$$F(z) = z^3 \int_0^{\infty} \frac{x^2 dx}{\exp [z \sqrt{1+x^2}] + 1},$$

and the series in (2.3), (2.6) and (2.8) have alternating signs.

For $z \gg 1$

$$F(z) \approx F^*(z) \approx z^2 \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \left(1 + \frac{15}{8} \frac{1}{z} \right),$$

$$\Phi(z) \approx \Phi^*(z) \approx z^3 \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \left(1 + \frac{27}{8} \frac{1}{z} \right),$$

$$G(z) \approx G^*(z) \approx z^3 \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \left(1 + \frac{35}{8} \frac{1}{z} \right).$$

In addition to pions, other particles may in general be formed in the system. As shown in ref. 7, the equilibrium number of mesons with masses exceeding that of the pion may in general be considerable when $T_b \approx \mu c^2$, but it appears that some of the heavy mesons interact only weakly with nucleons and so are not produced in stars. Other heavy particles are apparently formed in pairs in stars (e.g. Λ particles are formed together with K mesons[†]), but the proportion of these particles is relatively small and the experimental data are insufficient to draw any definite conclusions, and we shall therefore not discuss them here.

[†] The relatively long lifetimes of all the heavy particles are difficult to reconcile with their assumed strong interaction with nuclei unless we assume these particles to be formed in pairs.

It must also be borne in mind that nucleons are present in the system. Since at least two nucleons take part in high-energy collisions, the conservation of nuclear charge must also be taken into account⁹.

The expressions for the nucleon density n_{nn} and the antinucleon density n_{an} are

$$\begin{aligned} n_{nn} &= \frac{g_n}{2\pi^2} \left(\frac{T_k}{\hbar c} \right)^3 F_1(z_n, y_{nn}), \\ n_{an} &= \frac{g_n}{2\pi^2} \left(\frac{T_k}{\hbar c} \right)^3 F_1(z_n, y_{an}), \end{aligned} \quad (2.10)$$

where g_n is the number of possible states of a particle with given momentum; for nucleons, $g_n = 4$ (two charge states and two directions of spin);

$$F_1(z_n, y) = z_n^3 \int_0^\infty \frac{x^2 dx}{\exp[-y + z_n \sqrt{1+x^2}] + 1}, \quad (2.11)$$

$z_n = Mc^2/T$, M is the nucleon mass, $y = \mu_0/T$, and μ_0 is the chemical potential.

The condition of equilibrium with respect to pair production and annihilation is $y_{nn} + y_{an} = 0$. Putting $y_{nn} = y$, we have $y_{an} = -y$.

In cases of interest to us, $y \ll z$ and $z > 1$. In formula (2.11) we can therefore expand the denominator in a power series in $y - z\sqrt{1+x^2}$. Taking only the first term, we have

$$\begin{aligned} n_{nn} &= \frac{g_n}{2\pi^2} \left(\frac{T_k}{\hbar c} \right)^3 F_0(z_n) e^y, \\ n_{an} &= \frac{g_n}{2\pi^2} \left(\frac{T_k}{\hbar c} \right)^3 F_0(z_n) e^{-y}. \end{aligned} \quad (2.10a)$$

The energy densities ε_{nn} , ε_{an} and the entropy densities s_{nn} , s_{an} of the nucleons and anti-nucleons are

$$\begin{aligned} \varepsilon_{nn} &= T_k \left(\frac{g_n}{2\pi^2} \right) \left(\frac{T_k}{\hbar c} \right)^3 \Phi_0(z_n) e^y, \\ \varepsilon_{an} &= T_k \left(\frac{g_n}{2\pi^2} \right) \left(\frac{T_k}{\hbar c} \right)^3 \Phi_0(z) e^{-y}; \\ s_{nn} &= \left(\frac{T_k}{\hbar c} \right)^3 \left(\frac{g_n}{2\pi^2} \right) [G_0(z_n) - y F_0(z_n)] e^y, \\ s_{an} &= \left(\frac{T_k}{\hbar c} \right)^3 \left(\frac{g_n}{2\pi^2} \right) [G_0(z_n) + y F_0(z_n)] e^{-y}. \end{aligned} \quad (2.12) \quad (2.13)$$

The value of y (the chemical potential) is determined from the condition that the difference between the numbers of nucleons and anti-nucleons in the whole system should remain constant and equal to the initial number of nucleons N_0 .

The formulae (2.1–2.13) may be used to calculate various quantities. First of all, let us estimate the break-up temperature T_k of the system. Since the pion density is considerably greater than the nucleon density, the value of T_k is mainly determined by the pion density. In order that the system can break up at temperature T_k , the mean free path l of the particles at T_k must be of the order of the characteristic dimensions L of the system. A more detailed hydrodynamic calculation given in section 4 shows that L can be approximately written in the form $L \approx (\hbar/\mu c) (E/Mc^2)^{1/12}$, where E is the energy of the primary nucleon and M its mass. The mean free path is $l = 1/n\sigma$. As an estimate, let us assume that in order of magnitude $\sigma = \pi(\hbar/\mu c)^2$. If we take for n the expression given by (2.1), we get

$$\frac{l}{L} = \frac{2\pi}{3} \frac{z_\pi^3}{F(z_\pi)} \left(\frac{Mc^2}{E} \right)^{1/12}. \quad (2.14)$$

The results of calculation from this formula are shown in Table 2.

TABLE 2

$T_k/\mu c^2$	$l/L \dagger$
0.5	9
0.7	3
1.0	0.7
1.5	0.21
2	0.07

† Here l/L is taken for $E = 10^{12}$ eV.

Thus we see that the break-up temperature T_k is between 0.7 and 1.5 times μc^2 . For, when $l/L \ll 1$ break-up cannot occur, since the mean free path is extremely small, while when $l/L \gg 1$ the system cannot exist because of the smallness of its dimensions. It should be noted that the indeterminacy in our calculations due to the fact that both σ and L are known only in order of magnitude makes impossible a theoretical determination of the break-up temperature T_k .

The value of T_k does not depend greatly on that of l/L ; when l/L changes by a factor of nearly 100, T_k changes by a factor of 4. Since l/L depends only slightly on the initial energy, the dependence of T_k on the latter is extremely slight. Thus it is physically meaningful to use a quantity T_k which is practically independent of the properties of the system (i.e. of the initial energy).

Let us now calculate the number of antinucleons. We shall suppose that each part of the system breaks up at the same temperature of the system T_k , though the time of break-up may not be the same for different parts. Then the ratio of the total number of nucleons in the system to the total number of pions will be equal to the ratio of their densities. Formulae (2.1) and (2.10a) easily give

$$\sinh y = \frac{N_\pi^0}{N_n} \cdot \frac{N_0}{N_\pi}; \quad \cosh y = \frac{N_\pi^0}{N_n^0} \cdot \frac{N_n}{N_\pi}. \quad (2.15)$$

Here N_π^0 and N_n^0 are the total numbers of pions and nucleons (the latter including anti-nucleons) in the system if there are no nucleons initially. N_0 is the number of initial nucleons, N_π and N_n the total numbers of pions and nucleons (again including anti-nucleons) when there are some nucleons initially. From (2.15) we have

$$\frac{N_n}{N_\pi} = \sqrt{\left(\frac{N_\pi^0}{N_\pi^0}\right)^2 + \left(\frac{N_0}{N_\pi}\right)^2}. \quad (2.16)$$

Similarly we find the ratio of the total energy density ε_n of nucleons and anti-nucleons to the energy density ε_π of pions:

$$\frac{\varepsilon_n}{\varepsilon_\pi} = \sqrt{1 + \left(\frac{N_\pi^0}{N_\pi^0}\right)^2 \left(\frac{N_0}{N_\pi}\right)^2} \frac{\varepsilon_n^0}{\varepsilon_\pi^0}. \quad (2.17)$$

Here ε_n^0 and ε_π^0 are the nucleon and pion energy densities when the initial number of nucleons $N_0 = 0$. The ratio $\varepsilon_n/\varepsilon_\pi$ does not depend on the co-ordinate system in which the collision process is considered.

For, let a particle of mass M have energy E in a co-ordinate system in which a given element of matter is at rest. In another co-ordinate system moving relative to the former system with velocity v , this particle will have energy $E' = (E + p_x v)/\sqrt{1 - v^2/c^2}$, where p_x is the momentum component of the particle in the direction of the velocity v . Since the distribution of the in the original system of co-ordinates, on averaging over all directions we find $E' = E/\sqrt{1 - v^2/c^2}$. The energy of a particle with a different mass μ will be transformed in the same way. Thus the ratio of energies will not depend on the velocity v .

The ratios N_n^0/N_π^0 and $\varepsilon_n^0/\varepsilon_\pi^0$ are easily found from formulae (2.1), (2.5), (2.10) and (2.12), if we put $y = 0$ in the two latter formulae. The corresponding values are given in Table 3.

TABLE 3

$T_n/\mu c^2$	N_n^0/N_π^0	$\varepsilon_n^0/\varepsilon_\pi^0$
1.0	0.06	0.14
1.2	0.126	0.30
1.5	0.27	0.57
2.0	0.56	1.0

If the break-up temperature is of the order of 1.2–1.5 times μc^2 , the excitation of "isobaric" states of the system and the formation of Λ particles may be of importance. The excitation of "isobaric" states leads to some increase of the number of pions produced, which is small for $N_\pi > N_0$. The importance of Λ particles has been roughly estimated in refs. 7 and 9. However, since the spin of Λ particles is unknown and they are formed in pairs with K particles (a point not taken into account in the work just quoted), this estimate is not at all accurate.

Let us consider the following example. Let $T_k = 1.5 \mu c^2$. Then $N_n^0/N_\pi^0 = 0.27$, and the number of anti-nucleons is 0.135. Suppose that $N_0/N_\pi = 0.15$ (corresponding, for example, to three initial nucleons for 20 mesons formed.) Then formula (2.16) shows that $N_n/N_\pi = 0.3$. The total number of nucleons formed is 0.15, and the number of anti-nucleons 0.075. Thus in this case the number of anti-nucleons is reduced by a factor of almost 2 when the initial nucleons are taken into account.

Formula (2.8) gives the ratio of the energies carried away by the nucleons and by the pions. Let $T_k = 1.2 \mu c^2$. Then $\varepsilon_n^0/\varepsilon_\pi^0 = 0.3$, $N_n^0/N_\pi^0 = 0.13$. For $N_0/N_\pi = 0.15$, $\varepsilon_n/\varepsilon_\pi = 0.42$, but if $N_0/N_\pi \approx 1$, $\varepsilon_n/\varepsilon_\pi \approx 2.3$, i.e. the nucleons carry about 70 per cent of the total energy. It should be emphasised, however, that for $N_0 \approx N_\pi$ the number of particles formed is small and the theory developed here is only a rough approximation.

It may be noted that the number of anti-nucleons may, in general, be determined by a higher temperature than the break-up temperature; in this respect formula (2.16) gives a lower limit. On the other hand, formulae (2.16) and (2.17) involve two parameters, the break-up temperature T_k of the system and the quantity N_0/N_π . We can, however, derive a quantity independent of N_0/N_π and determined only by the break-up temperature T_k of the system, even if the number of nucleons is determined at a higher temperature. It is easy to see that the ratio of the energy per nucleon to the energy per pion depends only on the break-up temperature T_k , and the experimental determination of this ratio would apparently be the best method of finding that temperature. It may be noted that the ratio in question is independent of the frame of reference used; its values are given in Table 4.

TABLE 4

$T_k/\mu c^2$	$a = (\varepsilon_n/\varepsilon_\pi) (N_\pi/N_n)$
<1	6.8
0.5	3.76
0.7	3.28
0.8	2.92
1	2.65
1.5	2.16
2	1.83

It is important to find the relation between the entropy and the number of particles. It will be recalled that the expansion of the system (the second stage) is regarded as the motion of a perfect fluid, and is therefore adiabatic. This property might be destroyed by shock waves, but these are not formed in the expansion process. The entropy of the system, and of its parts, therefore varies during the first stage of the collision and remains unchanged throughout the hydrodynamic stage of the expansion, until the break-up into separate particles.

From equations (2.1), (2.8), (2.10) and (2.13) we obtain the following expression for the entropy of the system:

$$S = \left\{ \frac{G_0(z_n)}{F_0(z_n)} - y \tanh y \right\} N_n + \frac{G_0(z_\pi)}{F_0(z_\pi)} N_\pi. \quad (2.18)$$

If $N_0 = 0$, then $y = 0$ and (2.18) becomes

$$N^* = \alpha S, \quad (2.19)$$

where $N^* = N + N_\pi$, i.e. N^* is equal to the total number of particles formed (nucleons and pions), and

$$\alpha = \frac{8F_0(z_n) + 3F(z_\pi)}{8G_0(z_n) + 3G(z_\pi)}. \quad (2.20)$$

This function is given in Table 5.

TABLE 5

$T_k / \mu c^2$	α
0.17	0.113
0.5	0.198
0.67	0.215
0.83	0.222
1	0.223
1.43	0.216
2	0.213
$\gg 1$	0.25

This table shows that α depends only slightly on the break-up temperature T_k . It is seen that for T_k between 0.7 and 1.5 times μc^2 the coefficient α is almost constant.

For the case where the initial number of nucleons is not zero, we return to the equation of conservation of nuclear charge:

$$N_{nn} - N_{an} = \left(\frac{T_k}{\hbar c} \right)^3 \frac{g_n}{2\pi^2} F(z_n) [e^y - e^{-y}] V = N_0,$$

where V is the total volume of the system. The total number of nucleon

$$N_n = N_{nn} + N_{an} = \left(\frac{T_k}{\hbar c} \right)^3 \frac{g_n}{2\pi^2} F(z_n) [e^y + e^{-y}] V. \quad (2.)$$

Let us now consider equation (2.18). Instead of S let us take the ratio of the number N^* of particles formed when $N_0 = 0$ to the initial number of nucleons N_0 . To do this, we multiply both sides of (2.12) by α and divide by N_0 (see equation (2.21)). The result is

$$\frac{N^*}{N_0} = \left[\frac{G_0(z_n)}{F_0(z_n)} \coth y - y + \frac{g_\pi G(z_\pi)}{2g_n F_0(z_n) \sinh y} \right] \alpha. \quad (2.)$$

Using equations (2.1), (2.10) and (2.22), we find

$$\frac{N}{N_0} = \frac{g_n F_0(z_n)}{2g_n F_0(z_n) \sinh y} + \coth y. \quad (2.24)$$

From these equations we can find the ratio N/N_0 as a function of N^*/N_0 (with y as a parameter). The calculations show that, over a wide range of temperatures ($T_k = 0.5$ to 2 times μc^2) and up to $N^*/N_0 \approx 2$, N/N_0 is always less than N^*/N_0 . The difference between N/N_0 and N^*/N_0 increases as the temperature T_k decreases, but even for $N^*/N_0 = 3$ and $T_k = 0.5 \mu c^2$ we have $N/N_0 = 2.54$, i.e. $N/N^* = 0.85$. For $N^*/N_0 < 2$ the number of newly formed particles rapidly decreases, and $N/N_0 \rightarrow 1$. Thus the relation

$$N = \alpha S \quad (2.25)$$

is valid even when nucleons are initially present, up to a value of N^*/N_0 of the order of 2, N being the sum of the initial nucleons and the particles formed in the collision.

3. THE TOTAL NUMBER OF PARTICLES

For the hydrodynamic discussion of the system as used in the first two stages of the collision, it is necessary to know the equation of state of the matter.

As the equation of state of highly compressed matter at temperatures $T \gg \mu c^2$, we shall take⁶

$$p = \frac{1}{3} \varepsilon, \quad (3.1)$$

where p is the pressure and ε the energy density. The pressure and energy density of macroscopic bodies are always such that $p \leq \frac{1}{3} \varepsilon$, equality holding in the extreme relativistic case. This inequality, however, is derived by assuming electromagnetic interactions between particles, and there is at present no proof that it must be valid for any interaction. Nevertheless, the choice of the equation of state (3.1) appears very plausible. Since the number of particles in the system is not fixed, but is itself determined from the statistical equilibrium, the chemical potential is zero. Hence $\varepsilon - Ts + p = 0$, where s is the entropy per unit volume. Using the equation of state, we find $Ts = \varepsilon + p = 4\varepsilon/3$. Since for a fixed volume $d\varepsilon = Tds$, it follows that

$$s \approx \varepsilon^{3/4}, \quad T \approx \varepsilon^{1/4}. \quad (3.2)$$

These relations are the same as for black-body radiation, as of course we should expect.

It has already been mentioned that the entropy of the system remains constant during the hydrodynamic stage of expansion, and varies only during the first stage, at the initial instant of collision. The number of particles in the star is related to the entropy by (2.25). Hence it follows that to determine the total number of particles we must calculate the change in entropy at the

initial instant of collision. This change in entropy can most simply be calculated for a collision of two identical particles, for instance two nucleons. Let E' be the energy of the nucleons in the centre of mass system. In this system the matter is at rest at the instant immediately after the collision, by symmetry. The total entropy of the system is proportional to $\varepsilon^{3/4} V_1$, where ε is the energy density and V_1 the volume in which the energy is contained. Hence $\varepsilon = E'/V_1$; the entropy, and consequently the number of particles, are proportional to $E'^{3/4} V_1^{1/4}$. Since the volume V_1 , owing to the Lorentz contraction, transforms in inverse proportion to E' , and the energy E in the laboratory system is proportional to E'^2 , we have finally

$$N \approx E^{1/4}, \quad (3.3)$$

where E is the energy in the laboratory system. From dimensional considerations, (3.3) can be written in the form

$$N = k \left(\frac{E}{2Mc^2} \right)^{1/4} \quad (3.3a)$$

where k is a constant of the order of unity. Experiment shows that $k \approx 2$. The relation (3.3) is the same as that derived by Fermi, as is natural, since we started from the equation of state for black-body radiation, as Fermi did, and from the relation $N \approx S$, which is valid for any break-up "temperature" of the system. According to Fermi, the break-up temperature is determined at the instant after collision. This assumption has been shown⁴ to involve an internal contradiction, but it does not affect the validity of (3.3).

We have considered head-on collisions, when the particles pass at a distance comparable with the range of their forces. Let us now turn to peripheral collisions, when the passage distance is large compared with the range of the forces exerted by either particle. At first sight it might seem that the mean number of particles produced should decrease rapidly as the impact parameter increases. This is because the energy of the meson field of the colliding nucleons, concentrated in each part of the system, would seem to decrease rapidly (in a frame in which that part is at rest) with increasing distance from their centres. This view is put forward by Heisenberg¹⁰ and Bhabha¹¹ in their treatment of the multiple production of particles at high energies. It is, however, inconsistent with the indeterminacy relation of quantum physics, and is therefore erroneous⁶. (This point is discussed in more detail by Feinberg and Chervanskiĭ¹².) If the energy of a part of the system were to decrease, it would very soon be small in comparison with the indeterminacy $\Delta E \approx \hbar c/l$, where l is the thickness of that part as reduced by the Lorentz contraction.† Hence it follows that the true total energy of the system is not small, but its mathematical expectation is small. In other words, the total energy of the system and the total number of particles do not decrease, but only the probability that such a collision should occur at all. In a detailed analysis of non-head-on collisions a quantum treat-

† Since $l \approx (\hbar/\mu c) Mc^2/E$, $\Delta E/E \approx \mu/M$.

ment is necessary, since the classical treatment is rendered invalid because the indeterminacy relation is violated.

A consistent quantum treatment of the problem of non-head-on collisions involves considerable difficulties; it requires the use of a meson theory of nuclear forces, which has not yet been developed. It is not impossible that in a non-head-on collision (which is itself an event of low probability) only some fraction of the energy of the primary nucleon is transferred in the collision process. However, this question can not yet be decided by the theory. It seems to us that in the present state of the theory there is no particular significance in distinguishing central and peripheral collisions of two nucleons. The cross-sections for collision with formation of a multi-prong star are determined in order of magnitude by the "radius" of the nucleon $\hbar/\mu c$.

Heisenberg¹⁰, through an incorrect allowance for peripheral collisions, arrives at the erroneous conclusion that the collision cross section increases logarithmically with energy. Let us consider this point more closely¹². Heisenberg discusses the collision of very fast nucleons in the system of their common centre of mass. The meson fields of the nucleons decrease in the perpendicular direction y as $\exp(-y/r_0)$, where $r_0 = \hbar/\mu c$. In order to estimate the fraction γ of the initial energy E of the nucleon which is transferred to the particles formed, a region is considered in which the meson clouds of the two particles overlap when the impact parameter is b . We then have $\gamma \approx \exp(-b/r_0)$, and the number of particles formed is in order of magnitude

$$N \approx \frac{\gamma E}{\mu c^2} \approx \left(\frac{E}{\mu c^2} \right) \exp\left(\frac{-b}{r_0} \right). \quad (3.4)$$

The cross-section for the formation of, say, N mesons is πb_0^2 for the b_0 corresponding to N . When $N \approx 2$, for example, we have

$$\sigma_{N \geq 2} \approx \pi r_0^2 \left(\ln \frac{E}{\mu c^2} \right)^2. \quad (3.5)$$

Hence it follows that showers with a small number of particles must be predominantly formed, and the cross-section increases logarithmically with energy. As stated above, however, Heisenberg's treatment of non-head-on collisions is incorrect and formula (3.5) is therefore in error. For, when $\gamma E \lesssim \Delta E$ the derivation contradicts the indeterminacy relation, and this corresponds to impact parameters $b \approx r_0$.

Bhabha¹¹ attempts to derive definite conclusions regarding the structure of the nucleon on the basis of a study of multi-prong stars. The author gives a classical treatment of distant collisions, and so likewise comes into contradiction with the indeterminacy relation. As already mentioned, a quantum treatment is not yet possible. Hence no conclusions regarding the structure of the nucleon can be drawn from an analysis of multi-prong stars.

Collisions between nucleons and nuclei, and between nuclei, will be discussed in section 5.

4. ENERGY AND ANGLE DISTRIBUTION OF PARTICLES

In this section we shall consider the expansion stage in the collision of two nucleons. To examine this expansion we must use the equations of relativistic hydrodynamics:

$$\frac{\partial T_{ik}}{\partial x_k} = 0, \quad (4.1)$$

$$T_{ik} = \omega u_i u_k + p g_{ik}, \quad (4.2)$$

where $\omega = \varepsilon + p$ is the heat function per unit volume, u_i the four-velocity, $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$. In what follows we put $c = 1$, $x_i = (x_1, x_2, x_3, it)$.

The component of equation (4.1) in the direction of u_i gives the equation

$$\frac{\partial (s u_k)}{\partial x_k} = 0, \quad (4.3)$$

where s is the entropy density. This equation shows that the motion is adiabatic. In deriving it we have used the thermodynamic relations $d\varepsilon = T ds$ and $\omega = Ts$. (Since the number of particles in the system is not fixed, the chemical potential μ_0 is zero).

Let us now take the component of equation (4.1) in a direction perpendicular to u_i . This is evidently

$$\frac{\partial T_{ik}}{\partial x_k} + u_i u_k \frac{\partial T_{kl}}{\partial x_l},$$

since this gives zero on scalar multiplication by u_i . Calculations¹³ lead to the equation

$$\omega u_k \frac{\partial u_i}{\partial x_k} + u_i u_k \frac{\partial p}{\partial x_k} + \frac{\partial p}{\partial x_i} = 0. \quad (4.4)$$

Since in our case $\omega = Ts$ and $dp = s dT$ (since $d\omega = dp + T ds$), equation (4.4) gives

$$u_k \frac{\partial (T u_i)}{\partial x_k} + \frac{\partial T}{\partial x_i} = 0 \quad (4.5)$$

In the centre of mass system, the shape of the system at the instant of collision is that of a much flattened disk (whose transverse dimension a is initially $E'/2M$ times its thickness Δ). This type of motion continues through a considerable part of the expansion stage, and the motion of the matter may be regarded as one-dimensional. The solution for both the one-dimensional and the subsequent three-dimensional motion was first obtained by Landau⁶. Later, Khalatnikov¹⁴ derived a more exact solution for the one-dimensional stage, on the basis of an investigation of the general one-dimensional problem in relativistic hydrodynamics. In the present section we shall follow first ref. 14 and then ref. 6 in considering the one-dimensional case.

In that case we have only the co-ordinates x_1 and x_4 . It is easy to see that equation (4.5) can then be written in the form¹⁴

$$\frac{\partial(Tu_1)}{\partial x_4} = \frac{\partial(Tu_4)}{\partial x_1}, \quad (4.6)$$

and so there is a function φ such that

$$Tu_1 = \frac{\partial \varphi}{\partial x_1}, \quad Tu_4 = \frac{\partial \varphi}{\partial x_4} \quad (4.7)$$

The function φ is the potential of one-dimensional motion in relativistic hydrodynamics, and satisfies the differential relation

$$d\varphi = Tu_4 dx_4 + Tu_1 dx_1. \quad (4.8)$$

In the following discussion we shall use t instead of x_4 ($= it$), $u_0 = 1/\sqrt{1-v^2}$ instead of u_4 ($= i u_0$), and x instead of x_1 . Then equation (4.8) becomes.

$$d\varphi = -Tu_0 dt + Tu_1 dx. \quad (4.8a)$$

We define the variable α in terms of the velocities u_0 and u_1 by

$$u_1 = \sinh \alpha, \quad u_0 = \cosh \alpha, \quad (4.9)$$

and effect a Legendre transformation with respect to the variables ω and α . The potential χ is such that

$$\begin{aligned} d\chi &= d(\varphi + Tu_0 t - Tu_1 x) \\ &= (t \cosh \alpha - x \sinh \alpha) dT + (t \sinh \alpha - x \cosh \alpha) T d\alpha. \end{aligned} \quad (4.10)$$

From (4.10) it follows that

$$\frac{\partial \chi}{\partial T} = t \cosh \alpha - x \sinh \alpha, \quad \frac{\partial \chi}{\partial \alpha} = T(t \sinh \alpha - x \cosh \alpha), \quad (4.11)$$

$$t = \frac{\partial \chi}{\partial T} \cosh \alpha - \frac{1}{T} \frac{\partial \chi}{\partial \alpha} \sinh \alpha, \quad x = \frac{\partial \chi}{\partial T} \sinh \alpha - \frac{1}{T} \frac{\partial \chi}{\partial \alpha} \cosh \alpha. \quad (4.12)$$

Now let us return to equation (4.3), which for a one-dimensional motion becomes

$$\frac{\partial(su_0)}{\partial t} + \frac{\partial(su_1)}{\partial x} = 0.$$

Changing from the variables t and x to T and α , we have

$$\frac{\partial(t, x)}{\partial(T, \alpha)} \left\{ \frac{\partial(s \cosh \alpha, x)}{\partial(t, x)} - \frac{\partial(s \sinh \alpha, t)}{\partial(t, x)} \right\} = \frac{\partial(s \sinh \alpha, x)}{\partial(T, \alpha)} - \frac{\partial(s \sinh \alpha, t)}{\partial(T, \alpha)}.$$

Since in our case s is determined entirely by the temperature T , this gives

$$\begin{aligned} \frac{1}{s} \frac{ds}{dT} \left\{ \frac{\partial}{\partial \alpha} (x \cosh \alpha - t \sinh \alpha) - (x \sinh \alpha - t \cosh \alpha) \right\} \\ + \frac{\partial}{\partial T} (t \cosh \alpha - x \sinh \alpha) = 0. \end{aligned} \quad (4.13)$$

Using the relations (4.11) and (4.12), we can transform (4.13) to

$$\frac{1}{s} \frac{ds}{dT} \left\{ \frac{\partial \chi}{\partial T} - \frac{1}{T} \frac{\partial^2 \chi}{\partial \alpha^2} \right\} + \frac{\partial^2 \chi}{\partial T^2} = 0. \quad (4.14)$$

But

$$\frac{s}{T} \frac{dT}{ds} = \frac{1}{T} \frac{dp}{d\varepsilon} \frac{d\varepsilon}{ds} = \frac{dp}{d\varepsilon} = c_0^2,$$

where c_0 is the velocity of sound in the medium.

Expressing the variable T in terms of $y = \ln T$, we find

$$\frac{\partial^2 \chi}{\partial \alpha^2} - c_0^2 \frac{\partial^2 \chi}{\partial y^2} + (c_0^2 - 1) \frac{\partial \chi}{\partial y} = 0. \quad (4.15)$$

Since the equation of state of the medium is $p = \frac{1}{3} \varepsilon$, $c_0^2 = \frac{1}{3}$. Substitution in (4.15) gives

$$3 \frac{\partial^2 \chi}{\partial \alpha^2} - \frac{\partial^2 \chi}{\partial y^2} - 2 \frac{\partial \chi}{\partial y} = 0. \dagger \quad (4.15a)$$

Thus the solution of the one-dimensional problem of relativistic hydrodynamics reduces to that of a linear differential equation with constant coefficients. The variables x and t are derived from the potential χ by means of formulae (4.12).

Let us now return to the problem of interest to us, that of the expansion of a flat disk of thickness Δ . In the centre of mass system, the disk expands symmetrically in both directions. We may therefore consider the expansion in one direction only. Let us choose $x = 0$ so that the expansion is symmetrical about the plane $x_1 = -l$, where $l = \frac{1}{2} \Delta$. First of all we must find the boundary conditions for the problem. Evidently the medium must be at rest in the plane $x_1 = -l$ (which, by symmetry, can be regarded as a fixed wall), and so $\alpha = 0$. Using equation (4.12), we write this equation in the form

$$\left(\frac{\partial \chi}{\partial \alpha} \right)_{\alpha=0} = l e^{-y}. \quad (4.16)$$

On the vacuum side the required solution must be fitted to a simple (Riemann) wave, for which we have the similarity solution

$$x = t \frac{v - c_0}{1 - v c_0}, \quad (4.17)$$

where v is the (ordinary) velocity of the motion. It can be shown¹³ that in the simple wave α and y are related by

$$\alpha = - \frac{y}{c_0}. \quad (4.18)$$

[†] Owing to errors in Khalatnikov's paper¹⁴ the coefficients in equations (4.15) and (4.15a) are not the same as those given by him.

In order to find the condition for joining the simple wave to the general solution, we must substitute the expressions (4.12) for x and t in equation (4.17). On making this substitution and using the condition (4.18), we find that $\chi = 0$.

Thus the second boundary condition is

$$\chi = 0 \quad \text{for} \quad \alpha = -\frac{y}{c_0}. \quad (4.19)$$

Instead of χ we use the potential χ_1 given by

$$\chi = \chi_1 e^{-y}, \quad (4.20)$$

and use as variables α and (instead of y)

$$z = -\alpha - \frac{y}{c_0}. \quad (4.21)$$

Equation (4.15) and the boundary conditions (4.16) and (4.19) become, in terms of the new variables,

$$\left(6 \frac{\partial^2}{\partial \alpha \partial z} - 3 \frac{\partial^2}{\partial \alpha^2} - 1 \right) \chi_1 = 0, \quad (4.22)$$

$$\chi_1 = 0 \quad \text{for} \quad z = 0, \quad (4.23)$$

$$\frac{\partial \chi_1}{\partial \alpha} - \frac{\partial \chi_1}{\partial z} = -l \exp\left(-\frac{2z}{\sqrt{3}}\right) \quad \text{for} \quad \alpha = 0. \quad (4.24)$$

A Laplace transformation with respect to the variable z ,

$$\psi = \int \chi_1(z) e^{-zq} dz, \quad (4.25)$$

gives from (4.22)

$$6q \frac{\partial \psi}{\partial \alpha} - 3 \frac{\partial^2 \psi}{\partial \alpha^2} - \psi = 0, \quad (4.26)$$

and the boundary conditions become

$$\frac{\partial \psi}{\partial \alpha} - q \psi = \frac{l}{q + \frac{2}{\sqrt{3}}} \quad \text{for} \quad \alpha = 0. \quad (4.27)$$

We seek a solution in the form $\psi = a(q) \exp[\alpha p(q)]$. From equation (4.26) we get a quadratic equation for p , and we take the solution which corresponds to ψ decreasing as $\alpha \rightarrow \infty$. The function $a(q)$ is given by (4.27). The result is

$$\psi = l \frac{\exp\{\alpha[q - \sqrt{(q^2 - \frac{1}{3})}]\}}{\sqrt{(q^2 - \frac{1}{3})(q + 2/\sqrt{3})}}. \quad (4.28)$$

Then the inversion formula gives the function χ_1 :

$$\chi_1 = \frac{1}{2\pi i} l \int_{\delta-i\infty}^{\delta+i\infty} \frac{\exp\{-\sqrt{(q^2 - \frac{1}{3})}\alpha - yq \cdot \sqrt{3}\}}{\sqrt{(q^2 - \frac{1}{3})(q + 2/\sqrt{3})}} dq, \quad \text{Re } q < \delta. \quad (4.29)$$

The integral in (4.29) is taken so that the poles of the integrand lie to the left of the line of integration.

Finally, the potential χ which satisfies the boundary conditions is

$$\chi = l\sqrt{3}e^y \int_{\alpha/\sqrt{3}}^y e^{2y'} J_0 \sqrt{y'^2 - \frac{1}{3}\alpha^2} dy', \quad (4.30)$$

where J_0 is the Bessel function of imaginary argument.†

Using the relations (4.12), which may conveniently be written

$$\left. \begin{aligned} t &= e^{-y} \left(\frac{\partial \chi}{\partial y} \cosh \alpha - \frac{\partial \chi}{\partial \alpha} \sinh \alpha \right), \\ x &= e^{-y} \left(\frac{\partial \chi}{\partial y} \sinh \alpha - \frac{\partial \chi}{\partial \alpha} \cosh \alpha \right), \end{aligned} \right\} \quad (4.12a)$$

we can now find y (that is, T) and α (that is, v) as functions of x and t , and so obtain the complete solution of the problem.

Let us consider the last stage of the one-dimensional break-up, when y and α are large and $y > \alpha$. These conditions signify that the temperature is relatively low (y negative) and the particle velocities are ultrarelativistic. For this stage we use the asymptotic expression for J_0 :

$$J_0 \sqrt{y^2 - \frac{1}{3}\alpha^2} \approx \exp \left[\sqrt{y^2 - \frac{1}{3}\alpha^2} \right].$$

Then the potential χ will be in order of magnitude

$$\chi \approx \Delta \cdot \exp \left[-y + \sqrt{y^2 - \frac{1}{3}\alpha^2} \right].$$

If we use only logarithmic accuracy, (4.12) easily gives‡

$$\left. \begin{aligned} \frac{t+x}{\Delta} &\approx \exp \left[\alpha - 2y + \sqrt{y^2 - \frac{1}{3}\alpha^2} \right], \\ \frac{t-x}{\Delta} &= \exp \left[-\alpha - 2y + \sqrt{y^2 - \frac{1}{3}\alpha^2} \right]. \end{aligned} \right\} \quad (4.31)$$

† See Appendix I, p. 694.

‡ Evidently $(t-x)/(t+x) \approx e^{-2} \ll 1$.

Hence

$$\left. \begin{aligned} \alpha &= \frac{1}{2} \ln \frac{t+x}{t-x}, \\ y &= -\frac{1}{3} \left[\ln \frac{t+x}{\Delta} + \ln \frac{t-x}{\Delta} - \sqrt{\left(\ln \frac{t+x}{\Delta} \right) \left(\ln \frac{t-x}{\Delta} \right)} \right] \end{aligned} \right\} \quad (4.32)$$

Using the notation

$$\ln \frac{t+x}{\Delta} = \tau, \quad \ln \frac{t-x}{\Delta} = \eta,$$

with $y = \ln T$ and the energy density $\varepsilon \approx T^4$, the second equation (4.32) can be written

$$\varepsilon = \varepsilon_0 \exp \left[-\frac{4}{3} \{ \eta + \tau - \sqrt{\tau \eta} \} \right]. \quad (4.33)$$

For $\alpha > 1$, $u_1 \approx u_0 \approx 1/\sqrt{1-v^2}$ (see equation (4.9)). Let

$$u = \frac{1}{\sqrt{1-v^2}}.$$

Then (4.9) shows that $\alpha = \operatorname{arsinh} u \approx \ln 2u$, and the first equation (4.32) gives

$$u^2 \approx \frac{t+x}{t-x}. \quad (4.34)$$

Equations (4.33) and (4.34) are the same as those previously derived by Landau⁶ using a different method (his equations (16) and (24)).

For the one-dimensional stage of expansion there is possible, on the basis of formula (4.30), a solution more exact than (4.33–4.34), which are of only logarithmic accuracy. However, since the solution for the subsequent three-dimensional stage is very inexact, there is no need for such further refinement at present.

Let us consider the distribution of energy and entropy through the thickness of the disk. The energy density is easily seen to be of the order of εu^2 .† For the energy dE corresponding to a layer of thickness $d\delta$, where $\delta = t-x$, we have $dE \approx \varepsilon a^2 u^2 d\delta$, a being the radius of the disk. Using formulae (4.33) and (4.34), we obtain

$$dE \approx \exp \left[-\frac{1}{3} (\sqrt{\tau} - 2\sqrt{\eta})^2 \right] d\eta. \quad (4.35)$$

This formula shows that the energy distribution has a maximum at $\eta = \frac{1}{4}\tau$. This means that the energy is mainly concentrated in a region $\delta \approx (t\Delta^3)^{1/4}$.

† For the energy density is given by the component T_{44} of the energy momentum tensor: $T_{44} \approx \varepsilon u^2$.

For $t \gg \Delta$, $\delta \ll t$. Thus the majority of the energy is concentrated in a thin layer $\delta \ll t$ close to the boundary of the region. Let us now find the entropy distribution. The entropy density is su .† Since $s \approx \varepsilon^{3/4}$, the entropy corresponding to a layer of thickness $d\delta$ is $dS \approx su a^2 d\delta$ or, using (4.33) and (4.34),

$$dS \approx \exp \left[-\frac{1}{2} (\sqrt{\tau} - \sqrt{\eta})^2 \right] d\eta. \quad (4.36)$$

This distribution has a maximum at $\eta = \tau$. Thus the entropy is concentrated in a region $L \approx t$. Hence we see (cf. equation (4.34)) that in the region of the maximum of the particles $u^2 \approx 1$, and therefore the condition $\alpha > 1$ is approximately valid. The matter in this region moves at speeds of the order of that of light, but not at ultra-relativistic speeds.

Let us now consider the three-dimensional stage of break-up. The solution obtained above is valid so long as the motion may be regarded as one-dimensional. For this to be so it is necessary that the angle of flight θ , i.e. the angle between the path of a given element of matter and the x -axis, should be small. More precisely, it is necessary that the distance travelled by that element should be less than the dimension a of the system. This condition may be written

$$t\theta \ll a. \quad (4.37)$$

To estimate the angle θ we go back to the transverse components of equation (4.1), which have not yet been used. We have

$$\frac{\partial T_{42}}{\partial t} \approx \frac{\partial T_{22}}{\partial y},$$

or, in order of magnitude, $T_{42}/t \approx T_{22}/y$. But $T_{42} \approx \varepsilon u^2 \theta$ and $T_{22} \approx \varepsilon$, whence $u^2 \theta \approx t/a$. Since by (4.34), $u^2 \approx t/\delta$, we have finally

$$\theta \approx \frac{\delta}{a}. \quad (4.38)$$

This means that the deviation increases with the distance of the particle from the front, i.e. as its energy decreases. Formulae (4.37) and (4.38) give

$$t \ll \frac{a^2}{\delta}. \quad (4.39)$$

The inequality gives the condition of applicability of the one-dimensional solution. The limit of applicability of that solution is more remote when the particle is nearer the front. From a time $t_1 = a^2/\delta$ onwards, the lateral deviation becomes considerable. The resulting motion can be discussed only approximately. Qualitatively, it may be regarded as a conical dispersal. The derivative $d\delta/dt = 1 - dx/dt = 1 - v \approx 1/2u^2$. In the second (three-dimensional) stage of the expansion, u is considerably greater than in the first stage. Hence δ remains practically constant for each element of matter. Further, it can be

† The entropy density is given by the component s^0 of the four-dimensional entropy flux density vector: $s^0 \approx u s^{3/4}$.

shown that we may neglect all derivatives of hydrodynamic quantities, both with respect to δ and in the transverse direction. This means that lateral forces are small and we have a kind of radial break-up under inertia. We may summarise the qualitative description of the break-up thus: the angular deviation of the path of each element reached in the first (mainly one-dimensional) stage of expansion remains unchanged in the subsequent three-dimensional stage.

Hence it follows that both the energy flux and the entropy flux transported within any cone of fixed aperture θ must remain constant. Since $\delta = t - x \ll t$, the cross-sectional area of this cone $\sim t^2$, and the constancy of the fluxes of energy and entropy signifies that

$$\varepsilon u^2 t^2 = \text{constant}, \quad s u t^2 \approx \varepsilon^{3/4} u t^2 = \text{constant}. \quad (4.40)$$

Hence

$$u \approx t, \quad \varepsilon \approx \frac{1}{t^4}. \quad (4.41)$$

These relations determine the variation with time of the quantities u and t in a conical expansion. In the first stage, (4.34) shows that $u \approx t^{1/2}$; in the second stage, $u \approx t$, and so the velocity in this case is closer to that of light than in the first case. The region of transition from the first to the second stage is very complex and will not be discussed in detail. Instead, our problem will be to "join" the solution obtained for the second stage to the one-dimensional solution at the point where the latter becomes inapplicable, viz. $t_1 = a^2/\delta$.

We use the notation

$$\frac{\delta}{a} = e^{-\lambda}, \quad \frac{\Delta}{a} = e^{-L}. \quad (4.42)$$

Since $\Delta/a \approx M/E' \approx \sqrt{(2M/E)}$, it follows that

$$L = \frac{1}{2} \ln \left(\frac{E}{2M} \right),$$

where E' is the energy in the centre of mass system, and E that in the laboratory system. Then

$$\left. \begin{aligned} \eta &= \ln \left(\frac{\delta}{\Delta} \right) = L - \lambda, \\ \tau_1 &= \ln \left(\frac{t_1}{\Delta} \right) = \ln \left(\frac{a^2}{\delta \Delta} \right) = L + \lambda. \end{aligned} \right\} \quad (4.43)$$

Substituting these expressions in formula (4.36) gives

$$dS = \exp[\sqrt{L^2 - \lambda^2}] d\lambda.$$

In the three-dimensional stage, as already explained, $\delta = \text{constant}$, and therefore $\lambda = \text{constant}$ also for each element of matter. The entropy of an element is therefore determined by the same formula until the system breaks up into

separate parts. We know that the number of particles is proportional to the entropy. For the distribution of particles, we therefore have

$$dN \approx \exp[\sqrt{L^2 - \lambda^2}] d\lambda, \quad (4.44)$$

$$\theta = \frac{\delta}{a} = e^{-\lambda}. \quad (4.45)$$

The angle θ remains constant, with δ , for each element of matter. Formulae (4.44) and (4.45) determine in parametric form the angular distribution of the particles in the centre of mass system. The majority of particles are formed at $\delta \approx t$, since in this range the entropy is a maximum. For these particles the one-dimensional stage ends at $t_k \approx a$. Hence the angle of flight for such particles $\theta \approx 1$. At small angles fewer particles are formed, but they have more energy. It follows from (4.44) and (4.45) that in a head-on collision the angular distribution in the centre of mass system is by no means isotropic, as Fermi assumed². It may be noted that the greatest value of λ which can meaningfully be considered is given by the condition

$$\int_{\lambda_{\max}}^L dN \approx 1.$$

Hence we find

$$\lambda_{\max} = \frac{1}{2} \sqrt{3L}. \dagger \quad (4.46)$$

Let us determine the energy distribution of the particles. In the stage of one-dimensional motion, $u \approx \sqrt{(t/\delta)}$, while in the stage of conical motion $u \approx t$. From the condition of joining these values at $t = t_1$ we have

$$u \approx \frac{t}{a}. \quad (4.47)$$

Next we join the expressions for the energy density ε . At $t = t_1$, formula (4.33) gives

$$\varepsilon = \varepsilon_0 \exp \left[-\frac{4}{3} \{2L - \sqrt{L^2 - \lambda^2}\} \right].$$

[†] Since $dN \approx \exp[\sqrt{L^2 - \lambda^2}] d\lambda$, the condition

$$\int_{\lambda_{\max}}^L dN \approx 1$$

gives $C \exp[\sqrt{L^2 - \lambda_{\max}^2}] \approx 1$ with logarithmic accuracy. But $N \approx \exp(\frac{1}{2}L)$, and so

$$\int_{\lambda=0}^{\lambda=L} dN \approx C e^{\lambda} \approx e^{L/2}.$$

Thus $C \approx e^{L/2}$, and $\lambda_{\max} = \frac{1}{2}\sqrt{3}L$.

On the other hand, in the stage of conical break-up $\varepsilon \approx 1/t^4$. The joining condition gives

$$\varepsilon = \varepsilon_0 \left(\frac{t_1}{t} \right)^4 \exp \left[-\frac{4}{3} \{2L - \sqrt{L^2 - \lambda^2}\} \right]. \quad (4.48)$$

The free flight of the particles begins at a time t_2 when the temperature of a part of the system reaches the "critical" value T_k (see section 2). This time corresponds to a definite energy density ε_k . From (4.48) we obtain an expression for the time t_k :

$$t_k \approx t_1 \left(\frac{\varepsilon_0}{\varepsilon_k} \right)^{1/4} \exp \left[-\frac{1}{3} \{2L - \sqrt{L^2 - \lambda^2}\} \right].$$

Hence, from (4.47), we find the value of u at the instant of break-up:

$$u_k \approx \frac{t_k}{a}. \quad (4.49)$$

The quantity u_k is simply related to the energy of the resulting particles. If the energy of a particle is E_0 in a frame of reference in which a given element of matter is at rest, then it is $E_0 u_k$ in the centre of mass system. From (4.49) we see that, the later the element breaks up, the greater will be the energy of the resulting particles, $u_k = \text{constant} \times \exp [\lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2}]$. The constant in this equation is determined from the equation $\int E_0 u_k dN = E'$, and hence the energy of the particle $\varepsilon = E_0 u_k$ is

$$\varepsilon \approx M \exp \left[-\frac{1}{6} L + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right]. \quad (4.50)$$

Equations (4.44) and (4.50) give the energy distribution in parametric form. If particles with different masses are formed, or the energy carried away by nucleons is important, allowance can be made on the basis of the considerations given in section 2.

From formulae (4.44), (4.45) and (4.49) we see that the energy of the particles is uniquely determined by the angle at which they emerge. The majority of particles ($\lambda = 0$) have relatively low energies. The particles which move at small angles, however, have considerable energy. Thus there is a concentrated flux of energy at small angles.

We must now go from the centre of mass system to the laboratory system. Without going into the details of the conversion (see Appendix II, page 696), we may give the final results:

$$dN = \frac{k}{\sqrt{2\pi L}} \exp \left[-\frac{1}{2} L + \sqrt{L^2 - \lambda^2} \right] d\lambda, \quad (4.51)$$

$$\varepsilon = \frac{5\sqrt{5}}{2\sqrt{3}} M \exp \left[\frac{5}{6} L + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right]. \quad (4.52)$$

The coefficients in these formulae are got from the relations $\int dN = N$ and $\int \varepsilon dN = E$.

The angle of flight in the laboratory system is

$$\chi = e^{-L-\lambda}. \quad (4.53)$$

The following is also a useful formula:

$$\overline{\ln \chi} = -L; \quad (4.54)$$

it holds good for a collision of two identical particles, and signifies that the geometric mean angle of flight is determined by the energy of the incident particle (in a collision of identical particles).

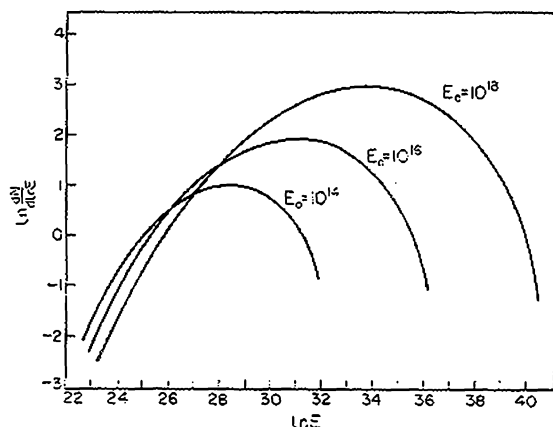


Fig. 1. Differential spectra of the energy of secondary particles created in high-energy nuclear collisions; E_0 is the energy of the initial particle in the laboratory system. (The calculations were performed by L. I. Sarycheva).

Figure 1 shows particle distributions calculated from formulae (4.42–4.54).

The distributions of particles both in angle and in energy are almost Gaussian when the logarithms of these quantities are used as variables. The distributions have long “tails” on each side of the maximum. It is shown in Appendix II that λ may conveniently be determined from the relation $\lambda = -\ln \tan \frac{1}{2} \theta$. Formula (4.44) then becomes

$$dN \approx \exp \left[\sqrt{L^2 - \ln^2 \tan \frac{1}{2} \theta} \right] \frac{d\theta}{\sin^2 \theta}. \quad (4.55)$$

This formula gives explicitly the angular distribution of the particles. Using (4.50), we find directly the relation between the particle energy and the angle θ (in the centre of mass system):

$$\varepsilon \approx M \exp \left[-\frac{1}{6} L - \ln \tan \frac{1}{2} \theta + \frac{1}{3} \sqrt{L^2 - \ln^2 \tan \frac{1}{2} \theta} \right]. \quad (4.56)$$

An important characteristic of the collision process is the angle within which half the energy is emitted. We give here the values of this quantity

calculated both according to Landau's theory (formulae (4.51-4.53)) and according to Fermi's theory, with allowance for non-head-on collisions (Table 6).

TABLE 6†

Energy (eV)	$\theta_{\frac{1}{2}}$ (Fermi)	$\chi_{\frac{1}{2}}$ (Fermi)	$\theta_{\frac{1}{2}}$ (Landau)	$\chi_{\frac{1}{2}}$ (Landau)
10^{14}	0.31	0.7×10^{-4}	0.10	1.3×10^{-4}
10^{15}	—	2.1×10^{-4}	0.0154	2.2×10^{-5}
10^{16}	—	6.7×10^{-5}	0.031	3.8×10^{-6}

† Taken from Rozental' and Chernavskii¹⁵

The value of the collision parameter r/R in Fermi's theory is 0.712. $\theta_{1/2}$ is the angle in the centre of mass system, and $\chi_{1/2}$ that in the laboratory system.

Despite the fact that Fermi's calculation of the angular distribution appears to us to be incorrect, we give his results for comparison, since they are often used in comparing with experimental results.

Table 6 shows that the present theory gives a considerably narrower energy flux distribution than that given by Fermi's theory.

5. COLLISIONS OF PARTICLES WITH DIFFERENT MASSES

So far we have considered only collisions between nucleons. Experimentally, however, nucleon-nucleus collisions are observed at high energies and, without a theoretical account of these we cannot, strictly speaking, compare the results of theory and experiment.

It would not be correct to regard nucleon-nucleus or nucleus-nucleus collisions as a series of collisions between nuclear protons and neutrons. Since the distance between particles in the nucleus is of the order of the range of the nuclear forces, and in each collision several particles are produced, the result must be the immediate formation of particles throughout the region of the nucleus traversed by the nucleon, or that of the larger nucleus traversed by the smaller one. The nucleon will not interact with the whole nucleus, but only with part of it, i.e. it will cut a "tunnel" out of the nucleus¹⁵. The number of particles formed in the collision will be related to the entropy by formula (2.25), N being the sum of the newly formed particles and the nucleons participating in the collision.

The simplest case is that of a head-on collision of two identical nuclei of atomic weight A . Here the arguments leading to formula (3.3) are entirely applicable, and it is not difficult to see how formula (3.3) should be modified⁶. Let the velocity of the incident nucleus be equal to that of the nucleon in the nucleon-nucleon collision. Then its energy is A times greater. Since the mass density in the nucleus is about the same as the mass density of the proton, relative to its range of action, the energy density immediately after the collision remains the same as in a collision between nucleons. Since the

Lorentz contraction also evidently remains unchanged, the change in entropy will be simply proportional to the volume of the nuclei, i.e. to A . Thus we have finally

$$N = k A \left(\frac{E}{2 A M c^2} \right)^{1/4} = k A^{3/4} \left(\frac{E}{2 M c^2} \right)^{1/4}. \quad (5.1)$$

The calculation is considerably more complex for a collision of a nucleon with a nucleus or for a collision between nuclei of different masses. We shall consider here a nucleon-nucleus collision. The calculation of the number of particles formed in the star amounts to calculating the entropy in the first stage of the collision, since the entropy does not vary in the process of hydrodynamic expansion. In a nucleon-nucleon collision it was not necessary to consider the mechanism of compression in calculating the entropy, since the result was obtained directly from considerations of symmetry. The situation is different in a nucleon-nucleus collision. This problem is considered in ref. 16.

We may extend the concept of nuclear matter as a continuous medium to the discussion of the first stage of the collision—the compression of nuclear matter. Such a treatment is, of course, highly approximate and is used only

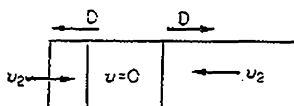


FIG. 2.

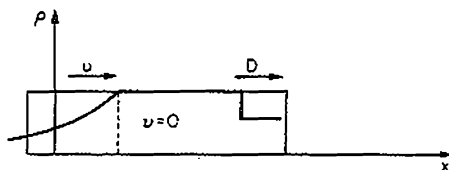


FIG. 3.

to obtain an estimate. One cannot speak of the "propagation of a shock wave or a travelling wave in the nucleon" except in a conventional sense.

Let us take a frame of reference in which the nucleon and the nucleus have equal and opposite velocities. Owing to the Lorentz contraction, the nucleon and the nucleus appear in this frame as two very thin disks. Consequently, the problem of the initial stage of the collision may be regarded as one-dimensional. Thus a nucleon-nucleus collision is a collision of the nucleon with a cylinder cut out of the nucleus, whose cross-section is equal to that of the nucleon and whose length varies from the diameter of the nucleus to the "diameter" of the nucleon. As the nucleon approaches this cylinder, shock waves are "propagated" in both directions through the matter in the cylinder and that in the nucleon (Fig. 2). In the chosen frame of reference the matter between the shock waves will be at rest. The shock wave travelling in the lighter

particle will reach the edge sooner than that in the cylinder. When the wave reaches the edge, an outflow of matter begins there (Fig. 3). A rarefaction wave travels through the nuclear matter at a speed equal to the speed of sound in the medium. At the same time a shock wave which has not yet reached the edge is travelling in the same direction. The entropy calculation will depend on whether the travelling wave overtakes the shock wave before the latter reaches the edge and the compression process ends. Since we use the equation of state $p = \frac{1}{3}\epsilon$, the speed of sound is $c_0 = \sqrt{(\partial p / \partial \epsilon)_S} = 1/\sqrt{3}$. Let us now calculate the velocity D of the shock wave. We use a co-ordinate system in which the shock wave is at rest. Then the velocity of matter beyond the shock wave (Fig. 2) is D , and in front of the shock wave it is $v'_2 = (v_2 + D)/(1 + v_2 D)$, where v_2 is the velocity of the particles in the "equal velocities" frame. By the continuity of energy and momentum fluxes through the discontinuity surface, we have¹³

$$\frac{p_1 + D^2 \epsilon_1}{1 - D^2} = \frac{p_2 + v_2'^2 \epsilon_2}{1 - v_2'^2}, \quad (5.2)$$

$$\frac{D(p_1 + \epsilon_1)}{1 - D^2} = \frac{v_2'(p_2 + \epsilon_2)}{1 - v_2'^2}. \quad (5.3)$$

Here p_1 and ϵ_1 are the pressure and energy density behind the shock wave, and p_2 and ϵ_2 those in front of the shock. Dividing (5.2) by (5.3), we have

$$\frac{p_1 + D^2 \epsilon_1}{D(p_1 + \epsilon_1)} = \frac{p_2 + v_2'^2 \epsilon_2}{v_2'(p_2 + \epsilon_2)}. \quad (5.4)$$

Since the velocity v_2 of the incident particles is very close to that of light, the velocity v_2' will be so as well. Putting $v_2' = 1$, we see that the right-hand side of equation (5.4) is equal to 1, and therefore our results are independent of the equation of state in front of the shock wave. Using the equation of state $p_2 = \frac{1}{3}\epsilon_2$ for matter behind the shock wave, we obtain for D the relation

$$\frac{1}{3} + D^2 = \frac{4}{3} D,$$

or

$$D = \frac{1}{3}. \quad (5.5)$$

We can now find the minimum length l_k of the "tunnel" for which the travelling rarefaction wave overtakes the shock wave. This is given by

$$\frac{l_k}{d} = \frac{D + c_0}{c_0 - D},$$

where d is the nucleon "diameter".

Substituting for D and c_0 , we find

$$\frac{l_k}{d} = 3.7. \quad (5.6)$$

For lengths $l < l_k$, the change of entropy is easily calculated. We have to find the entropy of the individual parts of the system immediately after the shock wave has passed, when they are at rest in the chosen frame of reference. It is not difficult to see that the change in the entropy of the whole system is then

$$\frac{S}{S_0} = \frac{1}{2} \left(\frac{l}{d} + 1 \right) \quad \text{for} \quad \frac{l}{d} \leq 3.7, \quad (5.7)$$

where S_0 is the change of entropy in a nucleon-nucleon collision.

Formula (5.7) is considerably inaccurate on account of the indeterminacy of the nucleon "diameter". If we apply it to a collision between a light and a heavy nucleus, it becomes more accurate. In that case d must be taken as the dimension of the light nucleus.

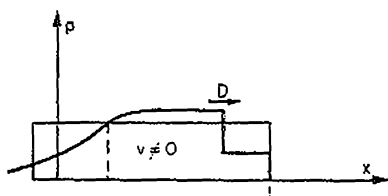


FIG. 4.

For lengths l exceeding $l_k = 3.7d$, the solution becomes more complex. The travelling wave in this case overtakes the shock wave, but cannot pass through the discontinuity, because the shock wave moves in the matter of the incident nucleus at a velocity exceeding that of sound. The travelling wave is reflected from the shock wave. A region is formed which is bounded on the right by the shock wave and on the left by the travelling wave (Fig. 4). In order to describe the motion of the medium in this region, we must return to equations (4.15) and (4.15a) for an arbitrary one-dimensional motion of a relativistic gas:

$$3 \frac{\partial^2 \chi}{\partial \alpha^2} - \frac{\partial^2 \chi}{\partial y^2} - 2 \frac{\partial \chi}{\partial y} = 0. \quad (4.15a)$$

The independent variables here are $\alpha = \operatorname{arctanh} v$ (where v is the velocity of the medium) and $y = \ln(T/T_0)$ (where T_0 is the temperature at $v = 0$). The co-ordinate x and the time t are expressed in terms of the function χ by formulae (4.12'). The relevant region of the solution is bounded on one side by the travelling wave and on the other by the shock wave. We shall derive the boundary conditions for the function χ .

At the boundary with the travelling wave we have a condition similar to (4.19):

$$\chi = 0 \quad \text{for} \quad \alpha = \frac{y}{c_0} = \sqrt{3y}. \quad (5.8)$$

Let us now consider the condition at the shock wave. Since in this case the matter behind the shock wave is not at rest, in a co-ordinate system in which

the velocity of the discontinuity is zero the velocity behind the discontinuity is

$$v'_1 = \frac{v_1 + D}{1 + v_1 D}, \quad (5.9)$$

where v_1 is the velocity behind the shock wave in a frame in which the velocities of the particles are equal before the collision. Formulae (5.2–5.4) remain valid in this case if D is replaced by v'_1 . Hence, in particular, it follows that $v'_1 = \frac{1}{3}$. Equation (5.9) gives the relation between v_1 and D (for $v'_1 = \frac{1}{3}$), which may be written

$$\frac{dx}{dt} = \frac{1 + 3 \tanh \alpha}{3 + \tanh \alpha}, \quad (5.10)$$

since $v_1 = -\tanh \alpha$. From (5.2–5.4) (with D replaced by v'_1) we can obtain the following relation at the shock wave:

$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{T}{T_0} \right)^4 = \frac{1 - v_1}{1 + v_1}. \quad (5.11)$$

In the variables α and y ($v_1 = \tanh \alpha$, $y = -\ln(T/T_0)$), this gives

$$\alpha = 2y. \quad (5.12)$$

Substituting in equation (5.10) the value of dx/dt calculated from formulae (4.12), and using (5.12), we have finally the following condition at the shock wave:

$$\left(3 \frac{\partial}{\partial y} + 5 \frac{\partial}{\partial \alpha} \right) \left(1 - \frac{\partial}{\partial \alpha} \right) \chi = 0 \quad \text{for } \alpha = 2y. \quad (5.13)$$

We are interested in the change of entropy during the passage of the shock wave. The change of entropy before the travelling wave overtakes the shock wave is given by (5.7). The subsequent change in entropy is

$$S_1 = \sigma_0 \int_{t'_1}^{t'_2} s u_1 dt', \quad (5.14)$$

where σ_0 is the cross-section of the "tunnel", S the entropy density behind the shock wave, $u_1 = v'_1/\sqrt{1 - v_1'^2}$, v'_1 the velocity of matter behind the shock wave, t'_1 the time when the travelling wave overtakes the shock wave and t'_2 the time when the shock wave reaches the right-hand boundary and so the compression process terminates; dt' is a time element in the frame in which the discontinuity is at rest: $dt' = dt\sqrt{1 - D^2}$.

Using formulae (4.12') and (5.12'), we have

$$S_1 = \frac{1}{9} \sigma_0 \int_0^{y_k} e^{2y} \left[\frac{\partial}{\partial \alpha} \left(\frac{\partial}{\partial y} - 1 \right) \chi \right]_{\alpha=2y} dy, \quad (5.15)$$

where y_k is the value of y when the shock wave reaches the edge of the system.

† See Appendix III, p. 698.

Effecting the fairly complex calculations, details of which are given in ref. 16, we find

$$\frac{S}{S_0} = 0.92 \left(\frac{l}{d} - \frac{1}{4} \right)^{3/4} \quad \text{for } \frac{l}{d} \geq 3.7, \quad (5.16)$$

where S_0 is the change of entropy in a nucleon-nucleon collision and S the total change of entropy in a collision of a nucleon with a "tunnel" of length l .

If we average formulae (5.7) and (5.16) over all possible collisions in the nucleus from a head-on collision to a collision between the incident nucleon and a peripheral one, the resulting expressions are

$$\frac{S}{S_0} = \frac{1}{3} \frac{A - (2A^{1/3} - 1)^{3/2}}{(A^{1/3} - 1)^2} + 0.5 \quad \text{for } A < 51, \quad (5.17)$$

$$\begin{aligned} \frac{S}{S_0} = & \frac{4}{(A^{1/3} - 1)^2} \left[0.167 (A^{11/12} - A_0^{11/12}) + \frac{1}{12} \{A_0 - (2A^{1/3} - 1)^{3/2}\} \right] \\ & - 0.6 \frac{A^{2/3} - A_0^{2/3}}{(A^{1/3} - 1)^2} + 0.5 \quad \text{for } A > 51, \quad A_0 = 51. \end{aligned} \quad (5.18)$$

The relation obtained can be approximated to within 5 per cent by the formula

$$\frac{S}{S_0} = A^{0.19} \quad (5.19)$$

(A is the atomic weight.) Hence the number of particles is

$$\frac{N}{N^n} = A^{0.19}, \quad (5.20)$$

where N^n is the number of particles formed in a nucleon-nucleon collision. The dependence of this number on the atomic weight is slight, in agreement with experiment¹⁵; the value for lead ($A = 207$) differs from that for nitrogen ($A = 14$) only by a factor of 1.7.

The angular and energy distributions in nucleon-nucleus collisions appear to differ only slightly, within the accuracy, of our calculations, from the corresponding distributions in a nucleon-nucleon collision.

APPENDIX I

Let us consider in more detail the derivation of formula (4.30) from (4.29). The latter formula may be written

$$h(-y) = \frac{\chi_1}{l\sqrt{3}} = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\exp\{-\sqrt{p^2-1}\alpha/\sqrt{3}-yp\}}{\sqrt{(p^2-1)(p+2)}} dp. \quad (I.1)$$

According to certain fundamental relations of operational calculus (ref. 17, p. 18),

$$\left. \begin{aligned} f(p) &= p \int_{-\infty}^{\infty} e^{-pt} h(t) dt, \\ h(t) &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{pt} \frac{f(p)}{p} dp. \end{aligned} \right\} \quad (\text{I.2})$$

Thus the function

$$f(p) = \frac{p \exp \{-\sqrt{(p^2 - 1)} \alpha / \sqrt{3}\}}{\sqrt{(p^2 - 1)}(p + 2)}$$

is the Laplace transform of the original function $h(-y)$.

We also make use of a rule of operational calculus called the convolution rule (ref. 17, p. 39). This states that if the transform $f_1(p)$ and $f_2(p)$ correspond to originals $h_1(t)$ and $h_2(t)$ respectively, then the transform $f(p) = f_1(p) f_2(p)/p$ corresponds to the original

$$h(t) = \int_{-\infty}^{\infty} h_2(\zeta) h_1(t - \zeta) d\zeta. \quad (\text{I.3})$$

Thus, if we know the originals $h_1(t)$ and $h_2(t)$, we can use the relation (I. 3) to find the original of the transform $f(p)$. We resolve the transform $f(p)$ into the factors

$$f_1(p) = \frac{p}{p + 2}, \quad f_2(p) = \frac{p \exp [-\sqrt{(p^2 - 1)} \alpha / \sqrt{3}]}{\sqrt{(p^2 - 1)}}. \quad (\text{I.4})$$

The transform $f_1(p)$ corresponds to the original

$$h_1(-y) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{p + 2} e^{-yp} dp = e^{2y} U(-y). \quad (\text{I.5})$$

The transform $f_2(p)$ corresponds to the original

$$\begin{aligned} h_2(-y) &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{-yp} \frac{1}{\sqrt{p^2 - 1}} \exp \left[\frac{-\alpha \sqrt{(p^2 - 1)}}{\sqrt{3}} \right] dp \\ &= I_0 \left[\sqrt{y^2 - \frac{1}{3} \alpha^2} \right] U \left(-y - \frac{\alpha}{\sqrt{3}} \right); \end{aligned} \quad (\text{I.6})$$

see (ref. 17, formula (XV, 15)). Here I_0 is the Bessel function of imaginary argument, and $U(t) = 1$ for $t > 0$, $\frac{1}{2}$ for $t = 0$, 0 for $t < 0$. Substitution of (I. 5)

and (I. 6) in (I. 3) gives

$$h(-y) = \int_{\alpha/\sqrt{3}}^{-y} e^{2(y+y_1)} I_0 \left[\sqrt{y_1^2 - \frac{1}{3} \alpha^2} \right] dy_1. \quad (\text{I.7})$$

This leads immediately to formula (4.30):

$$\chi = l\sqrt{3} e^y \int_{\alpha/\sqrt{3}}^{-y} e^{2y_1} I_0 \left[\sqrt{y_1^2 - \frac{1}{3} \alpha^2} \right] dy_1.$$

In Khalatnikov's paper¹⁴ a factor $l\sqrt{3}$ is omitted from his formula (4.36) for χ .

APPENDIX II.

In order to derive formulae (4.51) and (4.52) we change from the distributions in the centre of mass system to those in the laboratory system of co-ordinates. Let us first consider the transformation of angles. The angle of flight χ in the laboratory system is related to the angle θ in the centre of mass system by

$$\tan \chi \approx \chi = \frac{v' \sqrt{(1 - V^2)} \sin \theta}{v' \cos \theta + V}, \quad (\text{II.1})$$

where v' is the velocity of the particle in the centre of mass system and V that of the centre of mass in the laboratory system. The denominator in (II. 1) may be written as

$$v' \cos \theta + V \approx v' \cos \theta + v' + \frac{1}{2} (V^2 - v'^2)$$

since

$$V^2 - v'^2 = (V + v')(V - v') \approx 2(V - v').$$

Because V is closer than v' to 1, we have

$$v' \cos \theta + V \approx (1 + \cos \theta) + \frac{1}{2} (1 - v'^2).$$

Using the relation $u' = 1/\sqrt{(1 - v'^2)}$ gives

$$v' \cos \theta + V = 1 + \cos \theta + \frac{1}{2 u'^2} \quad (\text{II.2})$$

The last term can be of importance only for angles θ which are close to π . In order for that term to be small in this case also, we must have

$$\theta \gg \frac{1}{u'}. \quad (\text{II.3})$$

According to (4.45), $\theta \approx e^{-\lambda}$, and (4.50) shows that

$$u' \approx \exp \left[-\frac{1}{6} L + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right].$$

Thus the inequality (II. 3) signifies that

$$\exp \left[\frac{1}{6} L - \frac{1}{3} \sqrt{(L^2 - \lambda^2)} \right] \ll 1. \quad (\text{II.4})$$

This inequality is always satisfied, since the left-hand side equals $e^{-L/6}$ for $\lambda = 0$ and becomes equal to unity only for $\lambda = \lambda_{\max}$. Putting $v' = 1$ in the numerator of (II. 1) and replacing $v' \cos \theta + V$ by $1 + \cos \theta$, we obtain

$$\chi = \sqrt{(1 - V^2)} \tan \frac{1}{2} \theta. \quad (\text{II.5})$$

The break up is symmetrical in the centre of mass system for any collision of two identical particles. Hence angles θ occur as frequently as $\pi - \theta$. But $\tan \frac{1}{2} (\pi - \theta) = 1/\tan \frac{1}{2} \theta$. If we average $\ln \chi$ over all particles, the result is

$$\overline{\ln \chi} = \ln \sqrt{1 - V^2} = -L, \quad (\text{II.6})$$

i.e. formula (4.54).

In order to take account of angles of the order of unity, we define λ as

$$\lambda = -\ln \tan \frac{1}{2} \theta. \quad (\text{II.7})$$

If we substitute in (II. 5) $\tan \frac{1}{2} \theta = e^{-\lambda}$ and $\tan \frac{1}{2} (\pi - \theta) = e^{\lambda}$ for particles moving in opposite directions in the centre of mass system, it is easy to see that the change from particles moving to the right to those moving to the left (in the centre of mass system) is effected by changing the sign of λ . Hence we can put

$$\chi = e^{-L - \lambda}. \quad (\text{II.8})$$

Formulae (II. 8) and (4.44) thus gives the angular distribution of particles in the laboratory system, with λ taking positive and negative values. These formulae are essentially the same as (4.51) and (4.53).

Let us now consider the transformation of energy from the centre of mass system to the laboratory system. Let ε be the energy of a particle in the centre of mass system. If the particle is moving to the right its energy in the laboratory system is

$$\varepsilon' = \frac{(\varepsilon + P_x V)}{\sqrt{1 - V^2}} \quad (\text{II.9})$$

where P_x is the momentum component in the direction of the velocity of the centre of mass:

$$P_x = \sqrt{\varepsilon^2 - (u c^2)^2} - P_y, \quad P_y = P \sin \theta,$$

and P is the magnitude of the momentum, μ the mass of the particle. If the velocity of the particles in the centre of mass system approaches that of light, then

$$\varepsilon' = \frac{\varepsilon(1 + \cos\theta)}{\sqrt{1 - V^2}}.$$

Thus

$$\varepsilon' = \varepsilon e^L \cdot \frac{2}{1 + e^{-2\lambda}} \approx \varepsilon e^L. \quad (\text{II.10})$$

If the particles are moving to the left, then

$$\varepsilon' = \frac{\varepsilon - P_x V}{\sqrt{1 - V^2}} = \frac{1 - \cos\theta}{\sqrt{1 - V^2}} \varepsilon,$$

since

$$\theta \gg \frac{1}{u'} \approx \frac{\mu c^2}{\varepsilon}.$$

Thus

$$\varepsilon' = \frac{\varepsilon e^{L-2\lambda} \cdot 2}{1 + e^{-2\lambda}} \approx \varepsilon e^{L-2\lambda}. \quad (\text{II.11})$$

Substituting these expressions in equation (4.50), we obtain

$$\varepsilon' \approx M \exp \left[\frac{5}{6} L + \lambda + \frac{1}{3} \sqrt{L^2 - \lambda^2} \right]. \quad (\text{II.12})$$

This formula has the property of describing in the laboratory system particles moving both to the right and to the left in the centre of mass system, if λ given both signs.

APPENDIX III

We shall derive the relation (5.11).

Let the velocity behind the shock wave be v'_1 in a co-ordinate system in which the shock wave is at rest, and that in front of the shock be v'_2 . Then equations (5.2) and (5.3) become

$$\frac{p_1 + v_1'^2 \varepsilon_1}{1 - v_1'^2} = \frac{p_2 + v_2'^2 \varepsilon_2}{1 - v_2'^2}, \quad (\text{III.1})$$

$$\frac{v'_1(p_1 + \varepsilon_1)}{1 - v_1'^2} = \frac{v'_2(p_2 + \varepsilon_2)}{1 - v_2'^2}. \quad (\text{III.2})$$

The quantities v'_1 and v'_2 are related to the velocities v_1 and v_2 in a co-ordinate system where the velocities of the incident particles are equal by

$$v'_1 = \frac{(v_1 + D)}{(1 + v_1 D)}, \quad (\text{III.3})$$

$$v'_2 = \frac{(v_2 + D)}{(1 + v_2 D)}. \quad (\text{III.4})$$

Eliminating v'_1 or v'_2 from equations (III. 1) and (III. 2), we obtain

$$v_1'^2 = \frac{(p_1 - p_2)(p_1 + \varepsilon_2)}{(\varepsilon_1 - \varepsilon_2)(p_2 + \varepsilon_1)}, \quad (\text{III.5})$$

$$v_2'^2 = \frac{(p_2 - p_1)(p_2 + \varepsilon_1)}{(\varepsilon_2 - \varepsilon_1)(p_1 + \varepsilon_2)}. \quad (\text{III.6})$$

The last two formulae give

$$v'_1 v'_2 = \frac{(p_1 - p_2)}{(\varepsilon_1 - \varepsilon_2)}. \quad (\text{III.7})$$

To simplify the derivation we assume that the equation of state is the same ($p = \frac{1}{3}\varepsilon$) on both sides of the shock wave. It should be noted, however, that at high values of v_2 it is sufficient to assume this equation valid only behind the shock wave. From (III. 7) we have

$$v'_1 v'_2 = \frac{1}{3}. \quad (\text{III.7a})$$

Equation (III. 5) becomes

$$v_1'^2 = \frac{1}{3} \frac{p_1 + \varepsilon_2}{\frac{1}{3}\varepsilon_2 + 3p_1}, \quad (\text{III.5a})$$

whence

$$p_1 = \varepsilon_2 \frac{1 - v_1'^2}{9v_1'^2 - 1}. \quad (\text{III.8})$$

Substituting (III. 3) and (III. 4) in equation (III. 7), we have after some easy transformations

$$D = \frac{-(v_1 + v_2) + \sqrt{(v_1 + v_2)^2 - (3v_1 v_2 - 1)(3 - v_1 v_2)}}{3 - v_1 v_2}. \quad (\text{III.9})$$

Let us now find the ratio p_{1v_1}/p_{10} , where p_{1v_1} is the pressure for a velocity v_1 behind the shock wave and p_{10} the pressure for a velocity zero behind the shock wave. From (III. 8), this ratio is

$$\frac{p_{1v_1}}{p_{10}} = \frac{(1 - v_1'^2)}{(9v_1'^2 - 1)} \cdot \left[\frac{(9v_1'^2 - 1)}{(1 - v_1'^2)} \right]_0 \quad (\text{III.10})$$

Since, for large v_2 (tending to 1), $v'_2 \rightarrow 1$ also (see III. 4)), the relation (III. 7a) gives $v'_1 \rightarrow \frac{1}{3}$. The ratio (III. 10) becomes indeterminate. We must therefore effect a passage to the limit:

$$\left[\frac{p_{1v_1}}{p_{10}} \right]_{v_2 \rightarrow 1} \frac{[9v_1'^2 - 1]_{v_1=0}}{[9v_1'^2 - 1]_{v_1}} = \left\{ \frac{[\partial v'_1 / \partial v_2]_{v_1=0}}{[\partial v'_1 / \partial v_2]_{v_1}} \right\}_{v_2=1} \quad (\text{III.11})$$

But

$$\frac{\partial v'_1}{\partial v_2} = \left[\frac{\partial v'_1}{\partial D} \frac{\partial D}{\partial v_2} \right]_{v_2=1} \quad (\text{III.12})$$

The derivative $\partial v'_1/\partial D$ is determined from equation (III. 3), and $\partial D/\partial v_2$ from equation (III. 9):

$$\left(\frac{\partial v'_1}{\partial D}\right)_{v_2=1} = \frac{(3-v_1)^2}{9(1-v_1^2)}; \quad \left(\frac{\partial D}{\partial v_2}\right)_{v_2=1} = -\frac{3(v_1+1)^2}{2(3-v_1)^2}. \quad (\text{III.13})$$

Hence we easily obtain

$$\frac{p_{i0}}{p_{10}} = \frac{1-v_1}{1+v_1}. \quad (\text{III.14})$$

The equation of state immediately gives formula (5.11) from (III. 14).

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89. ON THE QUANTUM THEORY OF FIELDS

Contents. 1. The fundamental equations. 2. The asymptotical expression for the electron Green function. 3. The asymptotical expression for the photon Green function. 4. The mass of the electron in quantum electrodynamics. 5. The infra-red catastrophe. 6. The gauge transformations of the Green functions of charged particles. 7. The Green functions in the meson theory with weak pseudoscalar coupling.

1. THE FUNDAMENTAL EQUATIONS

As is well known, the main difficulties of the quantum field theory arise from the infinities. The existing methods of removing these infinities (regularisation) sometimes give quite correct results, but in general are of a very artificial nature and in many cases become inapplicable. Therefore it seems necessary to construct a theory without considering actually infinite quantities.

In accordance with the usual methods of theoretical physics, it is natural to regard the point interaction as a limit of a smeared out interaction with a finite decreasing radius. The order of magnitude of the radius we will denote by a . Then $\Lambda \sim 1/a$ is the upper limit of integration in momentum space. All the calculations will be made under the assumption, that the un-renormalised bare charge e_1 , which appears in the interaction term in the Lagrangian, satisfies the condition $e_1^2 \ll 1$.

The Green functions of the electron $G(p)$ and photon $D_{\mu\nu}(k)$ are connected with the vertex part $\Gamma_\mu(p, p-k; k)$ by the Dyson equations

$$G(p) \left\{ \hat{p} - m_1 - \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\mu D_{\mu\nu}(k) d^4 k \right\} = 1, \quad (1.1)$$

$$D_{\mu\nu}(k) = D_{\mu\nu}^0(k) - \frac{e_1^2}{\pi i} D_{\mu\sigma}(k) \text{Tr} \left[\int G(p) \Gamma_\sigma(p, p-k; k) G(p-k) \gamma_\tau d^4 p \right] \times D_{\tau\nu}^0(k). \quad (1.2)$$

Here and further the Feynman notation will be used. $D_{\mu\nu}^0$ is the photon Green function in zeroth approximation. Usually it is taken to be equal to $\delta_{\mu\nu}/k^2$. This choice is however inconvenient. Actually the longitudinal (proportional to $k_\mu k_\nu$) part of the tensor $D_{\mu\nu}$, in consequence of gauge invariance, can be taken arbitrarily and is not altered by perturbations (owing to the transversality of the Dirac current, the integral term in equation (1.2)).

L. D. Landau, A. Abrikosov and I. Halatnikov (= Khalatnikov), On the quantum theory of fields, *Nuovo Cimento, Supplement*, 3, 80 (1956).

We will take $D_{\mu\nu}^0$ in the form of $(1/k^2)(\delta_{\mu\nu} - (k_\mu k_\nu/k^2))$. Then the equation for the transversal part of the tensor $D_{\mu\nu}$ is

$$D_{\mu\sigma}^t \left(k^2 \delta_{\sigma\nu} + \frac{e_1^2}{\pi i} \text{Tr} \left[\int G(p) \Gamma_\sigma(p, p-k; k) G(p-k) \gamma_\nu d^4 p \right] \right) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \quad (1.3)$$

The unpleasant feature of the integrals in (1.1) and (1.3) is the integration over the pseudo-euclidean space. Let the k -vector in (1.3) have the time-like component equal zero. Then the whole integrated expression is a function of p_0 . After some simple operations we are left with a function of p_0^2 . As is well known, this function is regular in the upper half-plane. Owing to the smearing out it vanishes in the infinity. Therefore we have

$$\int_{-\infty}^{\infty} f(p_0^2) dp_0 = \int_0^{\infty} f(x) \frac{dx}{\sqrt{x}} = i \int_0^{\infty} f(-x) \frac{dx}{\sqrt{x}} = i \int_0^{\infty} f(-p_0^2) dp_0. \quad (1.4)$$

This corresponds to the fact that p_0 in the integral is changed to $i p_0$. Then the integration will be extended over a euclidean space. By this transformation G and D , for space vectors, will be expressed through the space-like Γ_μ . G and D , for time-like vectors, may be found by means of analytical continuation. The exception is G for p^2 nearing m^2 , but this case will be discussed later. As to the $\Gamma_\mu(p, p; k)$ which we obtain here, the analytical continuation will give only the cases when the p^2 , q^2 and k^2 are of the same order or, if one of these squares is small, the two others should be near to each other.

As is well known, the power of the logarithmic divergence never exceeds the implied order of the perturbation theory†. This means, that the main terms in the perturbation series are the terms proportional to $(e_1^2 \ln A^2/p^2)^n$ which are the only terms we will account for in calculations of the principal approximation.

First we will show that for the space-like Γ_μ a closed integral equation can be written. For this purpose we will use the fact that the addition of a virtual photon line surrounding two vertices will give only a convergent integral. Therefore in the diagrams representing successive approximations for the vertex part we may neglect the lines belonging to the interference of vertices, which contain powers of logarithms lower than the order of perturbation theory. Particularly we may neglect diagrams with crossing photon lines. In the same manner more complicated diagrams are also excluded. The diagrams which are left form ladders where every vertex has an additional ladder of the same kind, and the photon and electron lines are summed over all approximations,

† The quadratically divergent photon mass should be put equal zero. The presence of a finite photon mass would violate the charge conservation law.

‡ We should mention, that in general this is not true for not space-like Γ_μ . So when k^2 is large compared to p^2 and q^2 the formulae include $\ln k^2/q^2$, and their common power may be twice the power of e_1^2 . This question is discussed in more detail by V. V. Sudakov².

i.e. are represented by the true G and D . The summation of the described scheme with the assumptions implied may be easily performed. Then we arrive at the following equations

$$\Gamma_\sigma(p, p-l; l) = \gamma_\sigma + \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \Gamma_\sigma(p-k; p-k-l; l) \\ \times G(p-k-l) \Gamma_\nu(p-k-l, p-l; k) D_{\mu\nu}(k) d^4k. \quad (1.5)$$

2. THE ASYMPTOTICAL EXPRESSION FOR THE ELECTRON GREEN FUNCTION³

In the preceding section we have derived the equations for the fundamental functions in quantum electrodynamics. Since our purpose is the exact solution of these equations we cannot use the usual expressions for the quantities G , $D_{\mu\nu}$ and Γ_μ even with renormalised coefficients.

It was mentioned above that in view of the analytical properties of the functions G and $D_{\mu\nu}$, they may be determined only for negative values of p^2 and k^2 (space vectors). The main interest in solving the equations is the behaviour of all functions at large values of momenta $p \gg m$, $k \gg m$. We will suppose the functions G and $D_{\mu\nu}$ to have the form

$$G(p) = \frac{\beta(p^2)}{\hat{p}}, \quad (2.1)$$

$$D_{\mu\nu}(k) = \frac{1}{k^2} \left[d_t(k^2) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + d_l(k^2) \frac{k_\mu k_\nu}{k^2} \right], \quad (2.2)$$

where the functions $\beta(p^2)$, $d_t(k^2)$, $d_l(k^2)$ are slowly varying functions of p^2 and k^2 . As to the Γ_μ , we shall show that the equation (1.5) is satisfied by the following form:

$$\Gamma_\mu(p, q; l) = \gamma_\mu \alpha(f^2), \quad (2.3)$$

where $\alpha(f^2)$ is a slowly varying function and f^2 is any of p^2 , q^2 and l^2 , if they are of the same order of magnitude. If one of the squares is small compared to the other two, f^2 should be the largest square.

Let us substitute the expressions (2.1), (2.2) and (2.3) in the equation (1.5). It is easy to see that the integral on the right-hand side of the equation, excepting from the slowly varying functions, diverges logarithmically at $|k^2| \gg |p^2|$ and $|k^2| \gg |q^2|$. Hence this region plays the main role in the integration. Here we can neglect p , q and l as compared to k and the whole integrated expression becomes a function of k only. After averaging over the directions of k , the integral becomes equal to a scalar function multiplied by γ_μ . This confirms the supposed form (2.3). To transform the integration over the k space to a simple integration over k^2 we use the substitution $k_0 \rightarrow ik_0$. Then d^4k transforms to $i(2\pi)^{-2} d\Omega$. $d\Omega$ is a volume element in the four-dimensional

euclidean space and has the form $2\pi^2 R^3 dR$, where R is the radius equal to $\sqrt{-k^2}$. So we have

$$d^4k \rightarrow \frac{i}{4} (-k^2) d(-k^2). \quad (2.4)$$

Introducing a new variable $\xi = \ln(-p^2/m^2)$ we obtain the final equation

$$\alpha(\xi) = 1 + \frac{e_1^2}{4\pi} \int_{\xi}^{\infty} \alpha^3(z) \beta^2(z) d_l(z) dz. \quad (2.5)$$

It should be mentioned that d_l does not enter this equation.

Now we turn to the equation (1.1) for G . It can be seen that the integral in this equation diverges at $k \gg p$. Although this divergence at first sight is linear in k , the averaging over the directions gives again, owing to the vector character, of k , a logarithmic divergence. That means that by expansion of the integrated expressions in reciprocal powers of k we should account everywhere for the terms of the next order. The corresponding terms in G are found directly from the formula (2.1) where we can, of course, in spite of the weak dependence of β on p^2 , restrict ourselves to the expansion of the denominator, i.e.

$$G(p-k) \approx -\frac{\beta(k^2)}{\hat{k}} - \beta(k^2) \frac{1}{\hat{k}} \hat{p} \frac{1}{\hat{k}}. \quad (2.6)$$

However, the corrections of the order of p/k arising from $\Gamma_\mu(p, p-k; k)$ should be also taken into account. To find these corrections we will turn again to the equation for Γ_μ . Since we want to calculate small corrections occurring by passing from $p=0$ to finite p 's, we may consider the change of the integral on the right hand side of (1.5) as a sum of changes arising separately from each term of the integrated expression. The evaluation of the resulting expression shows that at $k \gg l$ the integrated function is of the order of $1/k^5$, i.e. the integral converges at large k . However, when k is in the interval $l \gg k \gg p$ the integral becomes logarithmic. So this interval is the main region of integration. At larger and smaller values of k the integral gives relatively small corrections which we may disregard within the accepted approximation.

In the mentioned region of integration it is enough to consider only corrections of the order of p/k , since they are considerably larger than p/l . It may be shown that the corrections in the expression for $\Gamma_\sigma(p, p-l; l)$ will have the form

$$[\Delta_1(p^2, l^2) \hat{l} \gamma_\sigma + \Delta_2(p^2, l^2) \gamma_\sigma \hat{l}] \frac{p}{\hat{l}}. \quad (2.7)$$

Indeed it is easy to verify that corrections of other types do not appear for G taken in the form (2.6), and if substituted in the integral give again expressions of the form (2.7), and therefore turn out to be zero by comparison of the right and left sides of the equation.

Substituting (2.6) and (2.7) in the equation for Γ_σ and using the transformation (2.4) we come to the following equations for Δ_1 , and Δ_2 :

$$\left. \begin{aligned} \Delta_1(\xi, \eta) &= \frac{e_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \{ \Delta_1(\xi, z) [d_l(z) - \bar{d}_l(z)] \\ &\quad + [\Delta_2(\xi, z) + \alpha(z)] [d_l(z) + \bar{d}_l(z)] \} dz \\ \Delta_2(\xi, \eta) &= \frac{e_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \{ \Delta_1(\xi, z) [d_l(z) + \bar{d}_l(z)] \\ &\quad + [\Delta_2(\xi, z) + \alpha(z)] [d_l(z) - \bar{d}_l(z)] \} dz \end{aligned} \right\} \quad (2.8)$$

where $\xi = \ln(-p^2/m^2)$, $\eta = \ln(-l^2/m^2)$; substituting (2.6) and (2.7) in the integral of the equation for G , where the main region of integration is $p \ll k < \infty$, we obtain, after similar transformations,

$$\frac{1}{\beta(\xi)} = 1 + \frac{e_1^2}{4\pi} \int_{\xi}^{\infty} [\alpha(z) \Delta_1(\xi, z) + \Delta_2(\xi, z)] \beta(z) d_l(z) dz. \quad (2.9)$$

Adding the two equations (2.8) we have

$$\Delta_1(\xi, \eta) + \Delta_2(\xi, \eta) = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) d_l(z) [\Delta_1(\xi, z) + \Delta_2(\xi, z) + \alpha(z)] dz. \quad (2.10)$$

It should be noted that the quantity d_l does not enter in the formulae (2.9) and (2.10). Therefore the equations for Γ_μ and G can be solved independently of calculating the photon Green function $D_{\mu\nu}$. The longitudinal part d_l , as mentioned previously, is an arbitrary quantity which we retain in the equations in order to prove their gauge invariance. Of course the dependence of G and Γ_μ upon d_l does not contradict the gauge invariance. The general gauge transforms of G and Γ_μ shall be found afterwards. Here we restrict ourselves to slowly varying $d_l(k^2)$.

Let us introduce the notation

$$q(\xi, \eta) = \frac{\Delta_1(\xi, \eta) + \Delta_2(\xi, \eta)}{\alpha^2(\eta) \beta(\eta)}. \quad (2.11)$$

The function $q(\xi, \eta)$ satisfies the equation

$$q(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d_l(z) q(\xi, z) dz + \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) d_l(z) dz. \quad (2.12)$$

The equation (2.9) can now be written in the form

$$\frac{1}{\beta(\xi)} = 1 + q(\xi, \infty). \quad (2.13)$$

The equations (2.5), (2.12), (2.13) can be solved for α and β . After differentiating equations (2.5) and (2.12) (the second with respect to η) we obtain

$$\frac{d\alpha}{d\xi} = -\frac{e_1^2}{4\pi} \alpha^2(\xi) \beta^2(\xi) d_l(\xi), \quad (2.14)$$

$$\frac{\partial q(\xi, \eta)}{\partial \eta} = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta^2(\eta) d_l(\eta) q(\xi, \eta) + \frac{e_1^2}{4\pi} \alpha(\eta) \beta(\eta) d_l(\eta). \quad (2.15)$$

There should be added the boundary conditions:

$$\alpha \rightarrow 1 \quad \text{when} \quad \xi \rightarrow 1,$$

$$q = 0 \quad \text{at} \quad \xi = \eta.$$

Introducing a new function $r(\xi, \eta) = \alpha(\eta) q(\xi, \eta)$ we have

$$\frac{\partial r(\xi, \eta)}{\partial \eta} = \frac{e_1^2}{4\pi} \alpha^2(\eta) \beta(\eta) d_l(\eta).$$

Hence

$$r(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta(z) d_l(z) dz,$$

$$\frac{\partial r(\xi, \eta)}{\partial \xi} = -\frac{e_1^2}{4\pi} \alpha^2(\xi) \beta(\xi) d_l(\xi).$$

From (2.13) we have that, when $\eta \rightarrow \infty$, $r(\xi, \infty) = (1/\beta(\xi)) - 1$. Hence:

$$\frac{d\beta}{d\xi} = \left(\frac{e_1^2}{4\pi} \right) \alpha^2(\xi) \beta^3(\xi) d_l(\xi).$$

From this equation and (2.14) it follows that $\alpha\beta = \text{const}$, and taking into account the boundary conditions we have:

$$\alpha(\xi) \beta(\xi) = 1. \quad (2.16)$$

It is easy to see that this formula follows immediately from Ward's theorem⁴

$$\Gamma_{\mu}(p, p; 0) = \frac{\partial G^{-1}(p)}{\partial p_{\mu}},$$

if we mention that the slowly varying function β gives terms of higher order by differentiation.

After the substitution of (2.16) into the equation (2.14) we obtain finally, making use also of the boundary condition

$$\alpha(\xi) = \exp \left[\frac{e_1^2}{4\pi} \int_{\xi}^{\infty} d_l(z) dz \right], \quad \beta(\xi) = \exp \left[-\frac{e_1^2}{4\pi} \int_{\xi}^{\infty} d_l(z) dz \right]. \quad (2.17)$$

The formulae express in fact the gauge transformation of G and Γ_μ (for slowly varying d_i). The most convenient choice is evidently $d_i = 0$ (contrary to the usual assumption $d_i = 1$). Then to the considered order Γ_μ and G have no divergences at all and these quantities (Γ_μ with the mentioned limitations) are equal to their zeroth approximation values. It should be mentioned that this calibration actually corresponds to the Lorentz condition $\partial A_i / \partial x_i = 0$.

3. THE ASYMPTOTICAL EXPRESSION FOR THE PHOTON GREEN FUNCTION⁵

Now we proceed to the calculation of the photon Green function $D_{\mu\nu}$. The integral standing in the equation for $D_{\mu\nu}$ (1.3) diverges at large k , this time quadratically. However, as the value of the integral at $k = 0$ should vanish in consequence of charge conservation (the transverse character of the Dirac current), we may consider only the difference between this integral and its value at $k = 0$. Taking in view that terms having p^3 in the denominator give zero being averaged over the angles, we arrive again to a logarithmic integral. According to the fact that the region $p \ll k$, as it is easily seen, gives nothing noticeable in the integral, it is enough to consider only the region $p \gg k$.

Calculating the logarithmic integral we have to take into account the corrections in $\Gamma_\mu(p, p - k; k)$ of the second order in (k/p) , as it was done in the preceding section. To find these corrections we will use the equation for Γ_σ (1.5) with the following transformation of the variable in the integral: $k' \rightarrow p - k$. Then the integral takes the form

$$\int D_{\mu\nu}(p - k') \Gamma_\mu(p, k'; p - k') \times G(k') \Gamma_\sigma(k', k' - l; l) G(k' - l) \Gamma_\nu(k' - l, p - l; k' - p) d^4 k'.$$

Here it is assumed $p \gg l$. It is evident that in the expansion of $G(k' - l)$ in powers of l , terms of the second order in l arise which give a logarithmic integral in the region $p \gg k' \gg l$. The terms of the first order in l cannot appear because they have k'^3 in the denominator and vanish after integration over the angles. The terms of considered order may come from $G(k' - l)$ and from $\Gamma_\sigma(k', k' - l; l)$. The l dependence of the last Γ_σ gives terms not of the order l^2/k^2 , but of the order of l^2/p^2 , which may be neglected.

We start with the calculation of corrections coming from $G(k' - l)$. After some calculations we find

$$\frac{e_1^2}{24\pi} \left\{ 2d_i(\xi) \frac{\hat{l} \gamma_\sigma \hat{l} - l^2 \gamma_\sigma}{p^2} + [d_i(\xi) - d_i(\xi)] \frac{l^2 \hat{p} \gamma_\sigma \hat{p} - \hat{p} \hat{l} \gamma_\sigma \hat{l} \hat{p}}{p^4} \right\} \times \alpha^2(\xi) \int_{\eta}^{\xi} \alpha(z) \beta^2(z) dz, \quad (3.1)$$

where $\xi = \ln(-p^2/m^2)$, $\eta = \ln(-l^2/m^2)$.

It turns out that the corrections to $\Gamma_\sigma(p, p-l; l)$ should be supposed to have the form

$$s(\xi, \eta) \alpha^2(\xi) \left\{ 2d_l(\xi) \frac{\hat{l} \gamma_\sigma \hat{l} - l^2 \gamma_\sigma}{p^2} + [d_l(\xi) - d_l(\xi)] \frac{l^2 \hat{p} \gamma_\sigma \hat{p} - \hat{p} \hat{l} \gamma_\sigma \hat{l} \hat{p}}{p^4} \right\}. \quad (3.2)$$

Substituting this expression in the integral and comparing terms on the left and right sides of the equation (1.5) for Γ_σ , we obtain the following equation for $s(\xi, \eta)$:

$$s(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\eta}^{\xi} \alpha^2(z) \beta^2(z) d_l(z) s(z, \eta) dz + \frac{e_1^2}{24\pi} \int_{\eta}^{\xi} \alpha(z) \beta^2(z) dz. \quad (3.2)$$

Substituting the expression (3.2) giving the correction to Γ_σ , into the equation for $D_{\mu\nu}$ (1.3) we obtain

$$d_l(\eta) = 1 - 8d_l(\eta) \left[\frac{e_1^2}{4\pi} \int_{\eta}^L \alpha^2(z) \beta^2(z) d_l(z) s(z, \eta) dz + \frac{e_1^2}{24\pi} \int_{\eta}^L \alpha(z) \beta^2(z) dz \right]. \quad (3.4)$$

In this formula L denotes the upper limit of the considered values of the quantity η , which is determined by the radius of smearing out of the interactions. As already said, according to our scheme of smeared out interaction, the magnitude of interaction should drop rapidly when the momentum exceeds the critical value $\Lambda \sim 1/a$, where a is the radius of interaction. The condition $p \ll \Lambda$ turns to $\xi \ll L$ on logarithmic scale, because considerable changes of p correspond to relatively small changes of $\ln(-p^2/m^2)$. Therefore the fact of smearing out produces an upper limit L in all integrals. It should be emphasised that, according to the general scheme, the constant e_1 is actually a function of the range of interaction i.e. of the quantity Λ . In calculating G and Γ_μ it was unnecessary to use the smearing out because of the divergence of $d_l(\xi)$, which is arbitrary and independent of the character of interaction. But if we introduce the smearing out we have to put the upper limit L , instead of the infinity, in the integrals standing in the exponents of α and β .

It should be mentioned that then we have $\beta(L) = 1$. This corresponds to the evident fact that at $p \gg \Lambda$ the electron must behave as free, not interacting with the photon field. Comparing the formula (3.4) with the formula (3.5) we obtain:

$$\frac{1}{d_l(\eta)} = 1 + 8s(L, \eta). \quad (3.5)$$

Differentiating the equation (3.3) with respect to ξ and taking into account the relations $\alpha = 1/\beta$ and $d\beta/d\xi = (e_1^2/4\pi) d_l(\xi) \beta(\xi)$ we have

$$\frac{\partial s(\xi, \eta)}{\partial \xi} = s(\xi, \eta) \frac{1}{\beta(\xi)} \frac{d\beta(\xi)}{d\xi} + \frac{e_1^2}{24\pi} \beta(\xi). \quad (3.6)$$

Hence

$$\frac{\partial}{\partial \xi} \left(\frac{s(\xi, \eta)}{\beta(\xi)} \right) = \frac{e_1^2}{24\pi}.$$

Noting that $s = 0$ for $\xi = \eta$ in accordance with (3.3) we have

$$s(\xi, \eta) = \frac{e_1^2}{24\pi} (\xi - \eta) \beta(\xi). \quad (3.7)$$

Substituting here $\xi = L$ and remembering $\beta(L) = 1$ we come to the final expression

$$d_i(k^2) = \frac{1}{1 + (e_1^2/3\pi) \ln(\Lambda^2/(-k^2))}. \quad (3.8)$$

It should be mentioned that, contrary to G and Γ_μ , this expression does not include the arbitrary function d_i , quite as it should be according to gauge invariance. At $k^2 \ll m^2$, $\ln(\Lambda^2/m^2)$ stands in place of $\ln(\Lambda^2/(-k^2))$. If particles other than electrons participate in vacuum polarisation, then the formula takes the form

$$d_i(k^2) = \frac{1}{1 + (\nu e_1^2/3\pi) \ln(\Lambda^2/(-k^2))}, \quad (3.9)$$

where the contribution of particles with spin $1/2$ and charge Ze equals Z^2 , and that of particles with spin 0 equals $Z^2/4$. Particles with higher spins give quadratically divergent expressions which we disregard here, the more so, as there is no indication on the existence of such particles in nature.

The analysis of physical effects shows that the physical charge e is expressed in terms of e_1 by means of the relation

$$e^2 = e_1^2 \lim_{k^2 \rightarrow 0} d_i(k^2). \quad (3.10)$$

From formula (3.9) it follows therefore

$$e^2 = \frac{e_1^2}{1 + (\nu e_1^2/3\pi) \ln(\Lambda^2/m^2)}, \quad (3.11)$$

or

$$e_1^2 = \frac{e^2}{1 - (\nu e^2/3\pi) \ln(\Lambda^2/m^2)}. \quad (3.12)$$

Expressing d_i through e^2 we have

$$d_i(k^2) = \frac{e^2}{e_1^2} \frac{1}{1 - (\nu e^2/3\pi) \ln(-k^2/m^2)}. \quad (3.13)$$

The formula (3.13) is evidently renormalizable, i.e. we can change in all the formulae the "bare" charge e_1 into the physical charge e multiplying simultaneously $D_{\mu\nu}$ by the renormalising factor $Z_3 = e_1^2/e^2$. Thereafter all the quantities will not depend upon the cut-off limit and the charge e_1 , which is connected with it.

From the formula (3.12) it follows that, for sufficiently large Λ , e_1^2 may become of the order of unity and the considered approximation fails. Hence the electrodynamics with weak coupling is a logically unclosed theory. It could be supposed that this theory should be supplemented by a strong-coupling theory at great energies. However I. Pomeranchuk and L. D. Landau suggested that there is a reason to understand the situation in a different way. According to their considerations, the formula (3.11) will be correct not only for small e_1^2 but also for $e_1^2 \gtrsim 1$. Then regarding the physical charge e as a function of e_1 and Λ , it may be seen that under a given Λ , however large e_1^2 may be, it cannot make e^2 exceed some limit, tending to zero with increasing Λ . This brings us to the conclusion that point interaction is impossible in pure electrodynamics.

It should be mentioned however that the whole question for electrodynamics has only a methodological sense, since at great energies other types of interactions may play a considerable role, in particular the gravitation.

Here we assumed everywhere that the radii of smearing out in the Dirac current and in the interaction of this current with electromagnetic potentials are equal. This assumption may be put aside. We will show how the results are changed if the radii are different. Let us introduce two different cut-off limits, Λ_p corresponding to the smearing out of the Dirac current, and Λ_A corresponding to the smearing out of the electromagnetic interaction of the current. It should be noticed that the Λ 's must satisfy the condition $\Lambda_A \lesssim \Lambda_p$. Then the formula (2.17) for $\alpha(\xi)$ and $\beta(\xi)$ will change to

$$\alpha(\xi) = \beta^{-1}(\xi) = \exp \left[\frac{e_1^2}{4\pi} \int_{\xi}^{\Lambda_A} d_l(z) dz \right] \quad L_A = \ln \left(\frac{\Lambda_A^2}{m^2} \right). \quad (3.14)$$

As to the d_l a simple calculation gives the following result

$$d_l = \frac{1}{1 + (v e_1^2 / 3\pi) \ln(\Lambda_p^2 / -k^2)} \quad \text{for } |k^2| < \Lambda_A^2 \quad (3.15)$$

$$d_l = 1 \quad \text{for } |k^2| > \Lambda_A^2. \quad (3.16)$$

It may be seen that in the case $\Lambda_p \gg \Lambda_A$, the d_l function undergoes a rapid change in the vicinity of $k^2 = \Lambda_A^2$.

4. THE MASS OF THE ELECTRON IN QUANTUM ELECTRODYNAMICS⁶

Now we will consider the mass of the electron, particularly the roles of electromagnetic and "bare" masses. The most important point in this respect is the behaviour of the electron Green function $G(p)$ when p^2 becomes of the order of m^2 . In section 2 we have obtained an asymptotic form of the electron Green function, justified for $p \gg m$. For the mass calculation it is insufficient to consider this expression of $G(p)$ at $p \sim m$, but we have to find corrections

to this expression. Let us suppose that $G(p)$ has the form

$$G(p) = \frac{\beta(p^2)}{\hat{p} - m(p^2)}, \quad (4.1)$$

where $\beta(p^2)$ is the function introduced in section 2 and $m(p^2)$ a new slowly varying function of p^2 . At large values of p , the ratio of the second term in the denominator to the first one becomes negligible, in any case considerably smaller than the inaccuracy in the calculated function $\beta(p^2)$. Nevertheless, the consideration of this term is justified because, contrary to the first term, it is an even function of the vector p . Therefore the inaccuracy in β cannot have any influence upon the value of $m(p^2)$. For large values of p , $G(p)$ may be written in the form

$$G(p) = \frac{\beta(p^2)}{\hat{p}} + \frac{\beta(p^2) m(p^2)}{p^2}, \quad (4.2)$$

where the first term is an odd and the second an even function of p .

Let us write the equation for the G function

$$G^{-1}(p) = \hat{p} - m_1 + \frac{e_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \gamma_\nu D_{\mu\nu}(k) d^4k \quad (4.3)$$

and separate from both sides the even function of p which must be equal to each other. Determining the even part of the integral on the right side we will change the sign of the variable k also. Then the even part of the integrated expression will consist of the product of the even part of Γ_μ (with respect to the change of sign of all variables) and the even part of G , and of the product of the odd part of Γ_μ and the odd part of G . In the first case we can use $\alpha(k^2) \gamma_\mu$ for Γ_μ , and the second term of (4.2) for G . In the latter case G is taken with the old form $\beta(p^2)/\hat{p}$, and the odd part of Γ_μ should be defined from the equation for Γ_σ (1.5).

Equating the odd parts of the expressions on either side of the equation, we will change the sign of the variable k together with the signs of \hat{p} and e . Thereafter the odd part of the integral expression will contain either the even part of one of the G 's or the odd part of one of the Γ 's. Other possibilities give terms of higher order. At first we will consider the influence of the even addition to G . Suppose $p \ll 1$. Then it is easy to see that the principal term of the expression comes from the even addition only to one of the G 's, namely to $G(p-k)$. The corresponding integral is logarithmic in the region $p \ll k \ll l$ and converges rapidly in the other regions. Direct calculation gives the following expression

$$- \frac{e_1^2}{4\pi} \frac{l_\sigma}{l^2} \alpha^2(\eta) \beta(\eta) \int_{\xi}^{\eta} \beta(z) \alpha(z) m(z) [3d_i(z) + d_s(z)] dz,$$

where, as before,

$$\xi = \ln\left(\frac{-p^2}{m^2}\right), \quad \eta = \ln\left(\frac{-l^2}{m^2}\right).$$

Now we will find the odd part of Γ_σ , which we suppose to have the form $(l_\sigma/l^2) \alpha^2(\eta) \beta(\eta) t(\xi, \eta)$.

One can easily verify that only the addition to the first $\Gamma_\mu(p, p-k; k)$ in (1.5) will give important terms. After substitution in (1.5), we obtain an integral equation for $t(\xi, \eta)$

$$t(\xi, \eta) = \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d_i(z) t(\xi, z) dz - \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) m(z) [3d_i(z) + d_i(z)] dz. \quad (4.4)$$

Substituting into equation (4.3) the expressions of the addition made to G and Γ_μ , we obtain from the even part of this equation

$$\begin{aligned} -\frac{m(\xi)}{\beta(z)} = & -m_1 + \frac{e_1^2}{4\pi} \int_{\xi}^L \alpha^2(z) \beta^2(z) d_i(z) t(\xi, z) \\ & - \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) m(z) [3d_i(z) + d_i(z)] dz. \end{aligned} \quad (4.5)$$

Then it follows

$$m(\xi) = \beta(\xi) [m_1 - t(\xi, L)]. \quad (4.6)$$

Differentiating the equation (4.4) with respect to η , and using the formulae $\beta\alpha = 1$ and $d\beta/d\xi = (e_1^2/4\pi) \beta d_i$, we obtain

$$\frac{\partial t(\xi, \eta)}{\partial \eta} = \frac{1}{\beta(\eta)} \frac{d\beta(\eta)}{d\eta} t(\xi, \eta) - \frac{e_1^2}{4\pi} m(\eta) [3d_i(\eta) + d_i(\eta)]$$

or

$$\frac{\partial}{\partial \eta} \left(\frac{t(\xi, \eta)}{\beta(\eta)} \right) = - \frac{e_1^2}{4\pi} \frac{m(\eta)}{\beta(\eta)} [3d_i(\eta) + d_i(\eta)].$$

Taking into account that, according to (4.4), $t(\xi, \eta) = 0$ at $\xi = \eta$, we have

$$\frac{t(\xi, \eta)}{\beta(\eta)} = - \frac{e_1^2}{4\pi} \int_{\xi}^{\eta} \frac{m(z)}{\beta(z)} [3d_i(z) + d_i(z)] dz. \quad (4.7)$$

Differentiating with respect to ξ we obtain

$$\frac{1}{\beta(\eta)} \frac{\partial t(\xi, \eta)}{\partial \xi} = \frac{e_1^2}{4\pi} \frac{m(\xi)}{\beta(\xi)} [3d_i(\xi) + d_i(\xi)].$$

Substituting $\eta = L$, and noting that $\beta(L) = 1$, we have

$$\frac{d}{d\xi} \frac{m(\xi)}{\beta(\xi)} = - \frac{e_1^2}{4\pi} \frac{m(\xi)}{\beta(\xi)} [3d_i(\xi) + d_i(\xi)],$$

or, using the relation $d\beta/d\xi = (e_1^2/4\pi) \beta d_i$

$$\frac{dm}{d\xi} = - \frac{3e_1^2}{4\pi} d_i(\xi) m(\xi).$$

Substituting d_i in formula (3.11) we arrive to the final expression

$$m(\xi) = m(0) \left(1 - \frac{e^2}{3\pi} \nu \xi \right)^{9/4\nu}, \quad (4.8)$$

where the constant $m(0)$ is the value of $m(\xi)$ at $\xi = 0$, i.e. the observable mass of the electron.

In the usual notation, (4.8) is

$$m(p^2) = m \left[1 - \frac{e^2}{3\pi} \nu \ln \left(\frac{-p^2}{m^2} \right) \right]^{9/4\nu}. \quad (4.9)$$

The formula (4.8) defines $m(p^2)$ for $|p^2| \gg m^2$. The case $|p^2| \ll m^2$ does not differ essentially from the case $p^2 \sim m^2$, i.e. in this case $m(p^2) = m$. At $p^2 = m^2$ the denominator of the formula (4.1) becomes zero, therefore the case of p^2 approaching m^2 to such an extent that $e^2 \ln(m^2/(p^2 - m^2)) \sim 1$ should be treated separately.

The formula (4.9) is gauge invariant and does not depend on the smearing out radius. It follows from (4.9) that the mass of the particle decreases by increasing k^2 . The quantity m_1 , according to (4.6), is equal to $m(L)$, i.e. corresponds to the value k^2 related to the smearing out radius

$$m_1 = m \left(\frac{e^2}{e_1^2} \right)^{9/4\nu}. \quad (4.10)$$

At the limit of applicability of the theory, $e_1 \sim 1$ and $m_1 \sim m e^{9/2\nu}$, that is a very small quantity. According to (4.10), m_1 decreases with increasing Δ . Therefore it may be supposed that if the theory were closed, m_1 would turn to zero. It means that the mass of the electron is of electromagnetic nature. So the concept stated above brings us back to the idea of a pure electromagnetic electron mass, which was abandoned long ago.

5. THE INFRARED CATASTROPHE⁷

Up to now we did not consider the behaviour of the electron Green function when the momentum p approaches m to such an extent that $e_1^2 \ln(m^2/(p^2 - m^2)) \gtrsim 1$. We were also not interested in the vertex parts Γ_μ whose argument did not satisfy the inequalities of a triangle.

We begin with the calculation of $G(p)$ for p approaching m . According to the general method, we will sum up the terms containing, for a given power of e , the greatest power of $\ln(m^2/(p^2 - m^2))$. Here we will use the physical charge together with the re-normalised D -function. In this case the considered terms are of the type $[e^2 \ln(m^2/(p^2 - m^2))]^n$. Analysing the diagrams for Γ_μ , we see that such terms appear only from diagrams with parallel virtual lines. So the equation (1.5) is completely applicable. Since p is essentially a time-like vector in the interesting region we cannot, in the integral of the equation (1.1), pass to the euclidean space in the same way as before, however, we may do

it by means of a rotation of space axes: $k_m \rightarrow -i k_m$ ($m = 1, 2, 3$). Then it turns out that again Γ_μ 's are needed whose arguments satisfy the triangle rule. These Γ_μ 's, together with G , form a closed system.

The G function at $p^2 \approx m^2$ is supposed to have the form

$$G(p) = \frac{p^2 + m}{p^2 - m^2} \beta(m^2) \sigma(m^2 - p^2), \quad (5.1)$$

where σ is a new slowly varying function. Analysing the integral in the equation (1.5) we come to the conclusion that the region where $p^2 - m^2 \ll (p - k)^2 - m^2 \ll m^2$ plays the main role in the integral. In this region we will suppose $\Gamma_\mu(p, p - l; l)$ to have the form

$$(\hat{p} + m) \Gamma_\mu(p, q; l) (\hat{q} + m) = (\hat{p} + m) \gamma_\mu(\hat{q} + m) \tau(m^2 - p^2) \alpha(m^2) \quad \hat{q} = \hat{p} - \hat{l}, \quad (5.2)$$

where τ is a new slowly varying function.

It can be seen that in all applications we use not Γ_μ itself, but its product with $\hat{p} + m$ on the left and $\hat{q} + m$ on the right. This is the quantity we have to find. In the integral of the equation for Γ_μ the main region is $q^2 - m^2 \ll (p - k)^2 - m^2 \ll m^2$. Transforming the equation for Γ_μ in this region we obtain the equation for τ

$$\tau(\eta) = 1 - \frac{3e^2}{2\pi} \int_0^\eta \tau^3(z) \sigma^2(z) dz, \quad (5.3)$$

where

$$\eta = \ln \frac{m^2}{m^2 - q^2}, \quad z = \ln \frac{m^2}{m^2 - (p - k)^2}.$$

Analysing the integral in the equation (1.1) we see that, as before, we have to find corrections to the expression (5.2) of Γ_μ proportional to $(p^2 - m^2)/(q^2 - m^2)$.

These corrections are found from the equation (1.5) where the main region of integration is $p^2 - m^2 \ll (p - k)^2 - m^2 \ll q^2 - m^2$. Omitting the detailed calculations we will write the final results.

$$\tau(\xi) \sigma(\xi) = 1, \quad (5.4)$$

$$\sigma(p) = \left(\frac{m^2}{m^2 - p^2} \right)^{(e^2/2\pi)(3-d_1^0)}, \quad \text{where} \quad d_1^0 = d_1(k^2 = 0). \quad (5.5)$$

So G appears to have, at $p^2 = m^2$, not a pole but in general a branch point.

The calculated τ belongs to a $\Gamma_\mu(p, q; l)$ with $p^2 \approx m^2$, $q^2 \approx m^2$ and $l^2 \ll m^2$. Γ_μ was determined also in a more general case: $(p, q) \gg p^2 - m^2$, $q^2 - m^2$. As before, only terms with highest powers of logarithms were accounted for, those corresponding to $[e^2 \ln(m^2/(p^2 - m^2))]^n$ and $(e^2 \ln^2(pq/m^2))^n$. To calculate the Γ_μ one has to sum up diagrams with all virtual lines surrounding

the original vertex. We will omit the calculations. The result is

$$(\hat{p} + m) \Gamma_\mu(p, q; l) (\hat{q} + m) = \alpha(m^2) (\hat{p} + m) \gamma_\mu (\hat{q} + m) \exp \left[-\frac{e^2}{2\pi} f \right], \quad (5.6)$$

where f is equal to

$$f = \frac{a}{\sqrt{a^2 - 1}} S + \frac{1}{2\sqrt{2}} (1 - d_i^0) (L_1 + L_2). \quad (5.7)$$

Here $a = (pq)/m^2$, S is the area on the plane λ and μ , restricted by the conditions

$$\left. \begin{aligned} 0 < \lambda < \ln \frac{(pq)}{p^2 - m^2}, \\ 0 < \mu < \ln \frac{(pq)}{q^2 - m^2}, \end{aligned} \right\} \quad (5.8)$$

$$\mu + \ln(a + \sqrt{a^2 - 1}) > \lambda > \mu - \ln(a - \sqrt{a^2 - 1}) \quad (5.9)$$

and L_1, L_2 are the lengths of the pieces of the lines $\lambda = \mu \pm \ln(a + \sqrt{a^2 - 1})$ which are cut out by the conditions (5.8).

A method analogous to that applied to calculation of Γ_μ was then used to investigate the question about the radiation of additional quanta in physical effects, the so called infrared catastrophe. The probability of such radiation appears to be represented by the Poisson formula

$$W(n) = \frac{e^{-\bar{n}}}{n!} (\bar{n})^n. \quad (5.10)$$

The factor $e^{-\bar{n}}$ in this formula comes from the integrals over virtual quanta and is essentially defined by the formula (4.6). So it becomes clear that the found deviation of Γ_μ from the zeroth approximation of perturbation theory comes from the infrared catastrophe.

The results represented here are essentially the same for the additional radiation in other physical effects.

So it occurs that at great energies the major department from the zeroth approximation of perturbation theory (effects of order $e^2 \ln^2(E/m)$) comes from the infrared catastrophe.

6. THE GAUGE TRANSFORMATIONS OF THE GREEN FUNCTIONS OF CHARGED PARTICLES

The theory describing the field of charged particles coupled to the electromagnetic field has the property of gauge invariance. By the gauge transformation of potentials of the electromagnetic field

$$A_\mu \rightarrow A_\mu + \frac{\partial \varphi}{\partial x_\mu} \quad (6.1)$$

(φ is an arbitrary scalar operator) the ψ operators of the particles transform in the following way

$$\psi \rightarrow \psi \exp[i e \varphi]. \quad (6.2)$$

Here we will consider how the Green functions of the particles are changed under the gauge transformation. The particle Green function, as is well known, is

$$G(x, x') = -i \langle (\psi(x) \bar{\psi}(x'))_+ \rangle. \quad (6.3)$$

Here the pointed bracket indicates the vacuum expectation value.

By the gauge transformation (6.1) the Green function will change. According to (6.2), this may be written in the form

$$G(x, x') = G_0(x, x') \langle (\exp[i e \varphi(x)] \exp[-i e \varphi(x')])_+ \rangle. \quad (6.4)$$

Here $G_0(x, x')$ is the Green function calculated for the particular case when the longitudinal (in a four-dimensional sense) part of the photon Green function is equal to zero.

In accordance with the transverse character of the Dirac current, the longitudinal part of the potential does not participate in the interaction of fields and therefore the task consists in averaging the expression $\exp[i e \varphi(x)] \times \exp[-i e \varphi(x')]$ over the vacuum state where the operators φ belong to a free field. Then we have

$$G(x, x') = G_0(x, x') \exp[i e^2 (\Delta_F(0) - \Delta_F(x, x'))], \quad (6.5)$$

where Δ_F is a Green function of the φ field

$$\Delta_F(x, x') = i \langle (\varphi(x) \varphi(x'))_+ \rangle. \quad (6.6)$$

The photon Green function in general contains a longitudinal part as well as a transverse part (formula (2.2)). It is easy to see that $d_l(k^2)$ is connected with the Fourier component $\Delta_F(k)$ of the function $\Delta_F(x, x')$ in the following manner

$$\Delta_F(k) = 4\pi \frac{d_l(k^2)}{k^4}. \quad (6.7)$$

The gauge transforms of Fourier components of the Green functions can be found only for infinitesimal gauge transformations of potentials.

Varying the relation (6.5) we have:

$$\delta G(x, x') = i e^2 G(x, x') (\delta \Delta_F(0) - \delta \Delta_F(x, x')). \quad (6.8)$$

Taking Fourier transforms of both sides and taking into account (6.7), we find:

$$\delta G(p) = -\frac{e^2}{\pi i} \int \frac{\delta d_l(k^2)}{k^4} \{G(p) - G(p-k)\} d^4 k. \quad (6.9)$$

For a particle having spin $\frac{1}{2}$ in the case when $d_l(k^2)$ is a slowly varying function this formula reduces to equation (2.17).

The gauge transform of Γ_μ may be obtained in an analogous manner. Omitting the detailed calculation we will write only the result

$$\begin{aligned} G(p) \delta \Gamma_\mu(p, p-k; k) G(p-k) &= \frac{e^2}{\pi i} \int \{G(p) \Gamma_\mu(p, p-k; k) \\ &\times [G(p-k) - G(p-k-r)] + G(p-r) [\Gamma_\mu(p-r, p-r-k; k) \\ &\times G(p-r-k) - \Gamma_\mu(p, p-k; k) G(p-k)]\} \frac{\delta d_l(r)}{r^4} d^4 r \end{aligned}$$

For a particle with spin $\frac{1}{2}$ and slowly varying $d_l(k^2)$ this gives (2.17). However the formulae (6.9) and (6.10) are applicable indeed for charged particles of any spin.

7. THE GREEN FUNCTIONS IN THE MESON THEORY WITH WEAK PSEUDOSCALAR COUPLING⁸

Now we will try to apply the methods used in the preceding sections for quantum electrodynamics to the meson theory with weak pseudoscalar coupling. As in electrodynamics, we will find the expressions of the fundamental functions: $G(p)$ the Green function of a nucleon, $D(k)$ the Green function of a meson and the related vertex part $\Gamma(p)$ of space vectors p , q and l . The fundamental equations are analogous to those derived in section 1, and have the form

$$G(p) \left\{ \hat{p} - M_1 - \frac{g_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \tau_\nu \gamma_5 D_{\mu\nu}(k) d^4 k \right\} = 1 \quad (7.1)$$

$$D_{\mu\nu}(k) \left\{ (k^2 - \mu_1^2) \delta_{\mu\nu} + \frac{g_1^2}{\pi i} \text{Tr} \left[\int G(p) \Gamma_\tau(p, p-k; k) G(p-k) \tau_\nu \gamma_5 d^4 p \right] \right\} = \delta_{\mu\nu} \quad (7.2)$$

$$\begin{aligned} \Gamma_\sigma(p, p-l; l) &= \gamma_5 \tau_\sigma + \frac{g_1^2}{\pi i} \int \Gamma_\mu(p, p-k; k) G(p-k) \Gamma_\sigma(p-k, p-k-l; l) \\ &\times G(p-k-l) \Gamma_\nu(p-k-l, p-l; k) D_{\mu\nu}(k) d^4 k. \quad (7.3) \end{aligned}$$

Here the following notations were used: $\gamma_5 = i \gamma_1 \gamma_2 \gamma_3 \gamma_0$ ($\gamma_{1,2,3} = \beta \alpha_{1,2,3}$, $\gamma_0 = \beta$); M_1 , μ_1 are the bare masses of the nucleon and meson, g_1 the bare coupling constant depending on the cut-off limit, the Greek indices here denote the components in isotopic space, τ_μ are the operators of isotopic spin. It should be mentioned also that the Tr in (7.2) is extended over the isotopic as well as over the Dirac matrices.

From equation (7.2) it is easy to see that $D_{\mu\nu}(k) = \delta_{\mu\nu} D(k)$. This gives us a possibility to eliminate the τ_μ operators from the equations, introducing $\Gamma_\mu = \tau_\mu \Gamma$.

After this, the equations take the form:

$$\Gamma(p, p-l; l) = \gamma_5 - \frac{g_1^2}{\pi i} \int \Gamma(p, p-k; k) G(p-k) \Gamma(p-k, p-k-l; l) \quad (7.4)$$

$$\times G(p-k-l) \Gamma(p-k-l, p-l; k) D(k) d^4(k),$$

$$G(p) \left\{ \hat{p} - M_1 - \frac{3g_1^2}{\pi i} \int \Gamma(p, p-k; k) G(p-k) \gamma_5 D(k) d^4k = 1, \quad (7.5)$$

$$D(k) \left\{ k^2 - \mu_1^2 + \frac{2g_1^2}{\pi i} \left(\text{Tr} \int G(p) \Gamma(p, p-k; k) G(p-k) \gamma_5 d^4p \right) \right\} = 1, \quad (7.6)$$

where the Tr sign belongs to the Dirac matrices.

We will find from these equations the asymptotic form of the functions G , D and Γ . We suppose the functions $G(p)$ and $D(k)$, for large values of momenta $p \gg M$, $k \gg M$, to have the form

$$G(p) = \frac{\beta(p^2)}{\hat{p} - M(p^2)}, \quad D(k) = \frac{d(k^2)}{k^2}. \quad (7.7)$$

As before, although $M(p^2) \gg p$, the function $M(p^2)$ can be derived because it is the even part of $G^{-1}(p)$. As to the function $\mu(k^2)$, the definition of this function is impossible in our consideration, because it cannot be separated from the much bigger k^2 . Therefore we will restrict ourselves to the form (7.7) of $D(k)$.

The Γ function in the case when p , q and l are of the same order, or two of them are much greater than the third, is given by the formula $\Gamma = \gamma_5 \alpha(f^2)$, where f^2 is the greatest of the squares. Noticing that the integral in (7.4) is logarithmic in the region $f^2 \ll k^2 \ll \infty$, the equation for α may be written in the form

$$\alpha(\xi) = 1 + \frac{g_1^2}{4\pi} \int_{\xi}^L d(z) \alpha^3(z) \beta^2(z) dz, \quad (7.8)$$

where $\xi = \ln(-p^2/M^2)$, $L = \ln(-\Lambda^2/M^2)$, Λ being the momentum corresponding to the smearing out radius.

In accordance with the fact that the expression in the integral of equation (7.5) has a k^3 in the denominator we have to account for terms of the first order with respect to p/k in G and Γ . These terms arise in G directly from the formula (7.7), and in Γ from the equation (7.4).

Finding the alterations occurring in the integral of the equation (7.4) from $G(p-k)$, it may be seen that the addition to Γ must have the form

$$\alpha^2(l^2) \beta(l^2) \left(\frac{\hat{p} \hat{l}}{l^2} \Delta_1(p^2, l^2) + \frac{\hat{l} M}{l^2} \Delta_2(p^2, l^2) \right) \gamma_5. \quad (7.9)$$

Here the first term in the brackets belongs to the even part, and the second term to the odd part of F . Equating the even and odd parts of equation (7.4) we find the equations for Δ_1 and Δ_2

$$\Delta_1(\xi, \eta) = \frac{g_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d(z) \Delta_1(\xi, z) dz - \frac{g_1^2}{8\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) d(z) dz, \quad (7.10)$$

$$\Delta_2(\xi, \eta) = \frac{g_1^2}{4\pi} \int_{\xi}^{\eta} \alpha^2(z) \beta^2(z) d(z) \Delta_2(\xi, z) dz - \frac{g_1^2}{4\pi} \int_{\xi}^{\eta} \alpha(z) \beta(z) d(z) dz, \quad (7.11)$$

where $\xi = \ln(-p^2/M^2)$, $\eta = \ln(-l^2/M^2)$.

Substituting the formula (7.9) in the equation (7.5), we note that there appear the same integrals as in the formulae (7.10) and (7.11), but with the limit L , i.e. correspondingly, $\Delta(\xi, L)$ and $\Delta_2(\xi, L)$. Separating the even and odd parts of the derived equation, we obtain two equations

$$\frac{1}{\beta(\xi)} = 1 - 3\Delta_1(\xi, L), \quad (7.12)$$

$$\frac{M(\xi)}{\beta(\xi)} = M_1 + 3\Delta_2(\xi, L). \quad (7.13)$$

The equations obtained: (7.8) and (7.10–7.13) do not form a complete system because they contain the quantity $d(\xi)$, the equation of which has not yet been written. But all the same we can use the equations obtained to find out the connection between the functions $\alpha(\xi)$, $\beta(\xi)$ and $M(\xi)$.

Differentiating (7.9) we obtain:

$$\frac{d\alpha(\xi)}{d\xi} = -\frac{g_1^2}{4\pi} \alpha^3(\xi) \beta^2(\xi) d(\xi). \quad (7.14)$$

Introducing the new variables $\tau_1(\xi, \eta) = \alpha(\eta) \Delta_1(\xi, \eta)$; $\tau_2(\xi, \eta) = \alpha(\eta) \Delta_2(\xi, \eta)$, differentiating the equations (7.10) and (7.11), and using the equation (7.14), we have:

$$\frac{\partial \tau_1(\xi, \eta)}{\partial \eta} = -\frac{g_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) d(\eta), \quad (7.15)$$

$$\frac{\partial \tau_2(\xi, \eta)}{\partial \eta} = \frac{g_1^2}{8\pi} \alpha^2(\eta) \beta(\eta) d(\eta) M(\eta). \quad (7.16)$$

According to (7.10) and (7.11), $\tau_1(\xi, \eta)$ and $\tau_2(\xi, \eta)$ should turn to zero at $\xi = \eta$, i.e. they have the form of integrals from ξ to η . Therefore $\partial \tau_1(\xi, \eta)/\partial \xi$ and $\partial \tau_2(\xi, \eta)/\partial \xi$ are represented by the right-hand sides of equations (7.15) and (7.16), taken with the opposite sign and with the variable ξ instead of η .

Differentiating the equations (7.12) and (7.13) with respect to ξ and substituting the expressions of the derivatives of Δ (in consequence of $\alpha(L) = 1$, $\Delta_{1,2}(\xi, L) = \tau_{1,2}(\xi, L)$) we obtain:

$$\frac{d\beta(\xi)}{d\xi} = \frac{3g_1^2}{8\pi} \alpha^2(\xi) \beta^3(\xi) d(\xi), \quad (7.17)$$

$$\frac{d}{d\xi} \left(\frac{M(\xi)}{\beta(\xi)} \right) = - \frac{3g_1^2}{4\pi} \alpha^2(\xi) \beta(\xi) d(\xi) M(\xi). \quad (7.18)$$

Comparing the equations (7.17) and (7.14) and remembering that, according to (7.12), we have: $\beta(L) = 1$, we obtain

$$\alpha(\xi) = [\beta(\xi)]^{-2/3}, \quad (7.19)$$

$$\frac{d\beta}{d\xi} = \frac{3g_1^2}{8\pi} [\beta(\xi)]^{5/3} d(\xi). \quad (7.20)$$

Comparing the equation (7.17) and (7.18) and taking into account that, according to (7.13), $M(L) = M_1$, we obtain

$$M(\xi) = \frac{M_1}{\beta(\xi)}. \quad (7.21)$$

Now we find the equation for $d(\xi)$. For this purpose let us consider the equation (7.6). It is easy to see that the integrated expression contains p^2 in the denominator. Therefore it is necessary to account for terms appearing in the expansion of the integrated expression up to the order of k^2/p^2 . Then the terms of zeroth order give a quadratically divergent integral, which is independent of k and may be enclosed in the proper mass of the meson.

The terms of order k^2/p^2 in $G(p-k)$ are obtained by means of a direct expansion of (7.7). As to the addition coming from Γ it is obtained from equation (7.4), where the change of variable $k' = p - k$ should be done. Finding the change of the integral coming from the addition to G , we conclude that the addition to Γ should have the form:

$$\frac{l^2}{p^2} \alpha^2(p^2) d(p^2) \Delta(p^2, l^2). \quad (7.22)$$

Performing the further derivation with the same procedure described above we come to the relation

$$d(\eta) = [\beta(\eta)]^{8/3}. \quad (7.23)$$

Substituting (7.23) in the equation (7.20) and taking into account that $\beta(L) = 1$, we obtain

$$\beta(p^2) = \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-p^2} \right) \right]^{-3/10}. \quad (7.24)$$

The functions $\alpha(p^2)$ and $d(p^2)$ are obtained from (7.19) and (7.23)

$$\alpha(p^2) = \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-p^2} \right) \right]^{1/5}, \quad (7.25)$$

$$d(p^2) = \left[1 + \frac{5g_1^2}{4\pi} \ln \left(\frac{\Lambda^2}{-p^2} \right) \right]^{-4/5}. \quad (7.26)$$

Considering now a particular process (for example, the Compton effect) we conclude that $g_1^2 \alpha^2(0) \beta^2(0) d(0)$ plays the role of the physical coupling constant g^2 , i.e.

$$g^2 = \frac{g_1^2}{1 + (5g_1^2/4\pi) \ln(\Lambda^2/M^2)}, \quad g_1^2 = \frac{g^2}{1 - (5g^2/4\pi) \ln(\Lambda^2/M^2)}. \quad (7.27)$$

Substituting (7.24) in (7.21) and defining the physical mass as $M(\xi=0)$, we obtain

$$M_1 = M \left(\frac{g^2}{g_1^2} \right)^{3/10}. \quad (7.28)$$

The results are close to those obtained in electrodynamics. As in electrodynamics, the effective coupling constant g_1^2 increases with increasing Λ . So we could have made the conclusion that even weak coupling becomes strong at great energies. However there are certain considerations due to I. Pommeranchuk showing that the situation in this case is essentially the same as in electrodynamics, but here we will not discuss the matter in more detail.

It is worth to point out the following peculiar feature of the theory with weak pseudoscalar coupling of the meson and nucleon fields. V. B. Beresteckij considered charged nucleons interacting with electromagnetic and pseudoscalar meson fields, under the assumption that the coupling constant of the nucleons with mesons, although much greater than the electric charge, is at the same time smaller than unity ($e^2 \ll g^2 \ll 1$). Then he could use our method of calculation. It appeared that the asymptotic expression of the photon Green function $D_{\mu\nu}$ did not depend on the coupling constant g^2 . So the weak pseudoscalar interaction of the nucleons with mesons does not have any influence on the photon Green function.

Now we describe briefly how the expressions obtained will change under the assumption of a difference between the orders of magnitude of the cut-offs as it was done already in electrodynamics. Let Λ_p be the cut-off limit in the Dirac current, and Λ_φ that in the interaction of the current with the meson field (it must be of course $\Lambda_p \gtrsim \Lambda_\varphi$).

Then the expressions of α , β and d change to the following:

$$\beta(p^2) = Q(p^2)^{-3/10} \quad (7.29)$$

$$\alpha(p^2) = Q(p^2)^{1/5} \quad (7.30)$$

$$d(p^2) = \frac{Q(p^2)^{-4/5}}{1 + (g_1^2/\pi) \ln(\Lambda_p^2/\Lambda_\varphi^2)} \quad (7.31)$$

if $p^2 \ll \Lambda_2^2$, where

$$Q(p^2) = 1 + \frac{5g_1^2}{4\pi} \frac{\ln(\Lambda_\phi^2/p^2)}{1 + (g_1^2/\pi) \ln(\Lambda_\phi^2/\Lambda_\phi^2)} \quad (7.32)$$

and

$$\beta(p^2) = d(p^2) = 1 \quad (7.33)$$

if $p^2 \gg \Lambda_\phi^2$.

We see again that the d -function undergoes a rapid change in the vicinity of $p^2 \cong \Lambda_\phi^2$.

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90. THE THEORY OF A FERMI LIQUID

A theory of the Fermi liquid is constructed, based on the representation of the perturbation theory as a functional of the distribution function. The effective mass of the excitation is found, along with the compressibility and the magnetic susceptibility of the Fermi liquid. Expressions are obtained for the momentum and energy flow.

As is well known, the model of a Fermi gas has been employed in a whole series of cases for the consideration of a system of Fermi particles, in spite of the fact that the interaction among such particles is not weak. Electrons in a metal serve as a classic example. Such a state of the theory is unsatisfactory, since it leaves unclear what properties of the gas model correspond to reality and what are intrinsic to such a gas.

For this purpose we must keep in mind that the problem is concerned with definite properties of the energy spectrum ("Fermi type spectrum"), for whose existence it is necessary, but not sufficient that the particles which compose the system obey Fermi statistics, i.e. that they possess half-integer spin. For example, the atoms of deuterium interact in such a manner that they form molecules. As a result, liquid deuterium possesses an energy spectrum of the Bose type. Thus the presence of a Fermi energy spectrum is connected not only with the properties of the particles, but also with the properties of their interaction.

A liquid of the Bose type was first considered by the author of the present article in application to the properties of He II. It follows from the character of the spectrum of such a liquid that a viscous liquid of Bose particles necessarily possesses superfluid properties. The converse theorem that a liquid consisting of Fermi particles cannot be superfluid, in accord with the above, is in general form not true.

1. THE ENERGY AS A FUNCTIONAL OF THE DISTRIBUTION ENERGY

If we consider a Fermi gas at temperatures which are low in comparison with the temperature of degeneration, and introduce some weak interaction between the atoms of this gas, then, as is known, the collision probability for a given atom, which is found in the diffuse Fermi zone, is proportional not only to the intensity of the interaction, but also to the square of the temperature. This shows that for a given intensity of interaction, the "indeterminacy

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of the momenta", associated with the finite path length, is also small for low temperatures, not only in comparison with the size of the momentum itself, but also in comparison with the width of the Fermi zone proportional to the first power of the temperature.

As a basis for the construction of the type of spectrum under consideration, is the assumption that, as we gradually "turn on" the interaction between the atoms, i.e. in the transition from the gas to the liquid, classification of the levels remains invariant. The role of the gas particles in this classification is assumed by the "elementary excitations" (quasi-particles), each of which possesses a definite momentum. They obey Fermi statistics, and their number always coincides with the number of particles in a liquid. The quasi-particle can, in a well-known sense, be considered as a particle in a self-consistent field of surrounding particles. In the presence of a self-consistent field, the energy of the particle depends on the state of the surrounding particles, but the energy of the whole system is no longer equal to the sum of the energies of the individual particles, and is a functional of the distribution function.

We consider an infinitely small change in the distribution function of quasi-particles n . Then we can write down the change in the energy density of the system in the form

$$\delta E = \int \varepsilon \delta n d\tau, \quad (1)$$

where $d\tau = dp_x dp_y dp_z / (2\pi \hbar)^3$. The quantity $\varepsilon(p)$ is a function of the derivative of the energy with respect to the distribution function. It corresponds to a change in the energy of the system upon the addition of a single quasi-particle with momentum p , and it can be regarded as the Hamiltonian function of the added quasi-particle with given momentum in the self-consistent field.

However, we have not taken it into account in equation (1) that the particles possess spin. Since the spin is a quantum mechanical quantity, it cannot be considered by classical means. We must therefore consider the distribution function of the statistical matrices in regard to spin, and replace (1) by the following:

$$\delta E = \text{Tr} \int \varepsilon \delta n d\tau, \quad (2)$$

where Tr is the trace over the spin states. The quantity ε in the general case is also an operator which depends on the spin operators. If we have an equilibrium liquid, which is not in an external magnetic field, then, because of isotropic, the energy cannot depend on the spin operators. We limit ourselves to the consideration of particles with $s = \frac{1}{2}$.

We can show that just this energy ε enters into the formula for the Fermi distribution of the quasi-particles. Actually, it is reasonable to determine the entropy of the liquid by the following way:

$$S = -\text{Tr} \int \{n \ln n + (1 - n) \ln(1 - n)\} d\tau. \quad (3)$$

By means of a variation, subject to the additional conditions

$$\delta N = \text{Tr} \int \delta n \, d\tau = 0, \quad \delta E = \text{Tr} \int \varepsilon \delta n \, d\tau = 0,$$

we can obtain the Fermi distribution

$$n(\varepsilon) = [e^{(\varepsilon - \mu)/\theta} + 1]^{-1} \quad (4)$$

from this equation. We note that ε , being a functional of n , naturally depends on the temperature also.

In correspondence with (4) the heat capacity of a Fermi liquid at low temperatures will be proportional to the temperature. It is determined by the same formula as for the Fermi gas, with one exception, that in place of the real mass m of the particles therein, we place the effective mass of the quasi-particles

$$m^* = \frac{p}{\partial \varepsilon / \partial p} \Big|_{p=p_0}, \quad (5)$$

where p_0 is the limiting momentum of the Fermi distribution of quasi-particles at absolute zero.

Not only $\varepsilon(p)$ for a given distribution, but also the change in ε produced by a change in n , is of essential importance for the theory of the Fermi liquid:

$$\delta \varepsilon(p) = \text{Tr} \int f(p, p') \delta n' \, d\tau'. \quad (6)$$

Being a second variational derivative, the function f is symmetric relative to p and p' ; moreover, it depends on the spins.

If the principal distribution n is isotropic, then the function f in the general case contains terms of the form $\varphi_{ik}(p, p') \sigma_i \sigma'_k$, where σ_i is the spin operator, and if the interaction is exchange, only terms of the form

$$\varphi(p, p') (\sigma \cdot \sigma')$$

will appear.

We can consider the function f from the following point of view. The number of acts of scattering of quasi-particles per unit volume per unit time can be written in the form

$$dW = \frac{2\pi}{\hbar} |F(p_1, p_2; p'_1, p'_2)|^2 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon'_1 - \varepsilon'_2) n_1 n_2 \\ \times (1 - n'_1)(1 - n'_2) d\tau_1 d\tau_2 d\tau'_1, \quad (7)$$

where conservation of momentum is assumed: $p_1 + p_2 = p'_1 + p'_2$. The quantity f is nothing but $-F(p_1, p_2; p_1, p_2)$, i.e. the forward scattering amplitude (with opposite sign). Generally speaking, this amplitude is complex, its imaginary part being determined by the total effective scattering cross-section. Inasmuch as we assume that the real acts of scattering are highly improbable, we can neglect the imaginary part.

2. RELATIONS WHICH FOLLOW FROM THE PRINCIPLE OF GALILEAN RELATIVITY

If we deal with a liquid which is not in an external field, then it follows from the principle of Galilean relativity that the momentum arriving at a unit volume must be equal to the density of mass flow†. Inasmuch as the velocity of the quasi-particle is $\partial \varepsilon / \partial \mathbf{p}$, and the number of quasi-particles coincides with the number of real particles, we have

$$\text{Tr} \int \mathbf{p} n \, d\tau = \text{Tr} \int m \frac{\partial \varepsilon}{\partial \mathbf{p}} n \, d\tau. \quad (8)$$

Therefore, the variational derivatives with respect to n ought to be the same on both sides of this equation. Then

$$\frac{1}{m} \text{Tr} \int \mathbf{p} \delta n \, d\tau = \text{Tr} \int \frac{\partial \varepsilon}{\partial \mathbf{p}} \delta n \, d\tau + \text{Tr} \text{Tr}' \int \frac{\partial}{\partial \mathbf{p}} f(\mathbf{p}, \mathbf{p}') \delta n' n \, d\tau \, d\tau'.$$

Since the quantity δn is arbitrary, we obtain

$$\frac{\mathbf{p}}{m} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \text{Tr}' \int \frac{\partial f}{\partial \mathbf{p}'} n' \, d\tau' = \frac{\partial \varepsilon}{\partial \mathbf{p}} - \text{Tr}' \int f \frac{\partial n'}{\partial \mathbf{p}'} \, d\tau'. \quad (9)$$

(the left side is understood as the unit matrix in the spins).

If we deal with the isotropic case, then it is sufficient that equation (9) holds for the traces, i.e.

$$\frac{\mathbf{p}}{m} = \frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{1}{2} \text{Tr} \text{Tr}' \int f \frac{\partial n'}{\partial \mathbf{p}'} \, d\tau' \quad (10)$$

We note that this formula determines the function ε through the quantity f with accuracy to within a constant.

Let us consider equation (10) for momenta close to the boundary of the Fermi distribution. For low temperatures, the function $\partial n / \partial \mathbf{p}$ will differ slightly from the δ -function. For this reason, we can carry out the integration in (10) over the absolute value of the momentum, leaving only the integration over the angle. This gives the following relation between the real and the effective masses:

$$\frac{1}{m} = \frac{1}{m^*} + \frac{p_0}{2(2\pi\hbar)^3} \text{Tr} \text{Tr}' \int f \cos \theta \, d\Omega. \quad (11)$$

Inasmuch as, in this formula, both of the vector arguments in f correspond to the Fermi surface, the function f depends only on the angle between them.

† This conclusion does not apply in particular to electrons in a metal. For them, \mathbf{p} is not the momentum, but the quasi-momentum.

3. COMPRESSIBILITY OF THE FERMI LIQUID

Let us express the compressibility (at absolute zero) by the more appropriate quantity for us, $\partial\mu/\partial N$. For this purpose, we note that as a consequence of homogeneity, the chemical potential μ depends only on the ratio N/V . Consequently, we have

$$\frac{\partial\mu}{\partial N} = - \frac{V \partial\mu/\partial V}{N} = - \frac{V^2}{N} \frac{\partial p}{\partial V}. \quad (12)$$

For the square of the velocity of sound, we have

$$c^2 = \frac{\partial p}{\partial(m N/V)} = \frac{1}{m} \left(N \frac{\partial\mu}{\partial N} \right). \quad (13)$$

Thus the problem reduces to the calculation of the derivative $\partial\mu/\partial N$. Inasmuch as $\mu = \varepsilon(p_0) = \varepsilon_0$, the change in the chemical potential $\delta\mu$ which is brought about as a result of the change in the total number of particles δN , will be equal to†

$$\delta\mu = \frac{1}{2} \text{Tr Tr}' \int f \delta n' d\tau' + \frac{\partial\varepsilon_0}{\partial p_0} \delta p_0. \quad (14)$$

The second term is connected with the fact that for a change δN the limiting momentum p_0 changes by an amount δp_0 .

For the case of spin $\frac{1}{2}$, δN and δp_0 are connected by the relation

$$\delta N = 8\pi p_0^2 \delta p_0 V / (2\pi \hbar)^3. \quad (15)$$

The value of the function under the integral in equation (14) is appreciable only for values of momentum close to p_0 . Therefore, we can carry out integration over the absolute value of p , obtaining

$$\text{Tr Tr}' \int f \delta n' d\tau' = \frac{1}{8\pi V} \text{Tr Tr}' \int f d\sigma \delta N. \quad (16)$$

We get from equation (14), with the help of equations (15) and (16):

$$\partial\mu/\partial N = \text{Tr Tr}' \int f \frac{d\sigma}{16\pi V} + \frac{(2\pi \hbar)^3}{8\pi p_0 m^* V} \quad (17)$$

Now let us make use of (11) and express the effective mass m^* in the expression that has been obtained by the mass of the particles, m . We have

$$\frac{\partial\mu}{\partial N} = \frac{1}{16\pi V} \int \text{Tr Tr}' f (1 - \cos\theta) d\sigma + \frac{(2\pi \hbar)^3}{8\pi p_0 m V}.$$

† Equation (14) is obtained as a result of taking the trace of the analogous expression which contains the spin operators.

Furthermore, multiplying the resultant equation by $N/m = (1/m) 8\pi p_0^3 V / 3(2\pi\hbar)^3$, we find an expression for the square of the sound velocity:

$$c^2 = \frac{p_0^2}{3m^2} + \frac{1}{6m} \left(\frac{p_0}{2\pi\hbar} \right)^3 \int \text{Tr Tr}' f(1 - \cos\theta) d\omega. \quad (18)$$

4. MAGNETIC SUSCEPTIBILITY

We calculate the magnetic susceptibility of a Fermi liquid. If the system is located in a magnetic field \mathbf{H} , then the additional energy of a free particle in this field is equal to $\beta(\boldsymbol{\sigma} \cdot \mathbf{H})$. Moreover, we must also consider the fact that the form of the distribution function also changes in the presence of a magnetic field. Consequently, in calculating the magnetic susceptibility, we must keep in mind that

$$\delta\varepsilon = -\beta(\boldsymbol{\sigma} \cdot \mathbf{H}) + \text{Tr}' \int f \delta n' d\tau', \quad (19)$$

i.e. it is impossible to neglect the effect of the term containing f . We write f in the form

$$f = \varphi + \psi(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'), \quad (20)$$

where the second term takes into account the exchange interaction between the particles. Furthermore, in calculating δn , which depends on the field, the change in the chemical potential $\delta\mu$ does not have to be considered. This change appears as a quantity of second order of smallness relative to the field \mathbf{H} , while $\delta\varepsilon$ is of first order with respect to the field. Therefore, we can substitute $\delta n = (\partial n / \partial \varepsilon) \delta\varepsilon$ in equation (19). We then have

$$\delta\varepsilon = -\beta(\boldsymbol{\sigma} \cdot \mathbf{H}) + \text{Tr}' \int f \frac{\partial n'}{\partial \varepsilon'} \delta\varepsilon' d\tau'. \quad (21)$$

We shall look for $\delta\varepsilon$ in the form

$$\delta\varepsilon = -\gamma(\boldsymbol{\sigma} \cdot \mathbf{H}). \quad (22)$$

The quantity γ is defined by equation (21)†

$$\gamma = \beta + \frac{1}{2} \int \psi \frac{\partial n'}{\partial \varepsilon'} \gamma' d\tau'. \quad (23)$$

Remembering the δ -character of $\partial n / \partial \varepsilon$, we then obtain

$$\gamma = \beta - \frac{1}{2} \overline{\psi_0} \gamma \left(\frac{\partial \tau}{\partial \varepsilon} \right)_0. \quad (24)$$

Here the index zero indicates that the values of all functions are taken at $p = p_0$; the bar over the symbol indicates averaging over the angles. On the other hand, the susceptibility is defined by the relation

$$\chi \mathbf{H} = \beta \text{Tr} \int n \boldsymbol{\sigma} d\tau$$

† Here we make use of the relation $\text{Tr}(\boldsymbol{\sigma} \boldsymbol{\sigma}') \boldsymbol{\sigma}' = (1/3) \boldsymbol{\sigma} \text{Tr}'(\boldsymbol{\sigma}' \boldsymbol{\sigma}') = (1/2) \boldsymbol{\sigma}$.

or

$$\chi \mathbf{H} = -\beta \operatorname{Tr} \int \frac{\partial \tau}{\partial \varepsilon} \gamma (\mathbf{H} \cdot \boldsymbol{\sigma}) \sigma d\tau = \frac{\mathbf{H}}{2} \beta \gamma \left(\frac{d\tau}{d\varepsilon} \right)_0. \quad (25)$$

Hence, we get finally,

$$\frac{1}{\chi} = \frac{2}{\beta \gamma_0 (\partial \tau / \partial \varepsilon)_0} = \frac{2}{\beta^2 (d\tau / d\varepsilon)_0} \left[1 + \frac{1}{2} \bar{\psi}_0 \left(\frac{d\tau}{d\varepsilon} \right)_0 \right]. \quad (26)$$

Further, we can replace $(d\tau / d\varepsilon)_0$ by the coefficient α in the linear heat capacity law. Then

$$\frac{1}{\chi} = \beta^{-2} \left\{ \frac{4\pi^2 k^2}{3\alpha} + \bar{\psi}_0 \right\}. \quad (27)$$

It is then evident that there does not exist in the liquid the relation between the heat capacity and the susceptibility that exists in gases. The term with $\bar{\psi}_0$ takes the exchange interaction into account and is large for liquids. Thus, for ^3He , analysis of the experimental data¹ shows that $\bar{\psi}_0$ is negative and amounts to about 2/3 of the first term.

5. THE KINETIC EQUATION

In the absence of a magnetic field and for a neglect of the magnetic spin-orbit interaction, ε does not depend on the operator σ and the kinetic equation in the quasi-classical approximation takes the form

$$\frac{\partial n}{\partial t} + \left(\frac{\partial n}{\partial \mathbf{r}} \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}} \right) - \left(\frac{\partial n}{\partial \mathbf{p}} \cdot \frac{\partial \varepsilon}{\partial \mathbf{r}} \right) = I(n). \quad (28)$$

The necessity of calculation of derivatives of the energy ε with respect to the co-ordinates in the absence of an external field is connected with the fact that ε is a functional of n , and the distribution function n depends on the co-ordinates.

We find the expression for the momentum flux. For this purpose, we multiply the left and right sides of the equation above by the momentum p_i and integrate over all phase space. We have

$$\frac{\partial}{\partial t} \operatorname{Tr} \int p_i n d\tau + \operatorname{Tr} \int p_i \left(\frac{\partial n}{\partial x_k} \frac{\partial \varepsilon}{\partial p_k} - \frac{\partial n}{\partial p_k} \frac{\partial \varepsilon}{\partial x_k} \right) d\tau = \operatorname{Tr} \int p_i I(n) d\tau. \quad (29)$$

As a consequence of the conservation of momentum for collisions, the right side of the equation is zero, while the left side yields, after simple transformations,

$$\frac{\partial}{\partial t} \int p_i n d\tau + \frac{\partial}{\partial x_k} \int p_i \frac{\partial \varepsilon}{\partial x_k} n d\tau - \int p_i \frac{\partial}{\partial p_k} \left(n \frac{\partial \varepsilon}{\partial x_k} \right) d\tau = 0. \quad (30)$$

Finally, integrating the three integrals by parts, we get

$$\frac{\partial}{\partial t} \int p_i n d\tau + \frac{\partial}{\partial x_k} \int p_i \frac{\partial \varepsilon}{\partial p_k} n d\tau + \int n \frac{\partial \varepsilon}{\partial x_i} d\tau = 0. \quad (31)$$

The integral $\text{Tr} \int n(\partial \varepsilon / \partial x_i) d\tau$ can be represented in the form (see equation (2))

$$\text{Tr} \int n \frac{\partial \varepsilon}{\partial x_i} d\tau = \text{Tr} \frac{\partial}{\partial x_i} \int n \varepsilon d\tau - \text{Tr} \int \varepsilon \frac{\partial n}{\partial x_i} d\tau = \frac{\partial}{\partial x_i} \left[\text{Tr} \int n \varepsilon d\tau - E \right].$$

Thus we finally obtain the law of conservation of momentum:

$$\frac{\partial}{\partial t} \text{Tr} \int p_i n d\tau + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \quad (32)$$

where the tensor of momentum flux is

$$\Pi_{ik} = \text{Tr} \int p_i \frac{\partial \varepsilon}{\partial p_k} n d\tau + \delta_{ik} \left[\text{Tr} \int n \varepsilon d\tau - E \right]. \quad (33)$$

In a similar way we obtain the expression for the energy flow. We multiply the left and right sides of the kinetic equation (28) by ε and integrate over all phase space. We have

$$\text{Tr} \int \varepsilon \frac{\partial n}{\partial t} d\tau + \text{Tr} \int \varepsilon \left[\left(\frac{\partial n}{\partial \mathbf{r}} \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}} \right) - \left(\frac{\partial n}{\partial \mathbf{p}} \cdot \frac{\partial \varepsilon}{\partial \mathbf{r}} \right) \right] d\tau = \text{Tr} \int \varepsilon I(n) d\tau.$$

As a consequence of the conservation of energy under collisions, the right side is zero while the left side reduces without difficulty to the form

$$\int \varepsilon \frac{\partial n}{\partial t} d\tau + \left(\frac{\partial}{\partial \mathbf{r}} \int n \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} d\tau \right) = 0.$$

Taking equation (2) into account, we have finally,

$$\frac{\partial E}{\partial t} + \text{div } Q = 0, \quad (34)$$

where the energy flow is

$$Q = \text{Tr} \int n \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} d\tau. \quad (35)$$

In the solution of concrete kinetic problems it is necessary to keep in mind the following circumstances. For such a solution we usually write down the function n in the form of a sum of equilibrium functions n_0 and correction δn . In this case the departure of the tensor of momentum flow Π_{ik} and the vector of energy flow Q from their equilibrium values will result as a consequence of the direct change of the function n by the quantity δn , as well as from the change in ε which comes about as a result of the functional dependence of ε on n (equation (2)).

In conclusion, I express my gratitude to I. M. Khalatnikov and A. A. Abrikosov for fruitful discussions.

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91. OSCILLATIONS IN A FERMI LIQUID

Different types of waves that can be propagated in a Fermi liquid, both at absolute zero and at non-zero temperatures, are investigated. Absorption of these waves is also considered.

THE present paper is devoted to the study of the propagation of waves in a Fermi liquid, and proceeds from the general theory of such liquids developed by the author¹. These phenomena in a Fermi liquid should be distinguished by a large singularity, connected primarily with the impossibility of propagation in it of ordinary hydrodynamic sound waves at absolute zero. The latter circumstance is already evident from the fact that the path length, and therefore the viscosity of a Fermi liquid, tends to infinity for $T \rightarrow 0$, as a result of which the sound absorption coefficient increases without limit.

It is shown, however, that in a Fermi liquid at absolute zero other waves can be propagated; these differ in nature from ordinary sound, and we shall call them waves of "zero sound".

Initially, the problem of vibrations in a Fermi liquid was considered by Gol'dman² in application to an electron gas with Coulomb interaction between the particles. The problem of a gas with uncharged particles, considered in detail here for liquids, was first considered in the research of Klimontovich and Silin³, and later in a series of works of Silin⁴⁻⁶. There, the gas was considered to be slightly non-ideal, with an interaction satisfying the conditions of applicability of perturbation theory.

1. VIBRATIONS IN A FERMI LIQUID AT ABSOLUTE ZERO

We begin with the investigation of those vibrations at absolute zero which do not involve the spin characteristics of the liquid. This means that not only the equilibrium distribution function n_0 , but also the "perturbed" function

$$n = n_0 + \delta n(p) \quad (1)$$

is independent of the spin variables. At absolute zero, n_0 is a step function which is broken off at the limiting momentum $p = p_0$.†

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† To avoid excessive complication of our study, we limited ourselves to the simplest and most important case of an energy spectrum with an occupied region represented by a uniform sphere of radius p_0 .

The energy of the quasi-particles (elementary excitations) is a function of n , i.e. the form of the function $\varepsilon(p)$ depends on the form of $n(p)$. By analogy to (1), we write it in the form

$$\varepsilon = \varepsilon_0(p) + \delta\varepsilon(p), \quad (2)$$

where the function $\varepsilon_0(p)$ corresponds to the distribution $n_0(p)$. The value of $\delta\varepsilon$ itself is connected with δn by a formula of the form (see ref. 1):

$$\delta\varepsilon(p) = \text{Tr} \int f(p, p') \delta n' d\tau', \quad d\tau = \frac{d^3 p}{(2\pi \hbar)^3}. \quad (3)$$

Inasmuch as δn is assumed to be independent of the spin variable, the operation Tr is applied only to the scattering amplitude f . But the scalar function $\text{Tr} f$ can contain the spin operator σ only in the form of the product $(\sigma \cdot [p \wedge p'])$ of two axial vectors: σ and $[p \wedge p']$ (we do not consider expressions containing two products of components of σ , since for spin $\frac{1}{2}$, as is well known, they reduce to expressions containing σ in the zeroth or first degree). But this product is not invariant to a time reversal and therefore cannot enter into the invariant quantity $\delta\varepsilon$. Thus σ drops out completely and $\delta\varepsilon$ is shown to be independent of the spin variable.

The kinetic equation for a Fermi liquid has the form:

$$\frac{\partial n}{\partial t} + \left(\frac{\partial n}{\partial r} \cdot \frac{\partial \varepsilon}{\partial p} \right) - \left(\frac{\partial n}{\partial p} \cdot \frac{\partial \varepsilon}{\partial r} \right) = I(n), \quad (4)$$

where $I(n)$ is the integral of collisions between quasi-particles. The number of collisions is proportional to the square of the width of the diffusion zone, so that at absolute zero, $I(n) = 0$. Substituting (1) and (2) in (4), and considering that n_0 and ε_0 do not depend on r , we get

$$\frac{\partial \delta n}{\partial t} + \left(\frac{\partial \delta n}{\partial r} \cdot \frac{\partial \varepsilon_0}{\partial p} \right) - \left(\frac{\partial \delta \varepsilon}{\partial r} \cdot \frac{\partial n_0}{\partial p} \right) = 0,$$

and assuming δn and $\delta \varepsilon$ to be proportional to $e^{-i\omega t + i(k \cdot r)}$

$$[(k \cdot v) - \omega] \delta n = (k \cdot v) \frac{\partial n_0}{\partial \varepsilon} \delta \varepsilon, \quad (5)$$

where we have introduced the velocity of the quasi-particles $v = \partial \varepsilon_0 / \partial p$. In view of the presence of the δ -function $\partial n_0 / \partial \varepsilon$ on the right hand side of this equation, there actually enter in them only the values of all quantities taken at the limit $p = p_0$ of the (unperturbed) Fermi distribution. We introduce a new notation for what follows:

$$F = \text{Tr} f(p, p') 4\pi p^2 dp d\varepsilon / (2\pi \hbar)^3. \quad (6)$$

Then we can write equation (3) in the form:

$$\delta \varepsilon = \int \int F \delta n' d\varepsilon' d\omega' / 4\pi.$$

Here only the $\delta n'$ are functions changing rapidly with ε' . Therefore, we can rewrite this expression in the form:

$$\delta\varepsilon = \int F v' d\sigma'/4\pi, \quad (7)$$

where the function

$$v(n) = \int \delta n(p) d\varepsilon \quad (8)$$

has been introduced which depends only on the direction n of the vector p , and the function $F(p, p')$ is taken on the boundary of the (unperturbed) Fermi distribution; here F depends only on the angle χ between p and p' .

We note for what follows that the relation found in ref. 1, which connects the actual mass m of the particles with the effective mass m^* of the quasi-particles, can, with the help of the function $F(\chi)$ be written in the form

$$\overline{F \cos \chi} = \left(\frac{m^*}{m} \right) - 1, \quad (9)$$

where the bar denotes averaging over the directions (in the derivation of this relation, we assume in (6) that $\varepsilon = p^2/2m^*$). The equation for the velocity of ordinary sound c can be put in the form

$$\overline{F} = \frac{3mm^*c^2}{p_0^2} - 1. \quad (10)$$

Let us substitute (7) in (5) and integrate the latter over $d\varepsilon$. This gives

$$[(\mathbf{k} \cdot \mathbf{v}) - \omega] v = - (\mathbf{k} \cdot \mathbf{v}) \int \frac{F v d\sigma'}{4\pi}.$$

Let us take the direction of \mathbf{k} as the polar axis, and let the angles θ, φ define the direction of the momentum p (and the direction of v coinciding with it) relative to this axis. Also, we introduce the propagation velocity $u = \omega/k$ of this wave, and the notation $\eta = u/v$, so that we can finally write the resultant equation in the form

$$(\eta - \cos \theta) v(\theta, \varphi) = \cos \theta \int F(\chi) v(\theta', \varphi') d\sigma'/4\pi. \quad (11)$$

This integral equation defines the principal velocity of propagation of the waves and the form of the function $v(\theta, \varphi)$ in them. The latter has the following graphic meaning. The fact that δn is proportional (as is evident from (5)) to the derivative $\partial n_0/\partial \varepsilon$ means that the change of the distribution function for vibrations reduces to the deformation of the boundary of the Fermi surface (a sphere in the undisturbed distribution). The integral of (8) represents the magnitude of the displacement (in energy units) of this surface in the given direction n .

We at once note that it follows from the form of eq. (11) that the real (only the undamped vibrations are of interest to us) value of η ought to exceed 1, i.e. the propagation velocity of the waves satisfies the inequality

$$u > v. \quad (12)$$

As an example, let us investigate the case in which the function $F(\chi)$ reduces to a constant (we denote it by F_0). The integral on the right-hand side of equation (11) does not depend on the angles θ , φ in this case. Therefore the desired function ν has the form (we omit the exponential factor):

$$\nu = \text{const} \cdot \cos \theta / (\eta - \cos \theta). \quad (13)$$

The limiting Fermi surface has the form of a surface of revolution, elongated in the forward direction of the propagation of the wave, and flattened in the opposite direction. For comparison, let us point out that the ordinary sound wave corresponds to a function ν of the form $\nu = \text{const} \cdot \cos \theta$, which represents the displacement of the Fermi surface as a whole, without a change in shape.

For the determination of the velocity u , we substitute equation (13) in (11) and get

$$\frac{F_0}{4\pi} \int_0^\pi \frac{\cos \theta}{\eta - \cos \theta} 2\pi \sin \theta \, d\theta = 1.$$

Carrying out the integration, we find the following equation, which determines in implicit form the velocity of the wave for a given value of F_0 :

$$\dot{\varphi}(\eta) \equiv \frac{\eta}{2} \ln \frac{\eta + 1}{\eta - 1} - 1 = \frac{1}{F_0}. \quad (14)$$

The function $\varphi(\eta)$ decreases monotonically from $+\infty$ to 0 for a change of η from 1 to ∞ , always remaining positive. It then follows that the waves under consideration can exist only for $F_0 > 0$. Inasmuch as the function F is proportional to the scattering amplitude, taken with opposite sign (at the angle 0°), of the quasi-particles with one another (see ref. 1), then the latter must be negative, which corresponds to the mutual collision of quasi-particles. However, it must be emphasised that this conclusion applies only to the case $F = \text{const}$. If the function $F(\chi)$ is not constant (and at the same time is not small compared with unity; see below), then propagation of zero sound is in general possible, for both attractive and repulsive interactions of the quasi-particles.

For $\eta \rightarrow \infty$: $\varphi(\eta) \approx 1/3 \eta^2$. Therefore, large F_0 corresponds to $\eta = \sqrt{F_0/3}$. In the opposite case of $F_0 \rightarrow 0$, we find that η tends toward unity according to the relation

$$\eta - 1 \sim e^{-2F_0}. \quad (15)$$

The latter case has much more general value. It corresponds to zero sound in an almost ideal Fermi gas for arbitrary form of the function $F(\chi)$. Actually, an almost ideal gas corresponds to a function F which is small in absolute magnitude. It is seen from (11) that in this case η will be close to unity and the function ν will be significantly different from zero only for small angles θ . On this basis, and being concerned only with this range of angles, we can replace the function F in the integral on the right side of (11) by its value for $\chi = 0$ (for $\theta \rightarrow 0$ and $\theta' \rightarrow 0$, $\chi \rightarrow 0$ also). As a result, we again recover equations (13) and (15) with the constant F_0 replaced by $F(0)$ (this result coincides with that obtained earlier by Silin⁴).

We note that in a weakly non-ideal Fermi gas, the velocity of zero sound exceeds the velocity of ordinary sound by a factor of $\sqrt{3}$. Actually, for the former, we have $\eta \approx 1$, i.e. $u \approx v$. For the velocity of ordinary sound we get from equation (10) (neglecting the term F in it and setting $m^* \approx m$):

$$c^2 \approx p_0^2 / 3m^2 = v^2 / 3.$$

In the general case of an arbitrary dependence of $F(\chi)$, the solution of (11) is not well defined. In principle, it permits the existence of different types of zero sound, which are distinguished from one another by the angular dependence of their amplitude $\nu(\theta, \varphi)$, and which are propagated with different velocities. Along with the axially symmetric solutions of $\nu(\theta)$, asymmetric solutions can also exist. In these ν has an azimuthal factor $e^{\pm im\varphi}$ ($m = \text{integer}$).

Thus, for a function $F(\chi)$ of the form

$$F = F_0 + F_1 \cos \chi = F_0 + F_1 (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi')) \quad (16)$$

solutions can exist with

$$\nu \sim e^{\pm i\varphi}.$$

Actually, substituting equation (16) in (11) and carrying out the integration over $d\varphi'$ (assuming in this case that $\nu = f(\theta) e^{i\varphi}$), we obtain

$$(\eta - \cos \theta) f = \frac{F_1}{4} \cos \theta \sin \theta \int_0^\pi \sin^2 \theta' f' d\theta'.$$

Thence,

$$\nu = \text{const} \cdot \frac{\sin \theta \cos \theta}{\eta - \cos \theta} e^{i\varphi}. \quad (17)$$

Conversely, substituting this expression in the equation, we obtain the relation

$$\int_0^\pi \frac{\sin^3 \theta \cos \theta}{\eta - \cos \theta} d\theta = \frac{4}{F_1}, \quad (18)$$

which determines the dependence of the propagation velocity on F_1 . The integral on the left side of the equation falls off monotonically with increase in the function η . Therefore its maximum possible value is achieved for $\eta = 1$. Computing the integral, we find that the corresponding minimum value of F_1 is 6. Thus, propagation of the asymmetric wave of the form (17) is possible only for $F_1 > 6$.

Turning to a real Fermi liquid—the liquid ^3He —it is reasonable to attempt to approximate the unknown function $F(\chi)$ by the two term expression (16). We can determine the coefficients F_0 and F_1 entering into it by means of the relations

$$F_0 = \frac{3mm^*c^2}{p_0^2} - 1, \quad \frac{F_1}{3} = \frac{m^*}{m} - 1$$

(see equations (9) and (10)), knowing the values of the effective mass m^* and the velocity of ordinary sound c . We can derive the first from experimental data on the temperature dependence of the entropy (in the lowest temperature

region). From the data available at present⁷, we get $m^* = 1.43m$ (m is the mass of the ^3He atom). For the velocity c , we get 195 m/sec from the data of Walters and Fairbank⁵, on the compressibility of liquid ^3He ⁴. Finally, p_0 is obtained directly from the density of the liquid:

$$\frac{p_0}{\hbar} = 0.76 \times 10^8 \text{ cm}^{-1}$$

On the basis of these data, we obtain

$$F_0 = 5.4; \quad F_1 = 1.3. \quad (19)$$

From these values, we can draw a conclusion about the fact that in liquid ^3He the propagation of asymmetric zero sound is impossible. For symmetric zero sound, the solution of the equation with the function $F(\chi)$ from (16) and (19)[†] leads to the value $\eta = 1.83$, when we obtain $u = v = 1.83 p_0/m^* = 206 \text{ m/sec}$.

The possibility of the propagation of waves in a Fermi liquid at absolute zero means that its energy spectrum can automatically possess a "Bose branch" in the form of phonons with energy $\varepsilon = u p$. However, one must say that it would be incorrect to introduce corrections corresponding to this branch in the thermodynamic quantities of the Fermi liquid, inasmuch as it has a much higher power of the temperature (T^3 in the heat capacity) than the departures from the approximate theory developed in ref. 1.

2. VIBRATIONS OF A FERMI LIQUID AT TEMPERATURES ABOVE ZERO

For low, but non-zero, temperatures, mutual collisions of quasi-particles take place in the Fermi liquid. The number of these collisions is proportional to T^2 . The corresponding relaxation time (the free path time) is $\tau \sim 1/T^2$. The character of the waves propagated in the liquid naturally depends fundamentally on the relations between their frequency and the reciprocal of the relaxation time.

For $\omega\tau \ll 1$ (which is actually equivalent to the condition of the shortness of the free path length of the quasi-particles in comparison with the wave length λ), the collisions succeed in establishing thermodynamic equilibrium in each (small in comparison with λ) element of volume of the liquid. This means that we are dealing with ordinary hydrodynamical sound waves, propagated with a velocity c .

If $\omega\tau \gg 1$, then, on the contrary, the collisions do not play essential roles in the process of the propagation of the vibrations, and we will have the waves of zero sound considered in the preceding section.

In both these limiting cases, the propagation of waves is accompanied by a comparatively weak absorption. In the intermediate region, $\omega\tau \sim 1$, the absorption is very strong and isolation of the different types of waves as undamped processes is not possible here.

[†] These computations were carried out by A. A. Abrikosov and I. M. Khalatnikov.

One can easily obtain the temperature and frequency dependence of the absorption coefficient γ in the region of ordinary sound with the aid of the known formula for the absorption of sound (see ref. 9, for example), according to which γ is proportional to the square of the frequency and to the viscosity coefficient†. Inasmuch as the viscosity of a Fermi liquid¹⁰ is proportional to $1/T^2$, then we find that

$$\gamma \sim \frac{\omega^2}{T^2} \quad \text{for} \quad \omega \ll \frac{1}{\tau}. \quad (20)$$

Absorption in the region of zero sound differs essentially in its character from absorption of ordinary sound. In the latter, the collisions cannot lead to a dissipation of the energy "into the noise" of the distribution, which is changed only by the sound vibrations as such. This is connected with the circumstance already mentioned, that a distribution changed in this fashion remains in thermodynamic equilibrium in each element of the volume. Therefore, the absorption of ordinary sound is connected with the effect of the collisions on the distribution function itself.

In the region of zero sound the collisions lead to absorption "into the background" of the distribution which is changed only by the vibrations themselves, which in this case are not in thermodynamic equilibrium (inasmuch as the form of the limiting Fermi surface is deformed). This change in the distribution function does not depend on the frequency, and therefore the absorption coefficient will not depend on the frequency either. The dependence of γ on the temperature is determined by its proportionality to the number of collisions, i.e.

$$\gamma \sim T^2 \quad \text{for} \quad \frac{\kappa T}{\hbar} \gg \omega \gg \frac{1}{\tau}. \quad (21)$$

The upper limit of the region of applicability of this formula is determined by the inequality $\hbar \omega \ll \kappa T$ (κ is Boltzmann's constant), which allows a classical consideration of collisions. We recall that the inequality assumed here,

$$\frac{\kappa T}{\hbar} \gg \frac{1}{\tau},$$

$$\frac{\hbar}{\tau} \ll \kappa T$$

(smallness of the quantum uncertainty of the energy of quasi-particles in comparison with κT), must hold since it is the condition of applicability of everything generally developed in the theory of the Fermi liquid¹.

The determination of the absorption coefficient of zero sound in the frequency range $\hbar \omega \gtrsim \kappa T$ requires quantum consideration. The corresponding calculations can be simplified if we develop them in such a way that we express the

† The contribution to γ from second viscosity and thermal conductivity is proportional to a much higher power of T and is therefore inconsiderable.

desired "quantum" absorption coefficient in terms of the "classical" from equation (21).

The absorption of sound quanta takes place in the collisions of quasi-particles. If we denote by ε_1 and ε_2 the energies of the quasi-particles before and after collisions, then at a given frequency ω , they are connected by the law of conservation of energy

$$\varepsilon_1 + \varepsilon_2 + \hbar \omega = \varepsilon'_1 + \varepsilon'_2.$$

In addition to the collisions, we must also consider the inverse collisions, which are accompanied by the emission of sound quanta. Taking into consideration the well-known properties of the collision probabilities of Fermi particles, we find that the total rate of decrease of the number of sound quanta as a result of collisions is given by the expression

$$\begin{aligned} & \iiint \int w(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}'_1, \mathbf{p}'_2) \{n_1 n_2 (1 - n'_1)(1 - n'_2) - n'_1 n'_2 (1 - n_1)(1 - n_2)\} \\ & \times \delta(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \mathbf{p}_2 - \hbar \mathbf{k}) \delta(\varepsilon'_1 + \varepsilon'_2 - \varepsilon_1 - \varepsilon_2 - \hbar \omega) d\tau_1 d\tau_2 d\tau'_1 d\tau'_2. \end{aligned} \quad (22)$$

The delta functions in the integrand allow the satisfaction of the laws of conservation of energy and momentum.

In the integral (22), the essential values of the energy are only those in the region of diffuseness of the Fermi distribution. In this region, the expressions under the integral sign are changed strongly only by multipliers which contain $n(\varepsilon)$. Furthermore, it should be noted that the angular integrals in (22) are practically unchanged in the transition from the "classical" region

$$\hbar \omega \ll \kappa T$$

to the "quantum" region

$$\hbar \omega \gg \kappa T.$$

In view of this fact, it will be sufficient for us to calculate the integral

$$\begin{aligned} J = & \iiint \int \{n_1 n_2 (1 - n'_1)(1 - n'_2) - n'_1 n'_2 (1 - n_1)(1 - n_2)\} \\ & \times (1 - n_2) \delta(\varepsilon'_1 + \varepsilon'_2 - \varepsilon_1 - \varepsilon_2 - \hbar \omega) d\varepsilon_1 d\varepsilon_2 d\varepsilon'_1 d\varepsilon'_2, \end{aligned}$$

taken only over the energy. Then, substituting

$$n(\varepsilon) = [e^{(\varepsilon - \mu)/\kappa T} + 1]^{-1}$$

and introducing the notation

$$x = \frac{\varepsilon - \mu}{\kappa T}, \quad \xi = \frac{\hbar \omega}{\kappa T},$$

we get (omitting the factor T^3)

$$J = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(1 - e^{-\xi}) \delta(x'_1 + x'_2 - x_1 - x_2 - \xi) dx_1 dx_2 dx'_1 dx'_2}{(e^{x_1} + 1)(e^{x_2} + 1)(1 + e^{-x'_1})(1 + e^{-x'_2})}.$$

In view of the rapid convergence of the integral, the region of integration can be extended from $-\infty$ to $+\infty$.

For integration purposes, we transform to the variables x_1, x_2, y_1, y_2 , where $y = x - x'$.

Integration over x_1 and x_2 is elementary and gives

$$\begin{aligned} J &= (1 - e^{-\xi}) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\delta(y_1 + y_2 + \xi) dx_1 dx_2 dy_1 dy_2}{(e^{x_1} + 1)(e^{x_2} + 1)(1 + e^{-x_1 + y_1})(1 + e^{-x_2 + y_2})} \\ &= (1 - e^{-\xi}) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{y_1 y_2 \delta(y_1 + y_2 + \xi) dy_1 dy_2}{(1 - e^{y_1})(1 - e^{y_2})} \\ &= -(1 - e^{-\xi}) \int_{-\infty}^{+\infty} \frac{y(\xi + y) dy}{(e^y - 1)(e^{-y - \xi} - 1)} \\ &= \int_{-\infty}^{+\infty} y(\xi + y) \left\{ \frac{1}{e^y - 1} - \frac{1}{e^{y + \xi} - 1} \right\} dx. \end{aligned}$$

For calculating the resulting difference of two diverging integrals, we introduce as an intermediate the finite lower limit $-A$ and write:

$$\begin{aligned} J &= \int_{-A}^{\infty} \frac{y(\xi + y)}{e^y - 1} dy - \int_{-A + \xi}^{\infty} \frac{y(y - \xi)}{e^y - 1} dy \\ &= 2\xi \int_{-A}^{\infty} \frac{y dy}{e^y - 1} - \int_{-A + \xi}^{-A} \frac{y(y - \xi) dy}{e^y - 1}. \end{aligned}$$

Keeping in mind that we shall transform to the limit $A \rightarrow \infty$ we neglect e^y in the denominator of the second of the integrals. The first we re-write in the form

$$\begin{aligned} \int_{-A}^{\infty} \frac{y dy}{e^y - 1} &= \int_0^{\infty} \frac{y dy}{e^y - 1} + \int_{-A}^0 \frac{y dy}{e^y - 1} = \frac{\pi^2}{6} + \int_{-A}^0 \left(\frac{y}{1 - e^{-y}} - y \right) dy \\ &= \frac{\pi^2}{6} + \int_0^A \frac{y dy}{e^y - 1} + \frac{A^2}{2}. \end{aligned}$$

Carrying out reductions and then transforming to $\lambda \rightarrow \infty$, we finally obtain

$$J = \left(\frac{2\xi\pi^2}{3} \right) \left(\frac{1 + \xi^2}{4\pi^2} \right).$$

The desired absorption coefficient γ is proportional to J . The coefficient of proportionality between them is so determined that for $\xi \ll 1$, $\gamma = \gamma_{cl}$. We then obtain:

$$\gamma = \gamma_{cl} \left[1 + \left(\frac{\hbar \omega}{2\pi \kappa T} \right)^2 \right] \quad \text{for } \hbar \omega \gtrsim \kappa T. \quad (23)$$

Considering that $\gamma_{cl} \sim T^2$, we find that in the limit of high frequencies:

$$\gamma \sim \omega^2 \quad \text{for } \hbar \omega \gg \kappa T, \quad (24)$$

i.e. the absorption coefficient remains proportional to the square of the frequency, but does not depend on the temperature. We note that the transition from the formula for "low" to the formula for "high" frequencies takes place at

$$\hbar \omega \sim 2\pi \kappa T$$

(and not $\hbar \omega \sim \kappa T$).[†] The result of (24) refers, in particular, to the zero sound of all frequencies at the absolute zero temperature.

3. SPIN WAVES IN A FERMI LIQUID

In addition to a consideration of zero sound in section 1, which does not involve the distribution of spins, in a Fermi liquid at absolute zero, waves of other types can also be propagated. These we call spin waves.[‡]

In this section, we denote by K the function

$$K = f(\mathbf{p}, \mathbf{p}') 4\pi p^2 dp d\varepsilon / (2\pi \hbar)^3, \quad (25)$$

in which the operator Tr is not used. In the calculation of exchange interaction between the quasi-particles, this function contains terms which are proportional to the product $(\sigma \sigma')$, i.e. it has the form:¹

$$K = \frac{1}{2} F(\chi) + \frac{1}{2} G(\chi) (\sigma \sigma') \quad (26)$$

(F coincides with the function (6) used above).

[†] Considering the frequencies $\omega \gg \kappa T / \hbar$, we at the same time assume satisfaction of the inequality

$$\hbar \omega \ll \kappa T_0$$

(T_0 is the temperature of degeneration of the Fermi distribution). In the opposite case, particles from the "depth" of the Fermi distribution take part in the absorption and all the theory developed here would become inapplicable.

[‡] The equation for spin waves in weakly non-ideal Fermi gas was considered by Silin⁶.

In place of equation (11) we have now

$$(\eta - \cos\theta) \nu = \cos\theta \operatorname{Tr}' \int \frac{F \nu' d\sigma'}{4\pi}. \quad (27)$$

In addition to the solutions $\nu(n)$ considered earlier, which do not depend on the spin, this equation also has a solution of the form

$$\nu = (\mu(n) \cdot \sigma). \quad (28)$$

Substituting (26) and (28) in (27), completing the operation Tr and dividing both sides of the equation by σ , we get

$$(\eta - \cos\theta) \mu = \cos\theta \int \frac{G \mu' d\sigma'}{16\pi}. \quad (29)$$

We see that for each of the components of the vector μ , we obtain an equation which differs from (11) only by the replacement of F by $G/4$. Therefore, all the further calculations of section 1 can immediately be applied to the spin waves.

In the real liquid ^3He , we can determine from available experimental data on its magnetic susceptibility only the mean value of \bar{G} , which was pointed out previously—1.9. Inasmuch as this quantity is negative, then (in view of the results of section 2) it is most probable that the propagation of spin waves in liquid ^3He is not possible. Such a conclusion, however, is in no sense categorical.

In conclusion, I wish to express my thanks to A. A. Abrikosov, E. M. Lifshitz and I. M. Khalatnikov for useful discussions.

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92. ON THE CONSERVATION LAWS FOR WEAK INTERACTIONS

A variant of the theory is proposed in which non-conservation of μ is introduced without assuming asymmetry of space with respect to t .

Various possible consequences of non-conservation of parity are pertinent to the properties of the neutrino and in this connection several problems involving neutrinos are examined on the assumption that the neutrino mass is zero.

1. COMBINED PARITY

As is well known, the unusual properties of K -mesons have created a perplexing situation in modern physics. The correlation between π -mesons in τ -decay ($K^+ \rightarrow 2\pi^+ + \pi^-$) leads to the necessity of assigning a 0^- state to K^+ -mesons. This kind of system, however, cannot decay into two π -mesons ($K^+ \rightarrow \pi^+ + \pi^0$). We are thus faced with the dilemma of either assuming that two different K -mesons exist or that the conservation laws are violated in K -meson decay. In the first case one must then explain the identity of masses (which are equal to within two electron masses) and the near coincidence in lifetime of the θ and τ -decays. One may attempt to explain the equality of K -meson masses by postulating, as Lee and Yang¹ have done, the existence of some hitherto unknown symmetry property of nuclear forces which transforms the τ -meson into a θ -meson. If, however, decay involving a neutrino ($K^+ \rightarrow \mu^+ + \nu$, $K^+ \rightarrow \mu^+ + \nu + \pi^0$, $K^+ \rightarrow e^+ + \pi^0 + \nu$) is considered to be essentially the same for particles of various parity a difference in lifetime related to the different rate of τ and θ -decay (≈ 8 per cent and ≈ 25 per cent) should be anticipated. This discrepancy should be not less than 30-40 per cent, a result which seems to be inconsistent with experiment²).

Thus we come to the conclusion that the hypothesis of the existence of two different K^+ -mesons is contrary to the experimental facts and the only alternative is to assume that the generally accepted conservation laws are violated in K -decay. Since there is no reason to think that the law of conservation of angular momentum is untenable, we are apparently dealing here with a direct violation of the law of conservation of parity.

L. Landau, On the conservation laws for weak interactions, *Nuclear Physics*, 3, 127 (1957).

Л. Д. Ландау, О законах сохранения при слабых взаимодействиях, *Журнал Экспериментальной и Теоретической Физики*, **32**, 405 (1957).

L. D. Landau Conservation laws in weak interactions, *Soviet Phys.-JETP*, 5, 336 (1957).

Л. Д. Ландау, Об одной возможности для поляризационных свойств нейтрино, *Журнал Экспериментальной и Теоретической Физики*, **32**, 407 (1957).

L. D. Landau, Possible properties of the neutrino spin, *Soviet Phys. JETP* 5, 337 (1957).

It might seem at first glance that non-conservation of parity implies asymmetry of space with respect to inversion. If however, complete isotropy of space (conservation of angular momentum) is taken into account this type of asymmetry would seem to be extremely strange and in my opinion a simple rejection of parity conservation would create a difficult situation in theoretical physics. I would like to point out a solution of this problem which consists in the following. As is well known, both the law of conservation of parity and charge conjugation invariance undoubtedly hold in strong interactions. Let us now assume that each of these conservation laws does not hold separately in weak interactions. However, invariance with respect to the set of both operations (which we shall call combined inversion) will be assumed to exist. In combined inversion, space inversion and transformation of a particle into an antiparticle occur simultaneously.

It is easy to see that invariance of the interactions with respect to combined inversion leaves space completely symmetrical, and only the electrical charges will be asymmetrical. The effect of this asymmetry on the symmetry of space is no greater than that due to chemical stereo-isomerism.

On the other hand the law of conservation of parity of charged particles will not hold as the operator of combined inversion does not transform charged particles into themselves.

Furthermore, it is easy to see that the constants characterising the particles and anti-particles (masses, lifetimes) should be identical since, as a result of invariance with respect to combined inversion, all processes involving particles and antiparticles should differ from each other only in regard to space inversion. Graphically speaking, a K^- -meson is a mirror reflected K^+ -meson.

Truly neutral particles, that is, particles which are identical to their antiparticles, transform into themselves in combined inversion. Consequently, with respect to these particles combined inversion leads to a law of conservation of combined parity. It should be emphasised that conservable parity is the product of ordinary parity and charge parity of the particles. Evidently, in this sense the π^0 -meson is an odd particle; the K_1^0 (θ^0)-meson which decays into 2 π -mesons is an even particle and the K_2^0 -meson predicted by Gell-Mann and Pais³ and recently discovered experimentally⁴ is an odd particle. Combined inversion changes the sign of the magnetic field of a photon but does not change that of the electric field. The ordinary parities of electric and magnetic multipoles are reversed for combined inversion.

It is easy to show from the foregoing that despite the absence of ordinary parity the particles cannot possess dipole moments. Indeed, the only vector which can be constructed from ψ -operators for a particle at rest is its spin vector which is even with respect to inversion and odd with respect to charge. It is consequently odd with respect to combined inversion and, in accord with the foregoing regarding the electromagnetic field, it defines only a magnetic but not an electric moment.

Lee and Yang^{5†} have shown that non-conservation of parity leads to correlations in a number of hyperon production and decay processes. It can be

† I would like to sincerely thank the authors for sending me a preprint of their paper.

shown that a consequence of invariance with respect to combined inversion is that the weak interaction operators in the Lagrangian contain real coefficients. This circumstance, however, does not appreciably modify the qualitative picture which is obtained in the general case of non-conservation of parity. Therefore asymmetry of hyperon decay with respect to the plane of their creation, which has been predicted by Lee and Yang⁵, will also hold in this case.

I would like to express my deep appreciation to L. Okun, B. Ioffe and A. Rudik for discussions from which the idea of this part of the present paper emerged.

2. PROPERTIES OF THE NEUTRINO

Rejection of the law of conservation of parity entails the possibility of the existence of new properties of the neutrino. The Dirac equation for the case of zero mass splits into two independent pairs of equations. It will be recalled that in the usual theory one cannot confine oneself to a single pair of equations since both pairs transform into each other as a result of space inversion. If, however, we restrict our attention to combined inversion we arrive at the possibility of describing the neutrino by a single pair of equations. In the sense of the usual scheme this would signify that the neutrino is always polarised in the direction of its motion (or in the opposite direction). The polarisation of the antineutrino is correspondingly reversed. According to this model the neutrino is not a truly neutral particle and this agrees with the fact that double β -decay has not been observed experimentally and especially with the results of experiments on induced β -decay. We shall call this kind of neutrino a longitudinally polarised neutrino or briefly a longitudinal neutrino.

In the usual theory the neutrino mass is zero, so to say, accidentally. Thus, account of neutrino interactions automatically leads to the appearance of a definite, albeit vanishingly small, rest mass. The mass of the longitudinal neutrino, on the other hand, vanishes automatically and this situation cannot be altered by the existence of any type of interaction.

The longitudinal neutrino concept appreciably reduces the possible number of types of weak interaction operators. Consider, for example, the decay of a μ -meson into an electron and two neutrinos. In the usual manner we represent the interaction operator as the product of operators consisting of μ -meson and electron ψ -operators on the one hand and ψ -operators of the two neutrinos on the other. For the longitudinal neutrino only one combination can be made from the two neutrino operators—(a scalar with respect to rotation; the operation of ordinary inversion is not applicable), as it is well known that the tensor combination of two identical operators obeying Fermi statistics is equal to zero. In this case two combinations, scalar and pseudo-scalar (in the usual sense of the word), can be constructed for a μ -meson and electron.

If a neutrino and anti-neutrino are emitted in μ -meson decay the situation changes. Only a four-dimensional vector can then be constructed from the longitudinal neutrino and anti-neutrino operators. In this case two combinations—vector and pseudo-vector—can be made from the μ -meson and electron ope-

rators. Thus, despite the absence of invariance with respect to inversion in each of the two cases only two interaction operators are possible.

It is easy to calculate the energy spectrum of the μ -meson decay electrons. It is found to be exactly the same as that calculated by Michel⁶. The two-neutrino case thus yields for Michel's constant, ρ , the value $\rho = 0$ and for a neutrino and anti-neutrino $\rho = 0.75$. The former case is apparently inconsistent with the experiments, whereas the latter agrees with the results obtained in refs. 7 and 8 which yield $\rho = 0.64 \pm 0.10$ and $\rho = 0.57 \pm 0.14$. Thus μ -meson disintegration experiments do not contradict the longitudinal neutrino concept and in this case lead to a unique result, namely, that a neutrino and anti-neutrino are involved in μ -meson decay.

Consider now the reaction $\pi \rightarrow \mu + \nu$. Since the π -meson is spinless we are obliged to set up a scalar expression for the μ -meson and neutrino ν -operators in the $\pi \rightarrow \mu + \nu$ decay operator. This automatically yields that if the neutrino is longitudinal the μ -mesons produced in $\pi \rightarrow \mu + \nu$ decays will be completely polarised in the direction of their motion (or in the opposite direction).

As Lee and Yang⁵ have noted, a possible consequence of non-conservation of parity is the correlation between the directions of the μ -meson and electron involved in the $\pi \rightarrow \mu \rightarrow e$ decay. Simple calculations based on our scheme give the following energy and angular distribution for the emitted electrons:

$$\frac{dN}{N} = 2\varepsilon^2[(3 - 2\varepsilon) + \lambda \cos \vartheta(2\varepsilon - 1)] d\varepsilon. \quad (1)$$

Here ε is the ratio of electron energy to the largest possible energy, ϑ is the angle between the directions of motion of the μ -meson and electron and λ is a constant which depends on the relation between the vector and pseudo-vector parts in the combination of the μ -meson and electron ψ -operators,

$$\lambda = \frac{2ab}{a^2 + b^2}. \quad (2)$$

where a and b are coefficients of the respective terms and, according to the foregoing, are real. Evidently, λ varies between -1 and $+1$. It is possible that λ is in fact equal to zero. The integral electron distribution is obviously proportional to $(1 + \frac{1}{2}\lambda \cos \vartheta)$ and this means that the largest possible value of the forward-backward asymmetry is 2. It should be noted that even if λ appreciably differs from zero it may be difficult to observe $\mu \rightarrow e$ correlation because of depolarisation of the slowed down mesons and in particular for μ^+ -mesons because of formation of mesonium ($\mu^+ + e^-$ system).

Consider now the effect of longitudinality of the neutrino on β -decay. According to experiment the decay operator should be represented as the sum of the scalar and tensor variants. It can be shown that in either case the same electron polarisation in the direction of motion will arise, which is equal to v/c (or $-v/c$), the ratio of the electron velocity to that of light. Thus high energy electrons will be totally polarised in the direction of their motion.

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93. HYDRODYNAMIC FLUCTUATIONS

A GENERAL theory of hydrodynamic fluctuations can be constructed by introducing "outside" terms into the equation of motion of the liquid, as was done by Rytov¹ for the fluctuations of an electromagnetic field in continuous media; he introduced corresponding "outside" fields in Maxwell's equations.

The introduction of such additional terms can be accomplished in different equivalent ways. The most advantageous is the form in which the fluctuations of the "outside quantities" at the various points of the liquid are not correlated with one another. This is accomplished by the introduction of "outside stress tensor" s_{ik} in the Navier-Stokes equation and the "outside heat flow" vector q in the heat conduction equation (the equation of continuity remains unchanged). The system of hydrodynamic equations then takes the form

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0, \quad (1)$$

$$\varrho \frac{\partial v_i}{\partial t} + \varrho (\mathbf{v} \cdot \nabla) v_i = - \frac{\partial \varrho}{\partial x_i} + \frac{\partial \sigma'_{ik}}{\partial x_k}, \quad (2)$$

$$\varrho T \left[\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla) s \right] = \frac{1}{2} \sigma'_{ik} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) - \operatorname{div} \mathbf{q}, \quad (3)$$

$$\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) + \zeta \frac{\partial v_l}{\partial x_l} \delta_{ik} + s_{ik}, \quad (4)$$

$$\mathbf{q} = -\kappa \nabla T + \mathbf{g} \quad (5)$$

(the notation agrees with that used in our book². To these equations should be added the relations which define the mean values of the products of components s_{ik} and g_i . We do this by first assuming the fluctuations to be classical (i.e. their frequencies $\omega \ll kT/\hbar$), while the viscosity and the thermal conductivity of the liquid are non-dispersive.

The rate of change of the total entropy of the liquid S is given by the expression (see ref. 2, section 49)

$$\dot{S} = \int \left\{ \frac{\sigma'_{ik}}{2T} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) - \frac{(\mathbf{q} \cdot \nabla T)}{T^2} \right\} dV. \quad (6)$$

Л. Д. Ландау и Е. М. Лифшиц, О гидродинамических флуктуациях, *Журнал Экспериментальной и Теоретической Физики*, **32**, 618 (1957).

L. D. Landau and E. M. Lifshitz, Hydrodynamic fluctuations, *Soviet Phys.-JETP*, **5**, 512 (1957).

Following the general rules of fluctuation theory laid down in ref. 3, sections 117 and 120, we select as the values \dot{x}_a figuring in this theory the components of the tensor σ'_{ik} and the vector q .† It is then evident from equation (6) that the role of the corresponding quantities X_a will be played by

$$-\frac{1}{2T} \left(\frac{\partial v}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \Delta V \quad \text{and} \quad \frac{1}{T^2} \frac{\partial T}{\partial x_i} \Delta V,$$

while (4) and (5) play the role of the relations $\dot{x}_a = -\gamma_{ab} X_b + y_a$ (see ref. 3, section 120), where the s_{ik} and g_i correspond to the quantities y_a . The coefficients γ_{ab} in these relations determine directly the mean values

$$\overline{y_a(t_1) y_b(t_2)} = k(\gamma_{ab} + \gamma_{ba}) \delta(t_1 - t_2).$$

The final formulas have the form:

$$\overline{s_{ik}(r_1, t_1) s_{lm}(r_2, t_2)} = 2kT [\eta (\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) + \left(\zeta - \frac{2\eta}{3} \right) \delta_{ik} \delta_{lm}] \delta(r_2 - r_1) \delta(t_2 - t_1), \quad (7)$$

$$\overline{g_i(r_1, t_1) g_k(r_2, t_2)} = 2kT^2 \varkappa \delta_{ik} \delta(r_2 - r_1) \delta(t_2 - t_1), \quad \overline{g_i(r_1, t_1) s_{lm}(r_2, t_2)} = 0.$$

If use is made of the spectral components of the fluctuating quantities, which are defined by

$$x_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt, \quad \overline{x^2} = \int_{-\infty}^{\infty} \overline{x_\omega x_{\omega'}} d\omega d\omega',$$

then the factor $\delta(t_2 - t_1)$ in equations (7) is replaced by $\delta(\omega + \omega')/2\pi$.

These results are generalized without difficulty to the case of the presence of dispersion in the coefficients of viscosity or thermal conductivity and the quantum nature of the fluctuations with the aid of the general theory of Callen and others, in the form set forth in ref. 4. There appears only the factor $(\hbar\omega/2kT) \coth \hbar\omega/2kT$ in the expressions for the average values of the products of the spectral components s_{ik} and g_i , while the quantities η , ζ , \varkappa are to be replaced by their real parts.

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† An inessential difference, connected with the fact that we are dealing here with a continuous (values at each point of the liquid) as against a discrete set of fluctuating quantities (for which the formulas in ref. 3 were developed), can easily be removed formally by dividing the volume of the liquid into small but finite regions ΔV and carrying out the transition $\Delta V \rightarrow 0$ in the final equations.

94. THE PROPERTIES OF THE GREEN FUNCTION FOR PARTICLES IN STATISTICS

In an attempt to apply the methods which have recently been developed in quantum electrodynamics to statistics, one is faced with the problem of the properties that will be exhibited in that case by the fundamental quantities entering into these methods. We shall show here that some very general relations can be derived for the particle Green function G , which is defined in the well-known manner as

$$G = -i \langle T \psi_1 \psi_2^+ \rangle, \quad (1)$$

where the indices 1 and 2 indicate that the ψ operator must be taken at t_1 or t_2 . T is the symbol for the chronologically ordered product, while the averaging is taken over the actual state of the given macroscopic system. It is well known that chronological ordering means that

$$G = -i \langle \psi_1 \psi_2^+ \rangle \quad \text{for } t_1 > t_2, \quad G = \mp i \langle \psi_2^- \psi_1 \rangle \quad \text{for } t_1 < t_2, \quad (2)$$

where the upper and lower signs refer to the Bose and Fermi statistics respectively.

The space-time dependence of the matrix elements of the operator ψ is given by equations of the form

$$\psi_{nm}(t, \mathbf{r}) = \psi_{nm}^{(0)} \exp \{i[\omega_{nm}t - (\mathbf{k}_{nm} \cdot \mathbf{r})]\}, \quad \omega_{nm} = \frac{(E_n - E_m)}{\hbar}, \quad \mathbf{k}_{nm} = \frac{(\mathbf{P}_n - \mathbf{P}_m)}{\hbar},$$

where the indices n, m refer to the states of a closed system with total energy E and total momentum \mathbf{P} . From the definition of the adjoint operator, we have $(\psi^+)_{mn} = (\psi_{nm})^*$.

Using these matrix elements we can write (2) in the form

$$\begin{aligned} \text{for } t_1 > t_2: G &= -i \sum_m |\psi_{nm}^{(0)}|^2 \exp \{i\omega_{nm}t - i(\mathbf{k}_{nm} \cdot \mathbf{r})\}, \\ \text{for } t_1 < t_2: G &= \mp i \sum_m |\psi_{nm}^{(0)}|^2 \exp \{i\omega_{mn}t - i(\mathbf{k}_{mn} \cdot \mathbf{r})\}, \end{aligned} \quad (3)$$

where we use the notation $t = t_1 - t_2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and where the index n refers to the given state over which the averaging of equations (1) or (2) is performed.

Л. Д. Ландау, Свойства гринковской функции в статистике, *Журнал Экспериментальной и Теоретической Физики*, **34**, 262 (1958).

L. D. Landau, The properties of the Green function for particles in statistics, *Soviet Phys.-JETP*, **7**, 182 (1958).

We shall now proceed to average expressions (3) over a Gibbsian ensemble. According to the basic principles of statistics, this operation means that we express the quantity G as a function of the temperature T and the chemical potential μ instead of as a function of the energy E and the number of particles in the system N . We have for $t > 0$,

$$G = -i \sum_{n,m} \exp \{(\Omega + \mu N_n - E_n)/T\} |\psi_{nm}^{(0)}|^2 \exp \{i \omega_{nm} t - i(\mathbf{k}_{nm} \cdot \mathbf{r})\}$$

where the temperature is measured in energy units. Since we sum now over the two indices n and m we can interchange the indices in such a sum. We use this possibility in the expression for G for $t_1 > t_2$ and write for $t > 0$,

$$\begin{aligned} G &= \mp i \sum_{n,m} \exp \{(\Omega + \mu N_m - E_m)/T\} |\psi_{nm}^{(0)}|^2 \exp \{i \omega_{nm} t - i(\mathbf{k}_{nm} \cdot \mathbf{r})\} \\ &= \mp i \sum_{m,n} \exp \{(\Omega + \mu N_n - E_n)/T\} \\ &\quad \times \exp \{(\hbar \omega_{nm} + \mu)/T\} |\psi_{nm}^{(0)}|^2 \exp \{i \omega_{nm} t - i(\mathbf{k}_{nm} \cdot \mathbf{r})\}. \end{aligned}$$

In the last transformation we have used the fact that the matrix elements $\psi_{nm}^{(0)}$ are different from zero only if $N_m = N_n + 1$.

We now go over from the space-time representation of the Green function to its Fourier coefficients,

$$G(\omega, \mathbf{k}) = \int \int G(t, \mathbf{r}) e^{i[\omega t - (\mathbf{k} \cdot \mathbf{r})]} d\mathbf{r} dt.$$

Integration over space gives a delta function of $\mathbf{k} - \mathbf{k}_{mn}$. The integration over t must be performed separately over the interval from $-\infty$ to 0 and from 0 to $+\infty$, using the well-known formula

$$\int_0^\infty e^{i\alpha x} dx = \pi \delta(\alpha) + \frac{i}{\alpha}.$$

We get as a result

$$\begin{aligned} G(\omega, \mathbf{k}) &= -(2\pi)^3 \sum_{n,m} \exp \{(\Omega + \mu N_n - E_n)/T\} |\psi_{nm}^{(0)}|^2 \delta(\mathbf{k} - \mathbf{k}_{mn}) \\ &\quad \times \{i \pi \delta(\omega - \omega_{mn}) [1 \pm e^{(\mu - \hbar \omega_{mn})/T}] + \frac{1}{\omega_{mn} - \omega} [1 \mp e^{(\mu - \hbar \omega_{mn})/T}]\}. \end{aligned} \quad (4)$$

Comparing the two terms within the braces we see that there exists a certain relation between the real (G') and the imaginary (G'') part of the Green function. In the case of Bose statistics this relation is

$$G'(\omega, \mathbf{k}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tanh \frac{\hbar x - \mu}{2T} \cdot \frac{G''(x, \mathbf{k})}{x - \omega} dx, \quad (5)$$

where we take the principal value of the integral. We have always

$$G''(\omega, k) < 0. \quad (6)$$

as follows from equation (4).

In the case of Fermi statistics we have

$$G'(\omega, k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \coth \frac{\hbar x - \mu}{2T} \cdot \frac{G''(x, k)}{x - \omega} dx, \quad (7)$$

where the sign of $G''(\omega, k)$ is the opposite of the sign of the difference $\hbar \omega - \mu$ and both these quantities go through zero at the same time,

$$\frac{G''(\omega, k)}{(\hbar \omega - \mu)} < 0. \quad (8)$$

At the absolute zero of temperature both (5) and (7) go over into

$$G'(\omega, k) = \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{G''(x, k)}{x - \omega} dx, \quad (9)$$

where the plus sign refers to $\hbar x > \mu$ and the minus sign to $\hbar x < \mu$.

It is interesting to note that these formulae show that the function G is not an analytical function of the variable ω . We can construct two analytical functions (which have no singularities in the upper half-plane) as follows

$$G' + i \tanh \frac{\hbar \omega - \mu}{2T} \cdot G'' \quad \text{and} \quad G' + i \coth \frac{\hbar \omega - \mu}{2T} \cdot G''.$$

95. ON THE THEORY OF THE FERMI LIQUID

A study is made of the zero-angle scattering in collisions of quasi-particles in a Fermi liquid. It is shown that the scattering amplitude for zero angle depends on the limit approached by the ratio of the momentum and energy transfer in the collision as both these quantities go to zero. It is ascertained which of these limits is connected with the interaction energy of the quasi-particles that occurs in the general theory of the Fermi liquid developed earlier by the writer.

A GENERAL theory of the Fermi liquid† has been developed in previous papers by the writer^{1, 2}. One of the quantities that plays an important part in this theory in characterising the properties of the liquid is the function $f(\mathbf{p}, \mathbf{p}')$ which determines the interaction energy of the quasi-particles, i.e. the variation of the energy $\varepsilon(\mathbf{p})$ of the quasiparticles arising from a variation of their distribution function:

$$\delta\varepsilon(\mathbf{p}) = \text{Tr} \int f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}') d\tau' \quad (1)$$

(where $d\tau' = d^3p/(2\pi)^3$; here and below we take $\hbar = 1$).

In ref. 1 it was shown that the function $f(\mathbf{p}, \mathbf{p}')$ is related in a definite way to the scattering amplitude of the quasi-particles in the liquid for their mutual collisions. The formulation given in ref. 1 for this connection is not, however, quite accurate, as will be shown in the present paper.

We use below methods borrowed from quantum field theory; as is well known, these methods have recently been used with success by various authors in the study of the properties of quantum many-particle systems.

The main part in these methods is played by the Green function G and the "vertex part" Γ . Let us recall the definitions and basic properties of these functions.

The function G is defined as the average value in the ground state of the system of the chronological product of two ψ operators:

$$G_{12} = -i \langle T(\psi_1 \psi_2^\dagger) \rangle. \quad (2)$$

The indices 1, 2 denote sets of values of the three co-ordinates and the time, and also of the spin index. As usual, we shall use below instead of the space-time representation (2) the Fourier expansion of this function. The only

Л. Д. Ландау, К теории ферми-жидкости, *Журнал Экспериментальной и Теоретической Физики*, **35**, 97 (1958).

L. D. Landau, On the theory of the Fermi liquid, *Soviet Phys.-JETP*, **8**, 70 (1959).

† To avoid misunderstanding we emphasise that we are concerned not with simply a liquid composed of Fermi particles; it is also postulated that this liquid has an energy spectrum of the Fermi type, i.e. that it is not a superfluid.

components different from zero are those with identical values of the two momenta and the two energies (that is, of the wave vectors and frequencies): $P_1 = P_2 \equiv P$; we denote by P the "four-momentum", i.e. the combination of the momentum p and the energy ε . In respect to the spin indices (which we denote by Greek letters) the Fourier components $G_{\alpha\beta}(P) = \int G_{\alpha\beta}(X_1 - X_2) \times e^{-iP(X_1 - X_2)} d^4(X_1 - X_2)$ are proportional to $\delta_{\alpha\beta}$; we shall write

$$G_{\alpha\beta}(P) = G(P) \delta_{\alpha\beta}. \quad (3)$$

As is well known, the poles of the function $G(P)$ give the energies of the quasiparticles (the elementary excitations). In accordance with this, for $|p|$ close to the boundary momentum p_0 and ε close to the boundary energy μ , $G(P)$ has the form

$$G(P) \sim \frac{a}{\varepsilon - \mu - v(p - p_0) + i\delta} \quad (4)$$

(μ is the chemical potential of the gas, and v is the speed of the quasi-particles at the Fermi boundary). This expression has a pole at

$$\varepsilon - \mu = v(p - p_0), \quad (5)$$

and the small constant δ is introduced in the usual way to specify the rule for going around the singularity in integrating; the sign of δ agrees with the sign of $\varepsilon - \mu$ (or, what is the same thing near the pole, with the sign of $p - p_0$). The "re-normalisation" factor a is positive and, as has been shown by Migdal,³ is smaller than unity:

$$a < 1. \quad (6)$$

The vertex part Γ is defined by means of the four-particle average value

$$\Phi_{1234} = \langle T(\psi_1 \psi_2 \psi_3^+ \psi_4^+) \rangle. \quad (7)$$

The Fourier components of this function contain a part that is expressed in terms of functions $G(P)$ only, and a remainder that gives the definition of the Fourier component of the vertex part by the following formula:

$$\begin{aligned} \Phi_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4) &= (2\pi)^3 G(P_1) G(P_2) \{ \delta(P_1 - P_3) \delta(P_2 - P_4) \delta_{\alpha\gamma} \delta_{\beta\delta} \\ &\quad - \delta(P_1 - P_4) \delta(P_2 - P_3) \delta_{\alpha\delta} \delta_{\beta\gamma} \} + i G(P_1) G(P_2) G(P_3) G(P_4) \\ &\quad \times \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4) (2\pi)^4 \delta(P_1 + P_2 - P_3 - P_4). \end{aligned} \quad (8)$$

Here the values of the arguments are connected by the relation

$$P_1 + P_2 = P_3 + P_4. \quad (9)$$

On interchange of the indices 1 and 2 (or 3 and 4) the function (7) changes sign; thus it follows from the definition (8) that Γ has the symmetry property:

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4) = -\Gamma_{\beta\alpha,\gamma\delta}(P_2, P_1; P_3, P_4). \quad (10)$$

In the formation of the vertex part intermediate states occur that correspond to different values of the total number of particles in the system: the unchanged number N and the numbers $N \pm 2$. The latter arise from such arrangements of the ψ operators in the T -product as, for example, $\psi_1 \psi_2 \psi_3^+ \psi_4^+$; the former correspond to arrangements such as, for example, $\psi_1 \psi_3^+ \psi_2 \psi_4^+$. In accordance with this the contributions to the function Γ connected with these intermediate states have different characters in regard to their singularities. Namely, terms due to states that appear with the addition or removal of two particles have singularities with respect to the variables $P_1 + P_2$; terms corresponding to intermediate states with unchanged number of particles have singularities with respect to the variables $P_1 - P_3$ or $P_2 - P_4$.

The probability of scattering of quasi-particles with the transition

$$P_1 \alpha, P_2 \beta \rightarrow P_3 \gamma, P_4 \delta \quad (11)$$

is given in terms of the function Γ by the formula

$$dW_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4) = 2\pi |a^2 \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4)|^2 \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \times n_1 n_2 (1 - n_3)(1 - n_4) d\tau_1 d\tau_2 d\tau_3 \quad (12)$$

(where n_1, n_2, \dots are the values of the distribution function for $P_1 \alpha, P_2 \beta$, and so on, and a is the renormalisation constant from (4)). The sign of Γ is defined in such a way that it corresponds to a positive scattering amplitude for repulsion and a negative amplitude for attraction.

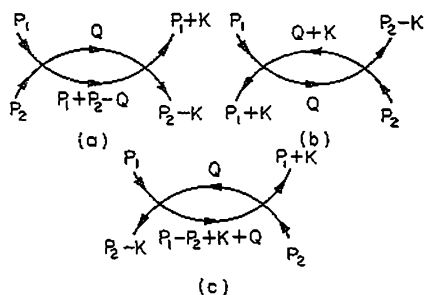


FIG. 1.

Below we shall consider the function Γ for nearly equal values of the pairs of variables P_1, P_3 and P_2, P_4 , i.e., we set $P_3 = P_1 + K, P_4 = P_2 - K$ with small K , and agree to write

$$\Gamma(P_1, P_2; P_1 + K, P_2 - K) \equiv \Gamma(P_1, P_2; K). \quad (13)$$

In terms of the scattering process (11) this means that we are considering collisions of quasi-particles giving nearly "forward scattering."

In the lowest order of perturbation theory contributions to the function $\Gamma(P_1, P_2; K)$ are made by the diagrams shown in Fig. 1 (a, b and c).

The internal parts of these diagrams correspond to the following propagation functions:

$$(a) G(Q) G(P_1 + P_2 - Q), \quad (b) G(Q) G(K + Q), \\ (c) G(Q) G(P_1 - P_2 + K + Q),$$

where Q is the intermediate four-momentum over which one integrates. With arbitrary P_1 and P_2 there is nothing to distinguish the value $K = 0$ for the functions (a) and (c), and for small K we can put $K = 0$. In the case (b), on the other hand, for $K \rightarrow 0$ the poles of the two factors come together, so that diagrams of this type require special consideration.

To calculate Γ one must sum the entire series of perturbation theory. Since in doing this our purpose is to separate out the parts having a singularity at $K = 0$, we must first single out the contribution from all the diagrams that do not have any parallel pairs of lines with nearly equal (differing by K) values of the four-momentum. We denote by $\Gamma^{(1)}$ this part of the function Γ , which has no singularity at $K = 0$; in it we can simply put $K = 0$, so that $\Gamma^{(1)}$ will be a function of the variables P_1 and P_2 only: $\Gamma^{(1)} = \Gamma^{(1)}(P_1, P_2)$. The entire series that has to be summed can be written symbolically in the form

$$(:\Gamma:) = (:I_1:) + (:I_1: I_1:) + (:I_1: I_1: I_1:) + \dots, \quad (14)$$

where the colons replace pairs of lines in the diagram with nearly equal values of the four-momentum, and I_1 denotes the set of all possible diagram elements that do not have such pairs.

The problem of summing this series (so-called "ladder" summation) reduces to the solution of an integral equation, to obtain which we "multiply" the series (14) by I_1 , i.e. replace it by the series

$$(:I_1: \Gamma:) = (:I_1: I_1:) + (:I_1: I_1: I_1:) + \dots$$

Comparison of this with (14) leads to the equation

$$(:\Gamma:) - (:I_1:) = (:I_1: \Gamma:),$$

which, when written out in explicit form, is the desired integral equation

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) = \Gamma_{\alpha\beta,\gamma\delta}^{(1)}(P_1, P_2) \\ - \frac{i}{(2\pi)^4} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^{(1)}(P_1, Q) G(Q) G(Q + K) \Gamma_{\zeta\delta,\epsilon\delta}(Q, P_2; K) d^4Q \quad (15)$$

(in the first factor of the integrand we should, strictly speaking, have $Q + K$ instead of the argument Q ; but in view of the absence of singularities in $\Gamma^{(1)}$ we can here set $K = 0$). To investigate this equation we examine the product $G(Q) G(Q + K)$ that occurs in the integrand. On substituting here $G(P)$ in the form (4) we get

$$a^2/[\varepsilon - \mu - v(q - p_0) + i\delta_1][\varepsilon + \omega - \mu - v(|q + k| - p_0) + i\delta_2]. \quad (16)$$

Here ε and q are the energy and momentum corresponding to the four-momentum Q , and $\varepsilon + \omega$ and $q + k$ are those corresponding to $Q + K$.

For small k and ω the expression (16), as a function of ε and q , behaves like δ functions of the arguments $\varepsilon - \mu$ and $q - p_0$; that is, it has the form

$$A \delta(\varepsilon - \mu) \delta(q - p_0), \quad (17)$$

where the coefficient A depends on the angle θ between the vectors k and q . Comparing (16) and (17), we see that this coefficient is given by the integral

$$A = \iint \frac{a^2 d\varepsilon dq}{[\varepsilon - \mu - v(q - p_0) + i\delta_1][\varepsilon + \omega - \mu - v(|q + k| - p_0) + i\delta_2]}. \quad (18)$$

Let us first carry out the integration with respect to $d\varepsilon$. The result of the integration depends essentially on the value of q . If the two differences $q - p_0$ and $|q + k| - p_0$ have the same sign, then we must also assign like signs to the quantities δ_1 and δ_2 . The poles of the integrand then lie in one half-plane of the complex variable ε , and by closing the path of integration through the other half-plane we can see that the integral vanishes. Thus the integral is non-vanishing only for opposite signs of the differences $q - p_0$ and $|q + k| - p_0$. Let us first suppose that $(q \cdot k) > 0$, i.e. $\cos\theta > 0$. Then the integral is non-vanishing for $q > p_0$, $|q + k| > p_0$, which, because of the smallness of k , is equivalent to the condition

$$p_0 - k \cos\theta < q < p_0. \quad (19)$$

In addition we must have for the quantities δ that $\delta_1 < 0$, $\delta_2 > 0$, so that the poles of the integrand lie in different half-planes. Closing the path of integration through one half-plane and calculating the integral from the residue at the corresponding pole, we find

$$A = \int \frac{2\pi i a^2 dq}{\omega - v(|q + k| - q)}.$$

Since by (19) q is nearly equal to p_0 and varies over a range $k \cos\theta$, we can put $|q + k| - q = k \cos\theta$, so that

$$A = \frac{2\pi i a^2 k \cos\theta}{\omega - v k \cos\theta}.$$

Let us note the peculiar character of this expression: its limit for $k \rightarrow 0$, $\omega \rightarrow 0$ depends on the limit approached by the ratio ω/k .

It is easy to show in the same way that for $\cos\theta < 0$ (in which case the integration must be taken over the region $q > p_0$, $|q + k| < p_0$) one gets the same expression for $A(\theta)$. Thus we have

$$G(Q) G(Q + K) = \frac{2\pi i a^2 (\mathbf{l} \cdot \mathbf{k})}{\omega - v(\mathbf{l} \cdot \mathbf{k})} \delta(\varepsilon - \mu) \delta(q - p_0) + g(Q), \quad (20)$$

where $(\mathbf{l} \cdot \mathbf{k})$ has been written instead of $k \cos\theta$ (\mathbf{l} is the unit vector in the direction of q), and $g(Q)$ does not contain any δ -function part (for small K), so that in it we can put $K = 0$.

Substituting (20) into (15), we get the fundamental integral equation in the form

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) &= \Gamma_{\alpha\beta,\gamma\delta}^{(1)}(P_1, P_2) \\ &- \frac{i}{(2\pi)^4} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^{(1)}(P_1, Q) g(Q) \Gamma_{\zeta\beta,\epsilon\delta}(Q, P_2; K) d^4Q \\ &+ \frac{\alpha^2 p_0^2}{(2\pi)^3} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^{(1)}(P_1, Q) \Gamma_{\zeta\beta,\epsilon\delta}(Q, P_2; K) \frac{(\mathbf{l} \cdot \mathbf{k})}{\omega - v(\mathbf{l} \cdot \mathbf{k})} d\omega. \end{aligned} \quad (21)$$

In the last term we have put $d^4Q = q^2 dq d\omega d\epsilon$, where $d\omega$ is an element of solid angle in the direction of \mathbf{l} , and have carried out the integration of the δ functions in the integrand with respect to $dq d\epsilon$. In the arguments of the functions $\Gamma^{(1)}$ and Γ in this term Q is taken on the Fermi surface, i.e. it consists of the momentum $\mathbf{q} = p_0 \mathbf{l}$ and the constant energy μ .

Because of the special character of the kernel of the integral equation as noted above, its solution also has just the same character: the limit of the function $\Gamma(P_1, P_2; K)$ for $K \rightarrow 0$ depends on the way in which k and ω go to zero, i.e. on the limit of the ratio ω/k .

Let us denote by $\Gamma^\omega(P_1, P_2)$ the limit

$$\Gamma_{\alpha\beta,\gamma\delta}^\omega(P_1, P_2) = \lim_{K \rightarrow 0} \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) \text{ for } k/\omega \rightarrow 0 \quad (22)$$

(we shall see below that it is just this quantity with which the function $f(\mathbf{p}, \mathbf{p}')$ of (1) is related). With this way of approaching the limit the kernel of the last term in (21) goes to zero, so that Γ^ω satisfies the equation

$$\Gamma_{\alpha\beta,\gamma\delta}^\omega(P_1, P_2) = \Gamma_{\alpha\beta,\gamma\delta}^{(1)}(P_1, P_2) - \frac{i}{(2\pi)^4} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^{(1)}(P_1, Q) g(Q) \Gamma_{\zeta\beta,\epsilon\delta}^\omega(Q, P_2) d^4Q. \quad (23)$$

We can eliminate $\Gamma^{(1)}$ from (21) and (23). The result of the elimination is

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) &= \Gamma_{\alpha\beta,\gamma\delta}^\omega(P_1, P_2) \\ &+ \frac{\alpha^2 p_0^2}{(2\pi)^3} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^\omega(P_1, Q) \Gamma_{\zeta\beta,\epsilon\delta}(Q, P_2; K) \frac{(\mathbf{l} \cdot \mathbf{k})}{\omega - v(\mathbf{l} \cdot \mathbf{k})} d\omega. \end{aligned} \quad (24)$$

In fact, if we formally write (23) in the form

$$\Gamma_{\alpha\beta,\gamma\delta}^{(1)}(P_1, P_2) = \hat{L} \Gamma_{\alpha\beta,\gamma\delta}^\omega(P_1, P_2), \quad (25)$$

then equation (21) is written

$$\begin{aligned} \hat{L} \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) &= \Gamma_{\alpha\beta,\gamma\delta}^{(1)}(P_1, P_2) \\ &+ \frac{\alpha^2 p_0^2}{(2\pi)^3} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^{(1)}(P_1, Q) \Gamma_{\zeta\beta,\epsilon\delta}(Q, P_2; K) \frac{(\mathbf{l} \cdot \mathbf{k})}{\omega - v(\mathbf{l} \cdot \mathbf{k})} d\omega; \end{aligned}$$

and substituting (25) and then applying the operator \hat{L}^{-1} to both sides, we get (24).

Let us now introduce the function Γ^k defined by

$$\Gamma_{\alpha\beta,\gamma\delta}^k(P_1, P_2) = \lim_{K \rightarrow 0} \Gamma_{\alpha\beta,\gamma\delta}^k(P_1, P_2; K) \quad \text{for } \omega/k \rightarrow 0. \quad (26)$$

This function (multiplied by the re-normalisation constant a^2) is the "forward" scattering amplitude (i.e. that for the transition $P_1, P_2 \rightarrow P_1, P_2$), corresponding to actual physical processes occurring with quasi-particles on the Fermi surface: collisions leaving the quasi-particles on this surface involve changes of momentum without change of energy, so that the passage to the limit of zero momentum transfer k must be made for energy transfer ω strictly equal to zero. On the other hand the function Γ^ω introduced above corresponds to the nonphysical limiting case of "scattering" with small energy transfer and momentum transfer strictly equal to zero.

Setting $\omega = 0$ in (24) going to the limit $K \rightarrow 0$, and multiplying both sides of the equation by a^2 , we get

$$a^2 \Gamma_{\alpha\beta,\gamma\delta}^k(P_1, P_2) = a^2 \Gamma_{\alpha\beta,\gamma\delta}^\omega(P_1, P_2) - \frac{p_0^2}{v(2\pi)^3} \int a^2 \Gamma_{\alpha\epsilon,\gamma\zeta}^\omega(P_1, Q) a^2 \Gamma_{\zeta\beta,\epsilon\delta}^k(Q, P_2) d\omega. \quad (27)$$

Thus there exists a general relation connecting the two limiting forms of the forward scattering amplitude.

Let us now turn to the study of the poles of $\Gamma(P_1, P_2; K)$ as function of K . As was already pointed out at the beginning of this paper, the poles with respect to the variable $K = P_2 - P_1$ are due to contributions to Γ associated with intermediate states in which the number of particles in the system is not changed. Therefore these poles correspond to elementary excitations of the liquid without change of the number of quasi-particles in it. It is obvious that these are the excitations which can be described as sonic excitations in the gas of quasi-particles (phonons of the "zeroth sound").

Near a pole of the function $\Gamma(P_1, P_2; K)$ the left side and the integral on the right side of the equation (24) are arbitrarily large; the term $\Gamma^\omega(P_1, P_2)$, on the other hand, remains finite and therefore can be dropped. We note further that the variables P_2 and also the indices β and δ are not affected by the operations applied to the function Γ in equation (24), i.e. they here play the role of parameters. Finally, we shall consider Γ close to the Fermi surface, i.e. we shall consider the energy of the quasi-particle, which is one of the variables P_1 , to be equal to μ , and the momentum to be equal to p_0 , so that we write it in the form $p_0 n$, where n is a variable unit vector. Keeping all this in mind, we conclude that the determination of the sonic excitations in the liquid reduces to the problem of the eigenvalues of the integral equation

$$\chi_{\alpha\gamma}(n) = \frac{a^2 p_0^2}{(2\pi)^3} \int \Gamma_{\alpha\epsilon,\gamma\zeta}^\omega(n, l) \chi_{\zeta\epsilon}(l) \frac{(l \cdot k)}{\omega - v(l \cdot k)} d\omega, \quad (28)$$

where $\chi_{\alpha\gamma}(n)$ is an auxiliary function.

We transform this equation, introducing instead of χ a new function, by the substitution

$$\nu_{\alpha\gamma}(n) = \frac{(n \cdot k)}{\omega - v(n \cdot k)} \chi_{\alpha\gamma}(n). \quad (29)$$

Then equation (28) takes the form

$$[\omega - v(n \cdot k)] \nu_{\alpha\gamma}(n) = (k \cdot n) \frac{p_0^2 a^2}{(2\pi)^3} \int \Gamma_{\alpha\delta, \gamma\zeta}^\omega(n, l) \nu_{\zeta\delta}(l) d\sigma. \quad (30)$$

This equation agrees precisely in form with equation (11) found in ref. 2 for the distribution function ν in the zeroth sound, and moreover a comparison of the two equations (using the definition of F by equation (6) of ref. 2) leads to the following correspondence between the function $f(p, p')^\dagger$ and the function Γ^ω :

$$f_{\alpha\beta, \gamma\delta}(n, l) = a^2 \Gamma_{\alpha\beta, \gamma\delta}^\omega(n, l). \quad (31)$$

This is the desired relation between f and the properties of the scattering of the quasi-particles. For clarity we point out that the four spin indices on this function correspond to the fact that $f(p, p')$, or more explicitly $f(p, \sigma; p', \sigma')$, depends on the spin operators (two-row matrices) σ and σ' of the two particles; thus to the two particles (momenta $p_0 n$ and $p_0 l$) there correspond the pairs of indices α, γ and β, δ (in the function $\Gamma_{\alpha\beta, \gamma\delta}^\omega(P_1, P_2; P_3, P_4)$ these pairs correspond to the pairs of nearly equal four-momenta P_1, P_3 and P_2, P_4).

Having thus found the connection of the function f with the properties of the scattering of the quasi-particles, let us return to the formula (27) and obtain with its explicit relations between the function f and the "physical" amplitude for zero-angle scattering on the Fermi surface, which we write in the form

$$A(n_1, \sigma_1; n_2, \sigma_2) = a^2 \Gamma^k(n_1, \sigma_1; n_2, \sigma_2). \quad (32)$$

On the Fermi surface the relation (27) takes the form

$$A(n_1, \sigma_1; n_2, \sigma_2) = f(n_1, \sigma_1; n_2, \sigma_2) - \frac{1}{4\pi} \frac{d\tau}{d\varepsilon} \text{Tr}' \int f(n_1, \sigma_1; n', \sigma') A(n', \sigma'; n_2, \sigma_2) d\sigma' \quad (33)$$

(where $d\tau/d\varepsilon = 4\pi p_0^2/v(2\pi)^3$). The scalar functions A and f depend on all scalar combinations of the four vectors $n_1, n_2, \sigma_1, \sigma_2$. If, however, the interaction between the particles is an exchange interaction, then the only admissible scalar products are $(n_1 \cdot n_2)$ and $(\sigma_1 \cdot \sigma_2)$. Then we can expand A and f as functions of $\cos\theta$ in terms of Legendre polynomials:

$$A(\cos\theta) = \sum_l A_l P_l(\cos\theta), \quad f(\cos\theta) = \sum_l f_l P_l(\cos\theta) \quad (34)$$

[†] In refs. 1 and 2 we did not write the spin indices explicitly.

Substituting this into (33) and performing the integration with respect to $d\sigma'$, we get

$$A_l(\sigma_1, \sigma_2) = f_l(\sigma_1 \sigma_2) - \frac{1}{2l+1} \frac{d\tau}{d\varepsilon} \text{Tr}' f_l(\sigma_1 \sigma') A_l(\sigma' \sigma_2). \quad (35)$$

In the case of an exchange interaction the spin dependence of the function reduces to a term proportional to $(\sigma_1 \cdot \sigma_2)$ (cf. ref. 1), so that

$$f_l = \varphi_l + \psi_l(\sigma_1 \cdot \sigma_2), \quad (36)$$

where φ_l, ψ_l do not depend on the spins. Corresponding to this we also set

$$A_l = B_l + C_l(\sigma_1 \cdot \sigma_2). \quad (37)$$

Substituting (36) and (37) into (35), we get without difficulty

$$\begin{aligned} B_l &= \varphi_l - \frac{2}{2l+1} \frac{d\tau}{d\varepsilon} B_l \varphi_l, \\ C_l &= \psi_l - \frac{1}{2(2l+1)} \frac{d\tau}{d\varepsilon} C_l \psi_l. \end{aligned} \quad (38)$$

These formulas give a simple algebraic connection between the coefficients of the expansions of f and A in spherical harmonics. We note that only terms of the same l are related to each other, and that B is related only to the φ 's and C only to the ψ 's.

In conclusion, I would like to thank A. B. Migdal, who called my attention to the dependence of the forward scattering amplitude on the ratio ω/k , and also E. M. Lifshitz and L. P. Gor'kov for a discussion of this work.

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96. POSSIBILITY OF FORMULATION OF A THEORY OF STRONGLY INTERACTING FERMIONS

A fermion field is investigated with the interaction Lagrangian density equal to $g(\bar{\psi} O_i \psi) \cdot (\bar{\psi} O_i \psi)$. This point interaction is considered as a limit of an extended one, where it is supposed that the interaction vanishes if the momentum of a particle exceeds Λ and/or if the momentum transferred in a collision of two particles exceeds λ .

The relation between Λ and λ is such as to make the quantity $\lambda^2/\ln(\Lambda/\lambda)$ arbitrarily small as $\lambda \rightarrow \infty$ and $\Lambda \rightarrow \infty$. This choice of the limiting procedure considerably simplifies the investigation of the theory. It is shown that in the limit $\lambda \rightarrow \infty$, $\Lambda \rightarrow \infty$, the physical interaction between particles vanishes in all types of four-fermion interactions. The case of two interacting fields ψ and χ with different isotopic spin is also considered. Going over to the local theory, the physical interaction vanishes in this case as well.

This result shows that in the cases considered no strongly interacting fermion theory can be constructed. In the case of the weak interaction, although no logically consistent theory can be built up, there does exist the perturbation theory, as in electrodynamics, which is valid for sufficiently small energies.

1. INTRODUCTION

The transition to the limit of a point interaction in electrodynamics¹ or meson theories² entails a difficulty which is connected with vanishing of the re-normalised charge and the disappearance of physical interactions between particles. It would seem to be of interest to ascertain whether this difficulty can be overcome by replacing the Yukawa-type interaction $g_0(\bar{\psi} O_i \psi) \varphi_i$ between fermions and bosons by other types of interactions, such as that between bosons alone or that between fermions alone. The difficulty created by the vanishing of the re-normalised charge remains in the case of a system of bosons whose interaction is determined by the operator

$$\frac{g_0}{4!} \int \varphi^4(x) dv.$$

In this case the dependence of g_c on g_0 and Λ (Λ is the cut-off momentum) has the form^{3†}

$$g_c = \frac{g_0}{1 + \frac{3}{2} g_0 \ln(\Lambda^2/\mu^2)}, \quad (1)$$

A. A. Abrikosov, A. D. Galanin, L. P. Gorkov, L. D. Landau, I. Ya. Pomeranchuk and K. A. Ter Martirosyan, Possibility of formulation of a theory of strongly interacting fermions, *Physical Review* 111, 321 (1958).

† This formula was derived for $g_0 < 1$; if, however, two cutoff momenta are introduced, it can be shown that it will be valid for an arbitrary positive value of g_0 .

and g_c vanishes for $\lambda \rightarrow \infty$ and any arbitrary dependence of g_0 on λ , providing that g_0 is positive. Negative values of g_0 (for which, in the limit $\lambda \rightarrow \infty$, g_c may not vanish) are in general inadmissible because no stationary states of a boson system exist for $g_0 < 0$. Indeed, for boson fields a classical limiting case exists in which each state may contain many particles. For $g_0 < 0$ the energy of the classical field φ ,

$$\frac{1}{2} \left[\left(\frac{\partial \varphi}{\partial x_\mu} \right)^2 + \mu^2 \varphi^2 \right] + \frac{g_0}{4!} \varphi^4(x),$$

is not positive definite and can decrease indefinitely with increase of the field amplitude φ . Physically this means that it should be energetically possible for an infinite number of particles to be created from vacuum. Thus the vacuum cannot exist for $g_0 < 0$.

The constant g_0 for the Fermi interaction

$$V = 2\pi^2 g_0 \int (\bar{\psi} O_i \psi) (\bar{\psi} O_j \psi) dv \quad (2)$$

(where the O_i are the ordinary spin and isotopic spin operators for fermions) can have any sign, since the occupation numbers cannot exceed unity and stationary states exist for any sign of g_0 . In the given case, the turning on of the interaction simply leads to a redistribution of the levels of negative and positive energy. The new stationary state with minimal energy which arises after the interaction is "turned on" is the one of a physical vacuum.

Hence if the relation between g_c and g_0 in this case were also determined by a formula analogous to (1), the re-normalised charge g_c would not vanish for $g_0 < 0$. Meson theories could be based on interaction (2) and the mesons from the very start would be similar (in the sense of the Fermi-Yang concept⁴) to non-local formations of fermions.

In the following we shall consider the possibility of setting up a theory of this type.[†]

2. EQUATION FOR THE VERTEX OPERATOR

If the interaction has the same form as (2), the matrix elements will contain quadratic and logarithmic divergences. We shall cut off the diverging integrals by assuming that interaction (2) is somewhat "smeared out": it will be assumed that the interaction vanishes in the momentum representation if momenta P and Q , directed along the fermion lines according to Fig. 1,[‡] exceed some limiting momentum λ or if the momentum p transferred from one line to

[†] An interaction of the same type as (2) has been discussed in a number of papers; see for example, ref. 5.

[‡] In the figures, line discontinuities signify that a large momentum (exceeding λ but smaller than λ) cannot be transferred at the point of discontinuity.

another exceeds λ , where $\lambda \ll \Lambda^\dagger$. In order to go over to the limit of a point interaction one should put $\Lambda \rightarrow \infty$, $\lambda \rightarrow \infty$, the relation between Λ and λ being arbitrary.

If the theory is internally consistent, one may expect the result of the limiting process to be independent of the nature of the transition. The latter, however, should not violate the general conditions required by any physically reasonable theory, such as the general theorems regarding the behavior of Green functions⁶ gauge invariance, etc. We shall restrict ourselves to the case of a limiting transition in which λ/Λ remains arbitrarily small, since in this case the analysis is considerably simplified.

It will be shown that if g_0 is assumed in general to depend on the cut-off limits, one finds that for any form of this dependence, the exact solution yields in the limit of a point interaction the result that no physical interaction exists between fermions.

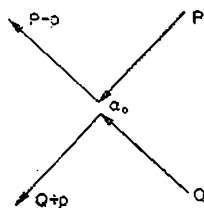


FIG. 1. Simplest four-fermion vertex.

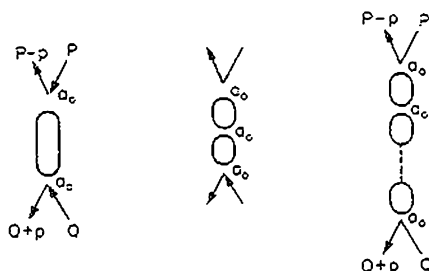


FIG. 2. Chains of closed loops, corresponding to the largest contribution to the vertex operator.

Of those graphs which define the vertex operator $\Gamma(P, Q, p)$ (at which the momentum arrangement corresponds to that drawn in Fig. 1), the largest contribution comes from those in which integration over virtual momenta is performed along closed loops (that is, up to Λ^2) and in which the degree of divergence of the integrals is maximal. This is the loop chain shown in Fig. 2.

† This corresponds to writing (2) with a form factor F_{AA} :

$$\int_{-\infty}^{+\infty} V(t) dt = 2\pi^2 g_0 \int (\bar{\Psi}(x) O_j \psi(x')) (\bar{\Psi}(y) O_j \psi(y')) F_{A,\lambda}(x-x', y-y', x-y) dx dx' dy dy',$$

in which the width of the distributions with respect to $x-x'$ and $y-y'$ are identical and equal $1/\Lambda$, whereas the width of the distribution with respect to $x-y$ is $1/\lambda$.

We shall now find the total contribution (Γ_1) of all of these graphs to the vertex operator and show, by using the value of the vertex operator thus obtained, that the contribution of the remaining graphs (Fig. 4) to Γ is, for sufficiently small λ/Λ , arbitrarily small, and that the Green function of a fermion is identical with the function $G_0 = (-i p - m)^{-1} \cong i/p$ for a free fermion.[†]

The infinite sum of quantities corresponding to the graphs in Fig. 2 satisfies the equation (see Fig. 3)

$$\Gamma_1(P, Q, p) = a_0 - i \int^{A^4} a_0 G(l - p) G(l) \Gamma_1(l, Q, p) d^4 l, \quad (3)$$

where

$$a_0 = g_0(O_j \times O_j) \quad (3a)$$

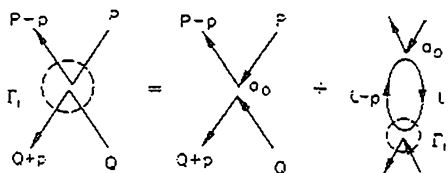


FIG. 3. Graphical representation of equation (3).

is a quantity which corresponds[‡] to the simplest graph in Fig. 1; $d^4 l = (2\pi)^{-2} \times dl_1 dl_2 dl_3 dl_4$. The spinor indices are arranged in the integral term in accord with the closed loop in Fig. 3; that is, the integral contains a trace. It is easy to verify that a consequence of this is that no interference occurs between the various interactions (3a) in (3).

3. SCALAR INTERACTION

Equation (3) possesses a solution which depends only on p ; in the simplest case of scalar ($O_j = 1$) or pseudo-scalar ($O_j = i\gamma_5$) theory the solution is $\Gamma_1 = \alpha(p) \times (O_j \times O_j)$, where

$$\alpha(p) = g_0 \left\{ 1 + g_0 i \int^{A^4} \text{Tr}[O_j G(l - p) O_j G(l) d^4 l] \right\}^{-1}.$$

Inserting $G(l) = G_0(l) = i/l$, we evaluate the quadratically divergent integral involved:

$$i \int^{A^4} \text{Tr}[O_j G(l - p) O_j G(l)] d^4 l = \frac{1}{4} \text{Tr}(O_j \gamma_\nu O_j \gamma_\mu) \left(\frac{1}{2} J_0 \delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} J_1 \right), \quad (4)$$

[†] Or is equal to βG_0 where β is a constant, the value of which can be determined from the equation for G (see below). We have employed the following notation here: $p = \gamma_\mu p_\mu$; $\gamma_\mu^\dagger = \gamma_\mu$.

[‡] Apart from a factor $(2\pi)^2 i$ which is neglected everywhere.

where†

$$J_0 = \frac{8}{3i} \int \frac{k^2 - (pk)^2/p^2}{(k - \frac{1}{2}p)^2(k + \frac{1}{2}p)^2} d^4k = \frac{8}{3\pi} \int (k^2)^2 dk^2 \int_0^1 \frac{(1-x^2)^{3/2} dx}{(k^2 + \frac{1}{4}p^2)^2 - p^2 k^2 x^2}$$

$$= \frac{1}{2} \Lambda^2 - \frac{1}{6} p^2 \ln(\Lambda^2/\xi_0^2 p^2),$$

$$J_1 = \frac{4}{i} \int \frac{\frac{4}{3} \{[(pk)^2/p^2] - \frac{1}{4}k^2\} - \frac{1}{4}p^2}{(k - \frac{1}{2}p)^2(k + \frac{1}{2}p)^2} d^4k$$

$$= \frac{4}{\pi} \int k^2 dk^2 \int_0^1 \frac{\frac{4}{3}k^2(x^2 - \frac{1}{4}) - \frac{1}{4}p^2}{(k^2 + \frac{1}{4}p^2) - p^2 k^2 x^2} (1-x^2)^{1/2} dx$$

$$= -\frac{1}{6} p^2 \ln\left(\frac{\Lambda^2}{\xi_1^2 p^2}\right),$$

and ξ_0 and ξ_1 are constants whose exact values are of no significance for the following. Substituting these values of J_0 and J_1 in (4), we get

$$\alpha(p) = g_0 \left\{ 1 + g_0 \left[\Lambda^2 - \frac{1}{2} p^2 \ln\left(\frac{\Lambda^2}{\xi p^2}\right) \right] \right\}^{-1} \quad (5)$$

where $\xi = (\xi_0^2 \xi_1)^{1/2}$. The quadratic divergence in the denominator cancels out if $g_0 = -1/\Lambda^2$ (or if $1 + g_0 \Lambda^2$ is a quantity which decreases with increase of Λ^2); if, however, $g_0 \neq -1/\Lambda^2$, $\alpha(p)$ will be practically independent of p^2 (it may be recalled that $p^2 \leq \lambda^2 \ll \Lambda^2$) and for $\Lambda \rightarrow \infty$ it vanishes as $1/\Lambda^2$. Postponing the analysis of this case, we shall first assume that

$$1 + g_0 \Lambda^2 = \frac{\mu^2}{\Lambda^2},$$

where μ is a quantity of the order of the lower cut-off limit. Neglecting μ^2/Λ^2 compared with unity, we get from (5)

$$\alpha(p) = -\frac{1}{(p^2 L + \mu^2)}, \quad (5a)$$

where $L = \ln(\Lambda/\lambda)$ (ξp^2 has been replaced by λ^2 as a result of which $\alpha(p)$ simply increases). With the aid of this value of $\alpha(p)$, we shall estimate the magnitude of the contribution from the graph in Fig. 4(a) which determines the difference $G - G_0$, where $G \equiv G_0 \beta$; that is, the difference between β and unity.

† The variable l has been replaced by $k = l + \frac{1}{2}p$ and the transition from pseudo-Euclidian to Euclidian metrics has been carried out: $(4/i) d^4k = (2/\pi) k^2 dk^2 (1-x^2)^{1/2} dx$, where $x^2 = (pk)^2/p^2 k^2$. Terms which remain finite for $\Lambda \rightarrow \infty$ have been neglected in the calculations of J_0 and J_1 .

The spinor indices can be neglected in estimating the integral corresponding to Fig. 4(a), since it is only the order of magnitude that matters. The integral for Fig. 4(a) has the following form (α being a function of f^2 in the given case):

$$g_0 \int \frac{\alpha(f^2) d^4 f}{p-f} \int \frac{d^4 l}{l(l+f)}. \quad (6)$$

Integration over l yields Λ^2 (inasmuch as $f^2 \leq \lambda^2 \ll \Lambda^2$ and the integral over l diverges and therefore $l+f \approx l$). For integration over f , the factor $(p-f)^{-1}$ should be expanded into a series in p :

$$(p-f)^{-1} = -\frac{1}{f} - \frac{1}{f} p \frac{1}{f}$$

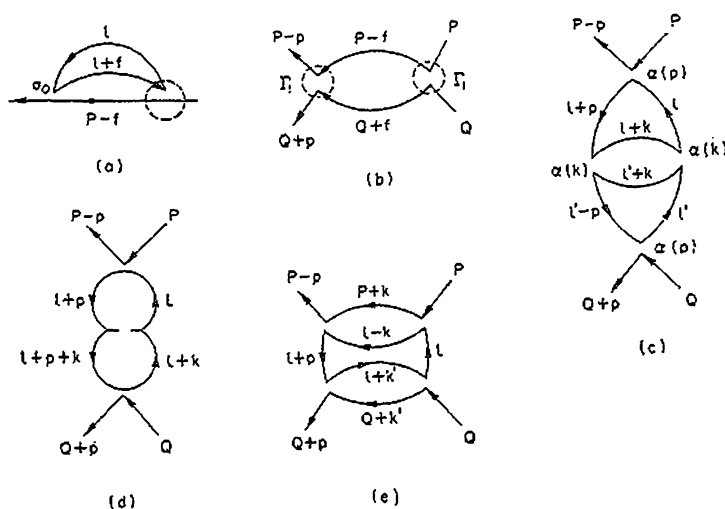


FIG. 4. (a) Graphs contributing to the deviation of β from unity. (b-e) Graphs contributing to the deviation of α from the values given by equation (5).

Integration of the first term yields zero; if (5a) is taken into account, the second term yields $(1/L) \ln(\lambda^2/p^2)$ after integration. Taking into account that $g_0 = -1/\Lambda^2$, we get

$$G^{-1} - G_0^{-1} = -ip \frac{\ln(\lambda^2/p^2)}{L},$$

or, for $L \gg \ln(\lambda^2/m^2)$,

$$\beta - 1 = -\frac{\ln(\lambda^2/p^2)}{L}.$$

Hence for sufficiently large values of L (and for $L \gg \ln(\lambda^2/m^2)$), the difference between β and unity is arbitrary small.

In a similar manner, we consider the ratio of the integrals corresponding to Figs. 4(b), (c), (d), and (e) (not included in equation (3)) to $\alpha(p)$. For Fig. 4(b) this ratio is

$$\frac{1}{\alpha(p)} \int \frac{\alpha(p) \alpha(f)}{(P-f)(Q+f)} d^4 f$$

Depending on the magnitude of the momenta P and Q , viz. $\lambda \ll P \sim Q \ll \Lambda$, $Q \ll \lambda \ll P \ll \Lambda$, or $p \sim P \sim Q \ll \lambda \ll \Lambda$, we obtain, respectively, taking into account (5a),

$$\frac{\lambda^2}{PQL}, \quad \frac{P}{Q} \frac{\ln(\lambda^2/p^2)}{L}, \quad \text{or} \quad \frac{\ln(\lambda^2/p^2)}{L}.$$

In any of the cases considered the ratio can be made arbitrarily small.

For the diagrams in Fig. 4(c), we get

$$\frac{1}{\alpha(p)} \int d^4 l \int d^4 l' \int d^4 k \frac{1}{l(l+p)(l+k)} \times \frac{1}{l'(l'+k)(l'-p)} \alpha^2(p) \alpha(k) \alpha(p-k).$$

To calculate the integral over l (and l') it should be expanded in k , and p should be set equal to zero. The result is

$$\alpha(p) \int d^4 k \alpha(k) \alpha(p-k) [\ln(\Lambda^2/k^2)]^2 d^4 k \cong \frac{1}{p^2 L} \int \frac{d^4 k}{k^2} \cong \frac{\lambda^2}{p^2 L}.$$

Figure 4(d) leads to the following integral:

$$\frac{1}{\alpha(p)} \int d^4 l \int d^4 k \frac{1}{l(l+p)(l+k)(l+p+k)} \alpha^2(p) \alpha(k).$$

Integration over l yields $\ln(\Lambda^2/k^2)$, and we obtain

$$\frac{1}{p^2 L} \int d^4 k \ln\left(\frac{\Lambda^2}{k^2}\right) d^4 k \cong \frac{\lambda^2}{p^2 L}.$$

Finally, we shall consider Fig. 4(e). The integration up to a small limit performed in Fig. 4(b) is repeated twice in this diagram. One may therefore expect that it will be smaller than Fig. 4(b). The corresponding ratio is

$$\frac{1}{\alpha(p)} \int d^4 l \int d^4 k \int d^4 k' \frac{\alpha(k) \alpha(k') \alpha(k-p) \alpha(k'-p)}{l(l+p)(l-k)(l+k')(P+k)(Q+k')}.$$

Depending on the magnitudes of P and Q , namely, $\lambda \ll P \sim Q \ll \Lambda$, $p \sim Q \ll \lambda \ll P$, or $p \sim P \sim Q \ll \lambda$, we correspondingly get

$$\frac{p^2}{PQ} \left(\frac{\ln \lambda^2/p^2}{L} \right)^2, \quad \frac{p}{P} \frac{\ln \lambda^2/p^2}{L^2}, \quad \text{or} \quad \frac{1}{L^2}.$$

These quantities can be made arbitrarily small if the limiting procedure is carried out in such a manner that $\lambda^2/m^2 L$ remains sufficiently small for $\Lambda \rightarrow \infty$. More complex graphs of the vertex operator not indicated in Fig. 4 are proportional to higher powers of the same ratio $\lambda^2/m^2 L$ (or of a smaller quantity) and hence are certainly small.

The expression (5a) obtained for $\alpha(p)$ is thus an exact solution.

The physical interaction between two fermions is determined by the product $\alpha \beta^2$. The results obtained above indicate that $\alpha \beta^2$ vanishes for $\Lambda \rightarrow \infty$; that is, physical interaction between point fermions is absent.

The case when $1 + g_0 \Lambda^2$ is negative should be excluded. Indeed, if $1 + g_0 \Lambda^2$ is negative and equal to -1 , by order of magnitude, the logarithmic term in the expression for Γ can be neglected. Evaluating the additional term in the Green function G with a negative Γ , we see that β would exceed unity, and this is inconsistent with the general theorems of field theory⁶.

If the absolute value of $1 + g_0 \Lambda^2$ is less than $g_0 \Lambda^2$, a pole in the space values of p^2 will arise in the formula for Γ . Since, in treating the interaction between fermions as an interaction between bosons, Γ has the meaning of a boson Green function, it is evident that this pole corresponds to an imaginary boson mass. Such bosons would yield a term $-|m|^2 \varphi^2$ in the Hamiltonian and this would point to nonstability of the vacuum in this case.

Returning to the case $g_0 \neq -1/\Lambda^2$, we note that (3) should be solved simultaneously with the equation for the Green function $G(p)$,

$$\left[-i p - \frac{2}{g_0} \int a_0 G(p-f) \Gamma_1(f) G(l) G(l+f) d^4 l d^4 f \right] G(p) = 1,$$

(the order of the spinor indices corresponds to Fig. 4(a); that is, the integral term contains an integral over l of the same type as (4)).

Neglecting terms of the form $p^2 \ln(\Lambda^2/p^2)$ compared with Λ^2 and taking into account that in this case the integral (4) is equal to Λ^2 , we obtain the following solution:

$$\Gamma_1(p) = (O_i \times O_j) \alpha; \quad G(p) = \frac{i \beta}{p},$$

where $\alpha = g_0 [1 + g_0 \beta^2 \Lambda^2]^{-1}$ and $\beta = [1 + \frac{1}{3} g_0 \lambda^2 \Lambda^2 \alpha \beta^3]^{-1}$ are quantities which do not depend on p . Inserting α in the formula for β , we obtain an equation of the fourth degree with respect to β . Its solution can easily be found in the cases when $g_0 \Lambda^2 \ll 1$ or $g_0 \Lambda^2 \sim 1$, $g_0 \lambda^2 \ll 1$. In the first case we get $\beta = (4/g_0 \lambda^2) \times [(1 + \frac{1}{2} g_0 \lambda^2)^2 - 1]$, and in the second, $\beta \cong 1$. In both cases (which include the complete range of values of g_0 which are of any interest), the quantity $\xi = \alpha \beta^2 \Lambda^2$, in the limit for $\Lambda \rightarrow \infty$, remains of the order of unity. Now it is easy to demonstrate that the unaccounted graphs of the type depicted in Fig. 4 are much less than α . For example, the ratio of the quantities corresponding to the graphs in Figs. 4(b) and (d) to α is $\alpha \beta^2 (\lambda^4/PQ) = \xi (\lambda^4/PQ \Lambda^2)$, and $\alpha^2 \beta^4 \Lambda^2 \lambda^3 = \xi^2 (\lambda^3/\Lambda^2)$. These quantities are arbitrarily small when $\lambda^2/\Lambda^2 \rightarrow 0$. It follows that the solutions obtained for Γ and G are exact ones,

and since $\alpha \beta^2 \Lambda^2 \sim 1$, the physical interaction $\alpha \beta^2$ vanishes as $1/\Lambda^2$ for $\Lambda^2 \rightarrow \infty$.

A point which was important for the foregoing analysis was that the integral J_0 contains Λ^2 . The magnitude of the quadratically divergent integral, however, may significantly depend on the form factor employed in computing this quantity. Suppose, for example, that the quadratically divergent integral is made to vanish by using an oscillating form factor. Then instead of (5) we get

$$\alpha(p) = \frac{g_0}{1 - \frac{1}{2} g_0 p^2 \ln(\Lambda^2/p^2)}. \quad (7)$$

If g_0 behaves as $1/m^2 L$ for $\Lambda \rightarrow \infty$, the difference between β and 1 will be of the order of $\lambda^2/m^2 L$. Indeed, in formula (6) the integral over l now does not contain a quadratically divergent part and hence is equal to $\sim f^2 \ln(\Lambda^2/f^2)$. The remaining integral over f yields $g_0 \lambda^2 p \sim (\lambda^2/m^2 L) p$, if one takes into account that $\alpha \sim 1/f^2 L$ and expands in a series in p .

If g_0 is not small, the equations for α and β should be solved simultaneously. The equation for β can be satisfied in this case by assuming that β is independent of p . Then, instead of (7), we get

$$\alpha(p) = \frac{g_0}{1 - \frac{1}{2} g_0 \beta^2 p^2 \ln(\Lambda^2/p^2)}. \quad (7a)$$

The integral in the equation for β has the following form:

$$g_0 \beta^3 \int \frac{\alpha(f)}{p-f} d^4 f \int \frac{d^4 l}{l(l-f)} \cong g_0 \beta^3 p \int \frac{\alpha(f)}{f^2} d^4 f \cdot f^2 \ln \frac{\Lambda^2}{f^2}.$$

For a sufficiently large L , when $g_0 \beta^2 \lambda^2 L \gg 1$, this integral equals $g_0 \beta \lambda^2 p$ and the following equation is obtained for β :

$$\beta \left[1 + \frac{1}{16} g_0 \beta \lambda^2 \right] = 1.$$

Hence

$$\beta = \frac{8}{g_0 \lambda^2} \left[\left(1 + \frac{1}{4} g_0^2 \lambda^4 \right)^{1/2} - 1 \right];$$

that is, β is independent of L . It therefore follows from (7a) that for a sufficiently large value of L ,

$$\alpha \beta^2 \sim \frac{1}{p^2 L}$$

(if $p^2 \rightarrow 0$, formula (7a) should be refined by taking into account the finiteness of the mass; then $\alpha \beta^2 \sim 1/m^2 L$).

It can readily be verified that in this case the diagrams of Fig. 4 and the more complicated ones are small compared with α in (7a).

Thus the conclusion drawn above regarding the vanishing of the interaction of point fermions does not depend on whether the quadratically divergent

integral is considered equal to Λ^2 or to zero. This confirms our viewpoint that the *physical* results of the theory (in the given case, the vanishing of the interaction) should not depend on the form of the cut-off factor.

4. VECTOR AND TENSOR INTERACTION

Consider now the vector interaction theories ($O_j = i\gamma_\mu$, or $O_j = i\gamma_5\gamma_\mu$). We start with the pseudo-vector theory. According to (4), insertion of Γ_1 in the form $\alpha(p) \times (i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\mu)$ into (3) leads to the appearance of a new spinor form, $(1/p^2)(i\gamma_\mu p \times i\gamma_5 p)$, in the right-hand part of the equation. Thus the solution of (3) should be sought in the form

$$\Gamma_1 = \alpha(p)(i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\mu) + \alpha_1(p)(i\gamma_5 p \times i\gamma_5 p) \frac{1}{p^2}. \quad (8)$$

Inserting (8) into equation (3) and separately equating the coefficients in each spinor form, we get

$$\begin{aligned} \alpha(p) &= g_0 - g_0(J_0 + J_1), \\ \alpha_1(p) &= 2g_0 J_1 \alpha(p) - g_0(J_0 - J_1) \alpha_1(p), \end{aligned} \quad (9)$$

or

$$\left. \begin{aligned} \alpha(p) &= \frac{g_0}{1 + g_0(J_0 + J_1)}, \\ \alpha(p) + \alpha_1(p) &= \frac{g_0}{1 + g_0(J_0 - J_1)}. \end{aligned} \right\} \quad (10)$$

Taking into account the values of the integrals J_0 and J_1 , we get

$$\left. \begin{aligned} \alpha(p) &= \frac{g_0}{1 + g_0[\frac{1}{2}\Lambda^2 - \frac{1}{3}p^2 \ln(\Lambda^2/p^2)]}, \\ \alpha(p) + \alpha_1(p) &= \frac{g_0}{1 + \frac{1}{2}g_0\Lambda^2}; \end{aligned} \right\} \quad (11)$$

that is $\alpha(p)$ has the same form as in (5) and decreases with increasing Λ^2 as $1/L$ for $g_0 \sim 1/\Lambda^2$, whereas $\alpha_1(p)$ does not contain $\ln(\Lambda^2/p^2)$ in the denominator and does not tend to zero for $\Lambda^2 \rightarrow \infty$.

The term containing $\alpha_1(p)$, however, does not correspond to any real interaction, as it can be excluded with help of a transformation similar to a gauge transformation. Let us examine, for example, a system consisting of fermions and of scalar bosons not interacting with the fermions. The Lagrangian of the system has the form

$$\mathcal{L} = \mathcal{L}_b^{(0)} + \mathcal{L}_f^{(0)} + \mathcal{L}_f^{(1)},$$

where $\int \mathcal{L}_f^{(1)} d^4x = -V$, $\mathcal{L}_b^{(0)}$ is the Lagrangian of the boson free field, and $\mathcal{L}_f^{(0)} = -\bar{\psi}(\gamma_\mu \partial/\partial x_\mu + m)\psi$ is the Lagrangian for free fermions.

The fermion system can be characterised by a wave field ψ_1 related to ψ by the relation

$$\psi_1 = \exp[i\gamma_5 \bar{\varepsilon} \varphi(x)] \psi; \quad \bar{\psi}_1 = \bar{\psi} \exp[i\gamma_5 \bar{\varepsilon} \varphi(x)], \quad (12)$$

where $\varphi(x)$ is a quantised boson field and $\tilde{\varepsilon}$ is an operator defined in the momentum representation by the function $\varepsilon(p^2)$ (i.e. $\tilde{\varepsilon} = \varepsilon(-\partial^2/\partial x_\mu^2)$). Under this transformation of the field $\psi(x)$, the Lagrangian $\mathcal{L}_f^{(1)}$ (i.e. interaction (2)) remains constant, whereas $\mathcal{L}_f^{(0)}$ changes:

$$\mathcal{L}_f^{(0)} = \mathcal{L}_f^{(0)} + \mathcal{L}_1 + \mathcal{L}_2,$$

where

$$\mathcal{L}_1 = -\bar{\psi}_1 i \gamma_\mu \gamma_5 \tilde{\varepsilon} \frac{\partial \varphi}{\partial x_\mu} \psi_1,$$

and

$$\mathcal{L}_2 = m \bar{\psi}_1 \{ \exp[-2i \gamma_5 \tilde{\varepsilon} \varphi(x)] - 1 \} \psi_1.$$

The Lagrangian \mathcal{L}_1 corresponds to the addition to (2) of a fictitious interaction

$$V' = \int \bar{\psi}_1 i \gamma_\mu \gamma_5 \tilde{\varepsilon} \frac{\partial \varphi}{\partial x_\mu} \psi_1 dv.$$

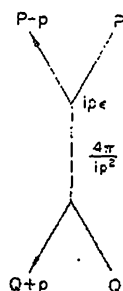


Fig. 5. Simplest graph corresponding to the second-order interaction arising from the term in \mathcal{L}' .

In second order, it leads to scattering of fermions by fermions, $(f_0/p_2) \times (i \gamma_5 \mathbf{p} \times i \gamma_5 \mathbf{p})$ corresponding to the simplest diagram in Fig. 5. Here $f_0 = \varepsilon^2(p^2)/\pi \geq 0$. The presence of this interaction (that is, of all the diagrams created from the simplest one in Fig. 5) can be taken into account in calculating Γ_1 if instead of (3a) we insert in (3) the expression

$$\alpha'_0 = g_0 (i \gamma_5 \gamma_\mu \times i \gamma_5 \gamma_\mu) + \left(\frac{f_0}{p_2} \right) (i \gamma_5 \mathbf{p} \times i \gamma_5 \mathbf{p}).$$

After simple transformations, the following results can be derived from (3):

$$\alpha(p) = \frac{g_0}{1 + g_0(J_0 + J_1)};$$

$$\alpha(p) + \alpha_1(p) = \frac{g_0 + f_0}{1 + (g_0 + f_0)(J_0 - J_1)}.$$

(11a)

Inserting the upper equation in the lower, we note that $\alpha_1(p)$ vanishes if

$$f_0 = -\frac{2g_0 J_1}{(1 + 2g_0 J_1)}. \quad (13)$$

The expression for $\alpha(p)$ is identical with (11); it yields the result that the interaction is maximal if $g_0 = -2/\Lambda^2$. From (13) we then obtain

$$f_0 = \left(\frac{p^2}{3\Lambda^4}\right)L.$$

Thus, in order to exclude the term $\alpha_1(p)$ in the expression for F_1 , it is sufficient to choose $\varepsilon(p^2)$ equal to $[(p^2/3\Lambda^4)L]^\dagger$; it is arbitrarily small for $\Lambda \rightarrow \infty$. It was thus sufficient to consider the fictitious interaction V' in the lower approximation, the Lagrangian \mathcal{L}_2 disappearing for $\Lambda \rightarrow \infty$.

Formula (11a) for $\alpha(p)$ is in all respects similar to the expression (5) obtained in scalar theories and therefore, as in the latter cases, it leads to the disappearance of physical interaction for $\Lambda \rightarrow \infty$.

We shall now consider the vector interaction theory. In this case the quadratically divergent integral can be considered to equal zero. Indeed, in vector theory the integral (4) can be written in the following form:

$$\int T \langle j_\nu(x) j_\mu(y) \rangle_0 e^{ip(x-y)} d^4x d^4y, \quad (14)$$

where $j_\nu(x) = \bar{\psi}(x) \gamma_\nu \psi(x)$ and $\dagger \partial j_\nu(x) / \partial x_\nu = 0$. If we put $p = 0$ in (14), we formally obtain a quadratically divergent integral. Its structure is the same as that of the integral which in electrodynamics determines the photon mass (since in electrodynamics j_ν satisfies the continuity equation), and it therefore must vanish.[‡]

In vector theory the term $\alpha_1(p)$ ($i p \times i p / p^2$) appears in F_1 , and formula (10) is valid as before; however, instead of (11), we have

$$\alpha(p) = g_0 \left[1 - \frac{1}{3} g_0 p^2 \ln \left(\frac{\Lambda^2}{p^2} \right) \right]^{-1}; \quad \alpha(p) + \alpha_1(p) = g_0. \quad (11b)$$

We now take notice of the fact that for real particles the interaction $\alpha_1(p)$ ($i p \times i p / p^2$) is identically equal to zero. Indeed, p is the difference

[‡] This equation is valid not only for free operators but for coupled operators as well.

[‡] A formal proof, not based on the analogy with electrodynamics, can be given, for example in the following way. Consider the integral

$$\begin{aligned} \int T \langle j_\nu(x) j_\mu(y) \rangle \frac{\partial \varphi(x)}{\partial x_\nu} d^4x d^4y &= \int \frac{\partial}{\partial x_\nu} T \langle j_\nu(x) j_\mu(y) \rangle \varphi(x) d^4x d^4y \\ &\quad - \int T \left\langle \frac{\partial j_\nu(x)}{\partial x_\nu} j_\mu(y) \right\rangle \varphi(x) d^4x d^4y. \end{aligned}$$

The first integral on the right-hand side is zero if $j_\nu(x)$ vanishes at infinity; the second integral is zero because $\partial j_\nu(x) / \partial x_\nu = 0$. If $\partial \varphi(x) / \partial x_\nu$ is a sufficiently slowly varying function of the co-ordinates, it can be written before the integral; and since the vector $\partial \varphi / \partial x_\nu$ is arbitrary, we obtain

$$\int T \langle j_\nu(x) j_\mu(y) \rangle d^4x d^4y = 0.$$

between the final and initial particle momenta. Since the wave functions of the initial and final states obey the free Dirac equation, the matrix element of $i \not{p}$ will vanish.† As $\alpha(p)$ in (11b) is practically identical with (7), the reasoning and also the results obtained in section 3 for scalar theory (in which the quadratic integral vanishes) are also valid in the given case.

It remains for us to consider tensor theory. Replacing O_j in (4) by $\sigma_{\mu\nu} = \frac{1}{2}i(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$, we see that

$$\text{Tr}(\sigma_{\alpha\beta} \gamma_\nu \sigma_{\lambda\epsilon} \gamma_\nu) = 0.$$

Thus no quadratic divergence is involved in tensor theory. The new spinor forms for Γ_1 arise, but in tensor theory $\alpha_1(p)$ contain $\ln \Lambda^2/p^2$ in the denominator. The considerations do not differ from those applied in the case of scalar theory (in which the quadratic integral is assumed to be zero).

Consequently, if only a single fermion field is considered, physical interaction will be absent in all types of four-fermion interaction.

5. INTERACTION OF SEVERAL FIELDS

The interaction between several fields will now be considered. If two types of neutral particles exist (the respective fields being designed by ψ and χ), we arrive at three types of interaction:

$$V = 2\pi^2 \int [g_1(\bar{\psi} O_j \psi)(\bar{\psi} O_j \psi) + g_2(\bar{\chi} O_j \chi)(\bar{\chi} O_j \chi) + 2g_3(\bar{\psi} O_j \psi)(\bar{\chi} O_j \chi)] dv, \quad (15)$$

the respective constants being g_1, g_2, g_3 . One may inquire whether these constants can be chosen in such a way so as to cancel out the logarithm in the denominator of the expression for $\alpha(p)$, which is of the same type as (5).

For convenience, the field of the two particles will be characterised by the column matrix

$$\varphi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}.$$

Then in interaction (15),

$$a_0 = \sum_{i=1}^3 g_i V'_i,$$

where

$$V'_1 = Q'_1 \times Q'_1; \quad V'_2 = Q'_2 \times Q'_2; \quad V'_3 = Q'_1 \times Q'_2 + Q'_2 \times Q'_1;$$

$$Q'_1 = O_j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad Q'_2 = O_j \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

will correspond to the simplest diagram in Fig. 1 and equation (3) will have a solution of the type

$$\Gamma_1 = \sum_{i=1}^3 \alpha_i(p) V'_i.$$

† See the similar arguments in a paper by R. Feynman⁷. At small p^2 the denominator of the expression $(i \not{p} \times i \not{p})/p^2$ will not vanish if finiteness of the mass is taken into account. Vanishing of the interaction α_1 can also be proved by employing a transformation similar to (12).

We shall consider scalar theory ($O_j = 1$, $O_j = i \gamma_5$), similar results obtaining in the other theories.

Inserting the foregoing expressions for α_0 and Γ_1 in (3), we obtain, after equating the coefficients before V'_i , the following set of equations for α_i :

$$\left. \begin{aligned} (1 + g_1 I) \alpha_1 + g_3 \alpha_3 I &= g_1, \\ (1 + g_2 I) \alpha_2 + g_3 \alpha_3 I &= g_2, \\ \frac{1}{2} g_3 \alpha_1 I + \frac{1}{2} g_3 \alpha_2 I + \left[1 + \frac{1}{2} (g_1 + g_2) I \right] \alpha_3 &= g_3, \end{aligned} \right\} \quad (16)$$

where $I = 2J_0 + J_1 \cong \Lambda^2 - p^2 L$. The solutions of these equations are

$$\alpha_1 = \frac{g_1 + f^2 I}{\Delta}, \quad \alpha_2 = \frac{g_2 + f^2 I}{\Delta}, \quad \alpha_3 = \frac{g_3}{\Delta}, \quad (17)$$

where $\Delta = 1 + 2gI + f^2 I^2$, $g = \frac{1}{2}(g_1 + g_2)$, and $f^2 = g_1 g_2 - g_3^2$. Taking into account only terms linear in $p^2 L$ (it can easily be seen that these are the leading terms if the $g_i \Lambda^2$ are of the order of unity), we obtain

$$\Lambda^2 \Delta = (1 + F_0) \Lambda^2 - F_1 p^2 L,$$

where $F_0 = 2\bar{g} + \bar{f}^2$ and $F_1 = 2(\bar{g} + \bar{f}^2)$. The dimensionless constants $\bar{g} = \Lambda^2 g$ and $\bar{f} = \Lambda^2 f$ have been introduced; in the case of interest these quantities are of the order of unity. In the expression $\Lambda^2 \Delta$ (that is, in the denominator of the $\alpha_i(p)$) there will be no term proportional to Λ^2 , and the quantity L will cancel out if $1 + F_0 = 0$ and $F_1 = 0$. Inserting the values of F_0 and F_1 , we get $\bar{g} = -1$ and $\bar{f}^2 = 1$; that is, $\bar{g}_1 + \bar{g}_2 = -2$ and $\bar{g}_1 \bar{g}_2 - \bar{g}_3^2 = 1$, where $\bar{g}_i = \Lambda^2 g_i$. Hence $g_1 = -2 - \bar{g}_2$ and $-\bar{g}_2^2 = (1 + \bar{g}_2)^2$.

Thus quadratic and logarithmic infinities can be removed from the denominator of the expressions for α_i only when the constant g_3 is imaginary (if the numerators of the expressions for α_i are multiplied by Λ^2 they will be of the order of unity, providing $g_i \sim 1$). Therefore, for Hermitian Lagrangians, physical interaction will vanish in this case also if $\Lambda^2 \rightarrow \infty$.

If the quadratic integrals vanishes (vector and tensor theories), the same type of reasoning as that used in section 3 for a single field should be applied. No physical interaction will arise in this case either.

We now turn to consideration of charged fields. Care should be exercised in using form factors to analyse the charged field. Thus if the form factor is introduced simply as the integration limit in momentum space, a contradiction with Ward's identity will arise when $\lambda^2 \ll \Lambda^2$ (in particular, because of the interaction, a neutral particle acquires a charge). In order to avoid this inconsistency, a form factor should be introduced which has, for example, the form

$$\left(\frac{\Lambda^2}{(p - \Lambda)^2 - \Lambda^2} \right)^2,$$

where A is the electromagnetic potential.† Gauge invariance will apply to the relation between λ^2 and A^2 , and hence the methods employed above may be used.

Consider now the interaction between two fields with isotopic spin $\frac{1}{2}$ (field ψ) and isotopic spin 1 (field φ). Instead of (15) we obtain

$$\begin{aligned} V = 2\pi^2 \int & [g_1(\bar{\psi}\psi)(\bar{\varphi}\varphi) + g_2(\bar{\Sigma}\Sigma)(\bar{\Sigma}\Sigma) + 2g_3(\bar{\psi}\psi)(\bar{\Sigma}\Sigma) \\ & + g_4(\bar{\psi}\tau_\alpha\psi)(\bar{\varphi}\tau_\alpha\varphi) + g_5(\bar{\Sigma}T_\alpha\Sigma)(\bar{\Sigma}T_\alpha\Sigma) \\ & + 2g_6(\bar{\psi}\tau_\alpha\psi)(\bar{\Sigma}T_\alpha\Sigma) + g_7(\bar{\Sigma}S_{\alpha\beta}\Sigma)(\bar{\Sigma}S_{\alpha\beta}\Sigma)] d\nu, \end{aligned} \quad (18)$$

where the τ_α are operators of isotopic spin $\frac{1}{2}$ and T_α of spin 1, and $S_{\alpha\beta} = T_\alpha T_\beta + T_\beta T_\alpha - (4/3)\delta_{\alpha\beta}$; for the sake of brevity, the operators of ordinary spin are not explicitly included. To compute the integral (4), the trace over the isotopic spin variables should also be taken. It can readily be verified in this case that interaction (18) splits into three independent sets: (1) g_1, g_2, g_3 ; (2) g_4, g_5, g_6 ; and (3) g_7 . The first two can be reduced to the case of interaction of two neutral fields, discussed above, by changing the notation. Thus, in the first case, the following substitution should be made: $2g_1 = g'_1$; $3g_2 = g'_2$; $6^{\frac{1}{2}}g_3 = g'_3$; $2\alpha_1 = \alpha'_1$; $3\alpha_2 = \alpha'_2$; and $6^{\frac{1}{2}}\alpha_3 = \alpha'_3$. In the second case, we obtain the previous equation (16) by performing the substitution $2g_i = g'_i$. In the third case, only one field is involved.

More complicated cases of interaction may be considered, such as the interaction between three different fields. Then, besides interactions of type (18), an interaction of the β -decay type $(\bar{\psi}\psi)(\bar{\varphi}\chi)$ can be set up. These cases may be handled by the same methods, but the problem becomes more unwieldy. There are no apparent physical reasons to expect other results to be obtained in these more complex cases, although a study of them would seem to be of interest.

Thus the expectation expressed in the introduction, that choice of the sign of the interaction constant might lead to a result differing from that in other cases of local interaction, is not confirmed.

In all of the simplest cases of local interaction considered up to the present, physical interaction between the fields is absent.

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† The authors are thankful to B. L. Ioffe for pointing out that a form factor of the indicated type removes the contradiction with Ward's identity.

97. NUMERICAL METHODS OF INTEGRATING DIFFERENTIAL EQUATIONS BY THE MESH METHOD

This paper describes some methods of numerical integration for hyperbolic and parabolic equations, which we developed in 1951-52.

The authors hope to publish shortly a longer article on the subject, and so only a brief outline description of the methods is given here. The steps in the space coordinate x and in time t are denoted by Δx and Δt . The value of the function $u(x, t)$ at the points of the mesh $(m\Delta x, n\Delta t)$, where m and n are integers, is denoted by u_m^n .

1. GENERAL CONSIDERATIONS

In replacing a differential equation by a difference equation, the problem immediately arises of the degree to which the solution of the difference equation approximates that of the differential equation. There are two fundamentally different sources of inaccuracy: (1) the inaccuracy of the solution of the difference equation, (2) the inaccuracies resulting from the replacement of the differential operator by the difference operator.

The significance of inaccuracies of the first kind is usually much exaggerated. With a stable difference system (the meaning of this expression will be explained below) errors of this kind can be followed, and by increasing the number of significant figures in the calculation these errors can be reduced.

The importance of inaccuracies of the second class has been demonstrated by one of us¹. It was also shown there how the concept of a stable difference system may be rigorously defined. The strict definition of stability does not allow the use of convenient tests and so is not a practical one.

We have used the same definition of stability as J. von Neumann. Let u_m^n be the function sought, determined from the difference equation and initial conditions. For constant coefficients, an initial perturbation of the form $\delta u = \delta^0 e^{ir_0^m}$ corresponds to a perturbation of the solution $\delta u_m^n = \delta^n e^{ir_0^m}$. The system is regarded as stable if the ratio of the perturbation δ^n at the n th layer to δ^0 , with $n = t/\Delta t$, remains bounded as $\Delta t \rightarrow 0$ for all r . This definition is modified in an obvious way when there are several unknown functions.

We shall not go into detail, but merely note that the test for stability thus defined consists of easily understood algebraic inequalities.

There is a natural distinction between two types of difference system, local and non-local. Local systems are such that the difference equation contains

Л. Д. Ландау, Н. Н. Мейман и И. М. Халатяников, Численные методы интегрирования уравнения в частных производных методом сеток, *Труды III Все-Союзного Математического Съезда (Москва 1956)*, Издательство Академии Наук СССР (1958), 92.

only one term of highest order in the time. In this case u_m^{n+1} is calculated directly from the values at adjoining points of the lattice in lower layers. It can be shown that for the model equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (1.1)$$

all stable local systems must satisfy the condition

$$\frac{\Delta t}{\Delta x} < \min \frac{1}{|a|},$$

which makes the computation very laborious and less accurate, since the region of influence of the initial conditions is artificially enlarged.

This stringent condition does not apply to non-local systems, where the difference operator contains two or more terms of highest order in the time. Here, however, stability in space is necessary as well as stability in time. Let us consider, for example, the following system for equation (1.1):

$$\begin{aligned} \frac{\partial u}{\partial t} &\approx \frac{1}{2\Delta t} [(u_{m+1}^{n+1} - u_{m+1}^n) + (u_m^{n+1} - u_m^n)], \\ \frac{\partial u}{\partial x} &\approx \frac{1}{2\Delta x} [(u_{m+1}^n - u_m^n) + (u_{m+1}^{n+1} - u_m^{n+1})]. \end{aligned} \quad (1.2)$$

The difference equation is

$$u_{m+1}^{n+1} + u_m^{n+1} - u_{m+1}^n - u_m^n + a \frac{\Delta t}{\Delta x} (u_{m+1}^{n+1} - u_m^{n+1} + u_{m+1}^n - u_m^n) = 0 \quad (1.3)$$

To examine stability in time we regard the coefficient a as constant and put $u_m^n = \zeta^n e^{i\tau m}$. Then

$$\zeta(r) = \frac{\cos \frac{1}{2}r - i a (\Delta t / \Delta x) \sin \frac{1}{2}r}{\cos \frac{1}{2}r + i a (\Delta t / \Delta x) \sin \frac{1}{2}r}. \quad (1.4)$$

For any value of $a \Delta t / \Delta x$, $|\zeta(r)| = 1$ and the system is stable in time. Regarding the terms of the n th layer in (1.3) as known, we can write the equation as

$$\left(1 + a \frac{\Delta t}{\Delta x}\right) u_{m+1}^{n+1} + \left(1 - a \frac{\Delta t}{\Delta x}\right) u_m^{n+1} = D_m^n, \quad (1.5)$$

where D_m^n is a known quantity. The solution consists of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation

$$\left(1 + a \frac{\Delta t}{\Delta x}\right) u_{m+1} + \left(1 - a \frac{\Delta t}{\Delta x}\right) u_m = 0. \quad (1.6)$$

This solution is $u_m = q^m$, where

$$q = - \frac{1 - a \Delta t / \Delta x}{1 + a \Delta t / \Delta x}. \quad (1.7)$$

In computing from left to right the stability condition in space is $|q| < 1$, i.e. $a > 0$; in computing from right to left the stability conditions is $|q| > 1$, i.e. $a < 0$.

If the computation is from left to right, the value of u_m^{n+1} at the extreme left-hand point must be determined from some condition, and similarly in computing from right to left.

The fundamental advantage of non-local systems is that they can be set up in such a way that the ratio of steps $\Delta t/\Delta x$ is not limited by conditions of stability. The steps in t and x are restricted only by the degree of homogeneity of the coefficients in the equation. Such systems are more accurate than local systems.

2. INTEGRATION OF THE EQUATIONS OF FLUID DYNAMICS

Let ρ be the density, p the pressure, v the velocity and S the entropy. The equations are

$$\frac{\partial S}{\partial t} + v \frac{\partial S}{\partial x} = g_1, \quad (2.1)$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = g_2, \quad (2.2)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_3, \quad (2.3)$$

where g_1, g_2, g_3 are known functions of all these quantities.

The quantities S, p and ρ must also satisfy the equation of state $S = F(\rho, p)$, with F a known function.

The problem is to be solved on the semi-axis $0 \leq x < \infty$ for $0 \leq t$. In the range $0 \leq x < \infty$ the initial values of the functions p, v and S are given, and on the boundary $x = 0$ the relations

$$\Phi_1(p, v, S) = 0 \quad \text{and} \quad \Phi_2(p, v, S) = 0$$

are given. Equations (2.2) and (2.3) can be replaced by the equivalent equations

$$\begin{aligned} \left(\frac{\partial p}{\partial t} + \rho c \frac{\partial v}{\partial t} \right) + (v + c) \left(\frac{\partial p}{\partial x} + \rho c \frac{\partial v}{\partial x} \right) \\ = c^2 \left(g_2 - g_1 \frac{\partial \rho}{\partial S} \right) + \rho c g_3 \equiv \varphi_1, \end{aligned} \quad (2.2a)$$

$$\begin{aligned} \left(\frac{\partial p}{\partial t} - \rho c \frac{\partial v}{\partial t} \right) + (v - c) \left(\frac{\partial p}{\partial x} - \rho c \frac{\partial v}{\partial x} \right) \\ = c^2 \left(g_2 - g_1 \frac{\partial \rho}{\partial S} \right) - \rho c g_3 \equiv \varphi_2, \end{aligned} \quad (2.3a)$$

where $c = \sqrt{\partial p / \partial \rho}$ is the local velocity of sound.

Equations (2.1), (2.2a) and (2.3a) have the property that each involves only one characteristic combination dS , $dp + \rho c dv$, $dp - \rho c dv$, which is transported with velocity v , $v + c$, $v - c$ respectively. The velocities v and $v + c$ remain positive, while $v - c$ changes from negative to positive values at the transition from subsonic to supersonic flow.

We may describe the principle of the calculation, without writing out the formulae. Let the equations be integrated up to a time $n \Delta t$, and let it be required to find the values of the functions at time $(n + 1) \Delta t$. We use the equations given above, with large steps in t , and determine the sonic point in the n th row, i.e. the point at which $v = c$. At this point the characteristic of equation (2.3a) is parallel to the t -axis. Let us integrate equation (2.3a) along a segment of this characteristic. Since $v - c = 0$, the integration amounts to calculating $\Delta p - \rho c \Delta v = \varphi_2 \Delta t$, where Δ signifies the increment in time Δt . Equation (2.3a) is regarded as an equation for $\Delta p_m - \rho c \Delta v_m$, and at the sonic point this quantity is already known. To the left of this point $v - c < 0$, to the right of it $v - c > 0$, and so, as shown above, the computation of the non-holonomic differential $\Delta p - \rho c \Delta v$ is stable in space both when computing to the left of the sonic point and when computing to the right of it. On moving to the left, we go to the point $x = 0$ and compute $\Delta p - \rho c \Delta v$ at the point $(0, n \Delta t)$. Together with the boundary conditions this enables us to find the values of $\Delta p + \rho c \Delta v$ and ΔS at the boundary point $(0, n \Delta t)$. Since $v + c$ and v are positive, the computation to the right is stable for equations (2.2a) and (2.1), and on moving to the right we find the values of $\Delta p + \rho c \Delta v$ and ΔS at all points in the n th row. Knowing $\Delta p - \rho c \Delta v$, $\Delta p + \rho c \Delta v$ and ΔS at all points in the n th row, we find the values of p , ρ , v and S at all points of the $(n + 1)$ th row. Since the values of the functions are known for $n = 0$, this completes the computation process.

3. DIFFERENCE SYSTEMS FOR THE EQUATION OF HEAT CONDUCTION

For a clearer elucidation of the fundamental aspect of the problem we shall first consider the one-dimensional equation of heat conduction with a constant coefficient of thermal conductivity:

$$\frac{\partial f}{\partial t} = \chi \frac{\partial^2 f}{\partial x^2}. \quad (3.1)$$

We may first discuss briefly computation systems in which the required value of the function at some point is expressed in terms of its values at neighbouring points (local systems). The simplest example of a stable local system for the equation of heat conduction is given by

$$\left. \begin{aligned} f_m^{n+1} - f_m^n &= \bar{\chi} [f_{m+1}^n - 2f_m^n + f_{m-1}^n], \\ \bar{\chi} &= \chi \frac{\Delta t}{(\Delta x)^2}. \end{aligned} \right\} \quad (3.2)$$

To find the limits of stability, we put

$$f_m^n = S^n e^{ikm}.$$

Then $S = 1 - 2\bar{\chi}(1 - \cos k)$ and the condition $|S| < 1$ gives the inequality $2\bar{\chi}(1 - \cos k) < 2$, whence

$$\bar{\chi} < \frac{1}{2}. \quad (3.3)$$

This condition may be called the natural condition of the problem; it is a very severe restriction on the steps.

This disadvantage does not arise in the "rhombic" system, which has the form

$$\frac{1}{2}(f_m^{n+1} - f_m^{n-1}) = \bar{\chi}[f_{m+1}^n - (f_m^{n+1} + f_m^{n-1}) + f_{m-1}^n] \quad (3.4)$$

or

$$(1 + 2\bar{\chi})f_m^{n+1} - (1 - 2\bar{\chi})f_m^{n-1} = 2\bar{\chi}(f_{m+1}^n - f_{m-1}^n).$$

The characteristic equation of stability for this finite-difference equation is

$$(1 + 2\bar{\chi})S^2 - (1 - 2\bar{\chi}) = 4\bar{\chi}S \cos k. \quad (3.5)$$

For a quadratic equation $aS^2 + bS + c = 0$ ($a > 0$) with real coefficients the condition for both roots to have moduli less than unity is $|b| < a + c$, $|c| < a$. The second condition is satisfied for equation (3.5), and the first gives $|\cos k| < 1$. This inequality becomes an equality at $k = \pi$ ($S = -1$), so that there is, in our terminology, weak stability. In actual computation this leads to a build-up of errors, to remove which we must use some method of averaging in space. In the rhombic system there is thus no condition which restricts the magnitude of the steps as regards stability. Its essential defect, however, is the considerable inaccuracy due to the use of the second time derivative. This inaccuracy is such that it practically imposes a condition identical with (3.3). Hence the rhombic system could be used for a computation with relatively large steps only by employing cumbersome iterative methods. This problem will be discussed further at the end of the section.

There are other methods, described below, which do not involve restrictions on the step due to requirements of stability or inaccuracies in the difference system (as compared with the differential equation). These methods correspond, in a sense, to the method which we have previously described of "marching" computation for the equations of fluid dynamics (hyperbolic equations). Unlike the local systems, these methods use the values of the function at previous instants and at distant points in space to calculate the required function at every point.

We start from the following difference system:

$$f_m^{n+1} - f_m^{n-1} = \bar{\chi}\{(f_{m+1}^n - 2f_m^n + f_{m-1}^n)^{n+1} + (f_{m+1}^n - 2f_m^n + f_{m-1}^n)^{n-1}\}. \quad (3.6)$$

This system is intrinsically stable for any step in time and is sufficiently accurate, since it does not involve the redundant second time derivative. Its

characteristic stability condition is

$$\left(S - \frac{1}{S}\right) = 2\bar{\chi}(\cos k - 1)\left(S + \frac{1}{S}\right),$$

whence

$$S^2 = \frac{1 - 2\bar{\chi}(1 - \cos k)}{1 + 2\bar{\chi}(1 - \cos k)},$$

so that we always have $S^2 < 1$, even for $k = \pi$. It should be borne in mind, however, that in computation with very large steps in time the stability parameter S is close to -1 . This may lead in practice to the necessity of averaging with respect to time. Such an averaging may, for example, be performed according to

$$\bar{f}^n = \frac{1}{4}(f^{n-1} + 2f^n + f^{n+1}). \quad (3.7)$$

Equation (3.6) is a second-order finite-difference equation, the solutions of which are uniquely determined by specifying the conditions at the two boundaries of the region of integration. This solution may be found by reducing equation (3.6) to a difference system of the "marching" type. We have developed two methods of doing this.

(a) *First type of marching computation.* We use the symmetrical sum

$$\Phi_m^n = f_m^{n+1} + f_m^{n-1}. \quad (3.8)$$

Then the system (3.6) becomes

$$\begin{aligned} \Phi_m - 2f_m^{n-1} &= \bar{\chi}[\Phi_{m+1} - 2\Phi_m + \Phi_{m-1}] \\ \text{or} \quad \bar{\chi}\Phi_{m+1} - (2\bar{\chi} + 1)\Phi_m + \bar{\chi}\Phi_{m-1} &= -2f_m^{n-1}, \end{aligned} \quad (3.9)$$

where the right-hand side is an expression involving the values of f only at an earlier time.

The characteristic equation is

$$\bar{\chi}q^2 - (2\bar{\chi} + 1)q + \bar{\chi} = 0.$$

We shall take q to be the root of this equation which is less than unity:

$$q = \frac{1}{2\bar{\chi}}[2\bar{\chi} + 1 - \sqrt{4\bar{\chi} + 1}]; \quad (3.10)$$

the other root is $1/q$. Then the difference equation (3.9) can be written in the operator form

$$\left(\frac{1}{q}T - E\right)\bar{\chi}(qT - E)\Phi_{m-1} = -2f_m^{n-1}, \quad (3.11)$$

where E is the unit operator and T the shift operator ($T\Phi_{m-1} = \Phi_m$). The form of this equation shows that it can be "factorised" by using the auxiliary

quantities obtained by the action on Φ_{m-1} of each of the two operator factors:

$$\begin{aligned}\varphi_m &= \bar{\chi}(E - qT)\Phi_{m-1} = \bar{\chi}(\Phi_{m-1} - q\Phi_m), \\ \psi_m &= -\bar{\chi}\left(E - \frac{1}{q}T\right)\Phi_{m-1} = -\bar{\chi}\left(\Phi_{m-1} - \frac{1}{q}\Phi_m\right).\end{aligned}\tag{3.12}$$

It then follows from equation (3.11) that these quantities satisfy the equations

$$\begin{aligned}\left(E - \frac{1}{q}T\right)\varphi_m &= \varphi_m - \frac{1}{q}\varphi_{m+1} = -2\bar{\chi}f_m^{n-1}, \\ (E - qT)\psi_m &= \psi_m - q\psi_{m+1} = 2\bar{\chi}f_m^{n-1},\end{aligned}\tag{3.13}$$

and, combining the two equations (3.12), we obtain

$$\psi_m + \varphi_m = \bar{\chi}\left(\frac{1}{q} - q\right)\Phi_m$$

or

$$f_m^{n+1} = f_m^{n-1} + \frac{\varphi_m + \psi_m}{\sqrt{4\bar{\chi} + 1}}.\tag{3.14}$$

Formulae (3.13) and (3.14) reduce the difference system to a "marching" computation of two auxiliary quantities, one of which (ψ) is computed from right to left (from large to small m) and the other (φ) from left to right, after which the required values of f^{n+1} are determined from the sums $\psi_m + \varphi_m$.

The usual conditions at the boundaries of the region of integration give a linear relation between f and $\partial f/\partial x$; such a condition is equivalent to a linear relation between the values of ψ and φ at the point concerned. The general sequence of calculations with such boundary conditions is therefore as follows. Having taken some value $\psi = \psi_a$ at the right-hand boundary of the region (point a), we "march" ψ from right to left to the point b (the left-hand boundary). With the value ψ_b obtained we use the boundary condition at the point b to determine the value of φ_b and then "march" from left to right. This brings us back to the point a with a definite value of φ_a , but in general φ_a and ψ_a will not satisfy the necessary boundary condition at the point a . Let ψ'_a be the value of ψ which meets this condition together with φ_a . Then, strictly speaking, in order to obtain the exact solution we must repeat the above "marching" starting from the value $\psi = \psi'_a$ at the point a . In practice, however, the effect of any change in ψ or φ dies away extremely rapidly in the process of "marching". It is therefore sufficient to make the corresponding correction only to the values at a few points adjoining the right hand boundary. The values of φ are unchanged.

(b) *Second type of marching computation.* Unlike the above method, this procedure does not use two independent auxiliary quantities which are "marched" in different directions; a direct solution of the finite-difference equation is obtained by "marching" quantities which themselves appear in it. This solution can be achieved by various methods, which are algebraically identical.

We write the original second-order finite-difference equation as two first-order equations. This corresponds to writing the differential equation of heat conduction as two first-order equations by using as an auxiliary quantity the heat flux:

$$\begin{aligned} q &= -\kappa \frac{\partial T}{\partial x}, \\ C \frac{\partial T}{\partial t} &= -\frac{\partial q}{\partial x} + K. \end{aligned} \quad (3.15)$$

These equations may be written as a difference system:

$$F_{m+1/2} - F_{m-1/2} = -\frac{\Delta x}{\kappa_m^n} I_m, \quad (3.16)$$

$$I_{m+1} - I_m = -\frac{\Delta x}{\Delta t} C_{m+1/2}^n F_{m+1/2} + 2\Delta x K_{m+1/2}^n + 2\frac{\Delta x}{\Delta t} C_{m+1/2}^n T_{m+1/2}^{n-1}, \quad (3.17)$$

where

$$\begin{aligned} F_{m+1/2} &= T_{m+1/2}^{n+1} + T_{m+1/2}^{n-1}, \\ I_m &= q_m^{n+1} + q_m^{n-1}. \end{aligned}$$

The values of I are calculated at integral points ($x = m \Delta x$) and those of F at half-integral points, in accordance with the "staggered" computation grid which we use. The boundary conditions are also written in difference form. We may take, for example, $I_0 = 0$ as the condition at the left-hand boundary.

Together with the inhomogeneous difference equations (3.16) we use the homogeneous equations

$$\begin{aligned} \tilde{F}_{m+1/2} - \tilde{F}_{m-1/2} &= -\frac{\Delta x}{\kappa_m^n} \tilde{I}_m, \\ \tilde{I}_{m+1} - \tilde{I}_m &= -\frac{\Delta x}{\Delta t} C_{m+1/2}^n \tilde{F}_{m+1/2}. \end{aligned} \quad (3.18)$$

If we construct solutions of the inhomogeneous and homogeneous systems which satisfy the boundary condition at the left-hand (or right-hand) end, then their sum with a suitable coefficient will give the required solution which satisfies also the boundary condition at the right-hand (or left-hand) end. These particular solutions are obtained as follows. We take the value $I_0 = 0$ at the point $m = 0$, and an arbitrary value of $F_{1/2}$ (preferably extrapolated from the preceding instant), and then "march" from left to right according to equations (3.16) and (3.18), successively determining I and F at every point. From the condition at the right-hand end we then determine the coefficient which must be used in combining the two solutions in order to obtain a solution satisfying the conditions at both boundaries. Similarly we can obtain the required solution from two particular solutions of the homogeneous and inhomogeneous equations which satisfy the boundary condition at the right-hand end and are found by "marching" from right to left.

The method just described is very simple, but has the disadvantage that in determining the required solution as a sum of two particular solutions there is in general a considerable cancelling of digits, especially when the value of $F_{1/2}$ with which the "marching" began has not been well chosen.

This disadvantage does not apply to a method in which the required solution of the inhomogeneous equation is constructed from two particular solutions of the homogeneous equation which satisfy respectively the conditions at the right-hand and left-hand boundaries. Let $\tilde{F}^{(1)}, \tilde{I}^{(1)}$ be the solution of the homogeneous equations which satisfies the condition at left-hand boundary with an arbitrary parameter $\tilde{F}_{1/2}^{(1)}$; $\tilde{F}^{(2)}, \tilde{I}^{(2)}$ the solution of the same equations which satisfies the boundary condition at the right-hand end with an arbitrary parameter $\tilde{I}_{\mu}^{(2)}$. The former is obtained by "marching" from left to right with equations (3.18), and the latter by "marching" from right to left. Let W denote the Wronskian

$$W = \tilde{F}_{m+1/2}^{(1)} \tilde{I}_m^{(2)} - \tilde{F}_{m+1/2}^{(2)} \tilde{I}_m^{(1)};$$

this quantity is independent of the suffix m . It is easy to see that the solution of the inhomogeneous system (3.16) which satisfies both boundary conditions is

$$\begin{aligned} F_{m+1/2} &= C_{m+1/2}^{(1)} \tilde{F}_{m+1/2}^{(1)} + C_{m+1/2}^{(2)} \tilde{F}_{m+1/2}^{(2)}, \\ I_m &= C_{m-1/2}^{(1)} \tilde{I}_m^{(1)} + C_{m-1/2}^{(2)} \tilde{I}_m^{(2)}, \end{aligned} \quad (3.19)$$

where

$$\left. \begin{aligned} C_{m-1/2}^{(1)} &= \frac{1}{W} \sum_{k=\mu-1}^m L_{k+1/2} \tilde{F}_{k+1/2}^{(2)}, \\ C_{m+1/2}^{(2)} &= \frac{1}{W} \sum_{k=0}^m L_{k+1/2} \tilde{F}_{k+1/2}^{(1)}, \\ C_{\mu-1/2}^{(1)} &= 0, \quad C_{\mu+1/2}^{(2)} = -\frac{1}{W} L_{\mu+1/2} \tilde{F}_{\mu+1/2}^{(2)}, \end{aligned} \right\} \quad (3.20)$$

where $L_{m+1/2}$ is given by (3.21) and μ is the number of the extreme right-hand point.

In this method of constructing the solution there is no cancellation of digits; its disadvantage is the greater complexity of the calculations.

The best method seems to be a modified "marching" suggested by I. M. Gel'fand. Here we begin directly from the second-order finite-difference equation obtained by eliminating I from (3.16);

$$\left. \begin{aligned} \kappa_m F_{m+1/2} + \kappa_{m-1} F_{m-3/2} - M_{m-1/2} F_{m-1/2} &= -L_{m-1/2}, \\ M_{m-1/2} &= \kappa_m + \kappa_{m-1} + C_{m-1/2} \frac{(\Delta x)^2}{\Delta t}, \\ L_{m-1/2} &= 2K_{m-1/2} (\Delta x)^2 + 2C_{m-1/2} f_{m-1/2}^{n-1}. \end{aligned} \right\} \quad (3.21)$$

This equation relates the values of F for three adjoining points in space. If we write the relation between two adjoining values of F as

$$F_{m+1/2} = A_m F_{m-1/2} + B_m, \quad (3.22)$$

then equation (3.21) becomes a recurrence relation between successive values of A_m and B_m : substituting (3.22) in (3.21), we obtain the relation between $F_{m-1/2}$ and $F_{m-3/2}$ with coefficients

$$\begin{aligned} A_{m-1} &= \frac{\kappa_{m-1}}{M_{m-1/2} - \kappa_m A_m}, \\ B_{m-1} &= \frac{L_{m-1/2} - \kappa_m B_m}{M_{m-1/2} - \kappa_m A_m}. \end{aligned} \quad (3.23)$$

The boundary condition at the right-hand end can be written

$$F_{\mu+1/2} = A_\mu F_{\mu-1/2} + B_\mu \quad (3.24)$$

with known A_μ and B_μ . The condition at the left-hand boundary can be written $F_{-1/2} = F_{1/2}$, whence $F_{1/2} = A_0 F_{1/2} + B_0$ or

$$F_{1/2} = \frac{B_0}{(1 - A_0)}. \quad (3.25)$$

The calculations proceed as follows. By marching from right to left with the recurrence relations (3.23), starting from the point μ , we successively determine all the coefficients A_m and B_m . From the resulting values of A_0 and B_0 we find the value of $F_{1/2}$ from (3.25) and then from the relations (3.22) we calculate the values of F at every point.

Independently of the method of marching, the solution is stable in space in the sense that (for given boundary conditions) any error is exponentially damped on both sides of the location of the error. As regards stability in time the discussion following equation (3.6) is entirely applicable, since the original finite-difference equation is the same. As already mentioned, it is in practice necessary (in integrating with large steps in time) to average according to (3.7). The definition (3.17) of \bar{F} must then be written

$$\bar{F}_{m+1/2} = \bar{F}_{m+1/2}^{n+1} + \bar{F}_{m+1/2}^{n-1}.$$

We shall now show that this averaging does not noticeably affect the accuracy. To do so, we may consider the model equation

$$\frac{\partial f}{\partial t} = e^{\lambda t}$$

with the difference system (putting $\Delta t = 1$)

$$\bar{f}^{n+1} - \bar{f}^{n-1} = 2e^{\lambda n},$$

$$\bar{f}^n = \frac{1}{4}(f^{n+1} + 2f^n + f^{n-1}),$$

which represents the main properties of our problem. The solution of these difference equations is

$$\bar{f}^n = e^{in} \frac{e^\lambda + 2}{2e^\lambda - 1 - e^{-\lambda}}.$$

The exact solution of the differential equation is $f = e^{\lambda t}/\lambda$. The ratio of the two solutions is $\lambda(2 + e^\lambda)/(2e^\lambda - 1 - e^{-\lambda})$, which for small λ is $1 + \lambda/6$. Thus even for $\lambda = 0.3$, corresponding to a increase in f by a factor of 1.35 per step, the error is only 5 per cent.

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98. ON ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

A general method of finding the singularities of quantum field theory values on the basis of graph techniques is evolved.

1. INTRODUCTION

In recent years many papers have been concerned with dispersion relations. As is known, the latter express the analytic properties of various quantities of quantum field theory. The problem of localising the singularities of these quantities is therefore highly important. As has become clear recently^{1, 2}, a direct study of graphs is the most effective method of investigating the location and nature of the singularities of vertex parts. It is often claimed that the graph technique is not sufficiently convincing, since it involves perturbation theory, unlike other, allegedly more rigorous methods. This view is actually based on misunderstanding. Since a rigorous theory which makes use of the Hamiltonian reduces interaction to zero, the sole completely rigorous dispersion relation in this theory is $0 = 0$. By posing the problem of analytic properties of quantum field values, we actually go beyond the framework of the current theory. An assumption is thereby automatically made that there exists a non-vanishing theory in which ψ -operators and Hamiltonians are not employed, yet graph techniques are retained. In evolving dispersion relations, therefore, the employment of the graph technique is, indeed, solely consistent, since the problem becomes meaningless if the graph technique is rejected.

The graph technique is by no means equivalent to perturbation theory, since all particles, stable as regards strong interactions, are considered, whether they are "simple" or "complex". In fact, first steps are thereby taken towards new graph techniques, which will be a generalisation of the previous methods and will lay the foundation of the future theory. It stands to reason that the applicability of the graph technique of this type is likewise hypothetical and a test of successive results will be a test of the hypothesis itself.

Unfortunately, expressions obtained in considering more complex graphs are very lengthy. It may be demonstrated, however, that a study may be

[¹ L. D. LANDAU, On analytical properties of vertex parts in quantum field theory, *Nuclear Physics* **13**, 181 (1959).

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carried out in general form and be simplified considerably with the help of a corresponding graphic representation.

2. GENERAL METHOD

An arbitrary graph represents a certain integral

$$\int \frac{B d^4 k d^4 l \dots}{A_1 A_2 A_3 \dots} \quad (1)$$

where

$$A_i = m_i^2 - q_i^2, \quad (2)$$

q_i is a certain four-momentum, corresponding to the given line in a graph, m_i the mass of the particle in question and B a certain polynomial of the vectors q_i . According to the well-known Feynman method, one may write

$$\frac{1}{A_1 A_2 A_3 \dots} = (n-1)! \int_0^1 \dots \int_0^1 \frac{d\alpha_1 d\alpha_2 \dots d\alpha_n \delta(\alpha_1 + \alpha_2 + \dots + \alpha_n - 1)}{(\alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_n A_n)^n}. \quad (3)$$

The expression $\alpha_1 A_1 + \alpha_2 A_2 + \dots$ in the denominator is a polynomial of second degree with respect to the integration variables k, l, \dots . The terms linear with respect to k, l, \dots may always be eliminated by transforming integration variables, after which we obtain

$$\alpha_1 A_1 + \alpha_2 A_2 + \dots = \varphi + K(k', l', \dots). \quad (4)$$

Here K is a homogeneous quadratic form of the new integration variables with coefficients depending only on the parameters α_i , and φ is a heterogeneous quadratic form of the vectors p_i , describing the free ends of the graph under consideration.

Let us confine ourselves to the case of real values of the squares and the scalar products of the vectors p_i . It is easy to verify that if the values of φ are positive the integral with respect to k, l, \dots is a real quantity (a self-conjugated spinor in the case of spinor functions), since the quadratic form K becomes positive definite (because all α_i are positive) when $k_4 \rightarrow i\kappa, l \rightarrow i\lambda$ are substituted. The vertex part is real, therefore, if $\varphi > 0$ for all values α_i ; if, on the other hand, $\varphi < 0$ for certain values α_i , the vertex part becomes complex. The nearest singularity of the vertex part is obviously located at the values p_i , for which φ vanishes for determined values of all α_i and is positive at all other α_i ; in other words, the singularities correspond to the vanishing of the minimum value of φ , treated as a function of α_i . If the singularities in the complex region are meant, any extremum of the function φ should be considered. It will be noted that since φ is a homogeneous function of the first order with respect to the variables α_i , the condition $\sum \alpha_i = 1$ may be excluded when the conditions of the existence of the vanishing extremum are being found.

Let us denote $\alpha_1 A_1 + \alpha_2 A_2 + \dots$ as f . Since K is a quadratic form of the variables k', l', \dots , it is clear that φ is the value of the function f at addi-

tional conditions

$$\frac{\partial f}{\partial k'} = \frac{\partial f}{\partial l'} = \dots = 0,$$

and, since k differs from k' by a constant vector, the latter conditions may be written

$$\frac{\partial f}{\partial k} = \frac{\partial f}{\partial l} = \dots = 0. \quad (5)$$

The positiveness of the values α_i should be taken into account in finding the conditions of the minimum. Hence it follows that either the condition $\partial\varphi/\partial\alpha_i = 0$ or $\alpha_i = 0$ should take place for every value α_i . In the latter case $\partial\varphi/\partial\alpha_i = 0$ should exist for the nearest singularity. According to the definition we have

$$\frac{\partial\varphi}{\partial\alpha_i} = \frac{\partial f}{\partial\alpha_i} + \frac{\partial f}{\partial k} \frac{\partial k}{\partial\alpha_i} + \frac{\partial f}{\partial l} \frac{\partial l}{\partial\alpha_i} + \dots$$

Since all $\partial f/\partial k$ equal zero according to (5), the condition $\partial\varphi/\partial\alpha_i = 0$ is equivalent to the condition $\partial f/\partial\alpha_i = 0$, i.e. according to the definition of f ,

$$A_i = 0. \quad (6)$$

Thus, the singularity of the vertex part may be obtained through a joint solution of the equations $A_i = 0$ (or $\alpha_i = 0$) with additional conditions

$$\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \dots = 0. \quad (7)$$

It is essential that these equations should have solutions with positive α_i .

Thus, it may be claimed with respect to every line of the Feynman graph that it either satisfies the condition $q_i^2 = m_i^2$ or passes out of consideration altogether (when $\alpha_i = 0$). In the latter case the singularity under consideration may be ascribed not to the graph in question but to a graph from which the i -line is absent, i.e. the vertices connected by it are fused. In analysing the singularities of the graph, it is sufficient therefore to consider the case when all $\alpha_i \neq 0$.

It can easily be seen that the condition (7) may be written as $\sum \alpha_i q_i = 0$, where summation is carried out not with respect to all lines of the Feynman graph, but with respect to the aggregates of lines forming closed contours, the direction of the vector q_i corresponding to the direction of the contour. The positiveness of the coefficients α denotes that if the vectors q_i are regarded as the directions of forces, the possibility of a solution of (7) denotes the possibility of picking such values of these forces as would enable them to preserve equilibrium.

3. GRAPHS FOR GREEN FUNCTIONS

It is comparatively easy to interpret the singularities as they arise with the help of the present method. Let us begin with considering the graphs for Green functions. In this case all vectors q_i obtained from the indicated equations are obviously parallel to the same vector p .

3.1. Vertex part with two external ends

Consider the graph in Fig. 1. Setting down the formula $\sum \alpha_i q_i = 0$ for the contour formed by any couple of lines and taking into account the positiveness of the quantities α_i , we may infer that all vectors q_i are equally directed with



FIG. 1.



FIG. 2.



FIG. 3.

respect to the vertices of the graphs. Noting that the vector lengths are equal to the corresponding masses, we easily obtain the obvious result $\pi^2 = (\sum m_i)^2$ for the singularities. It will be noted that in the case of Green functions the consideration of any other graphs, for example graphs of the type shown in Fig. 2, is superfluous. The number of equations for determining the values α_i is, indeed, equal to the number of independent contours in the graph, which is 2 in the example under discussion. Yet, the total number of values α_i equals 5 in this example; hence it is clear that one of them may be taken to equal zero, as a result of which this graph reduces to one of those considered above. It should be mentioned that these arguments likewise hold good for the graphs of the type shown in Fig. 3, where a sum of momenta acts as a momentum.

3.2. Simple vertex part with three external ends

Let us proceed to a vertex part with three external ends. Since these represent three vectors lying in the same plane, according to the conservation laws, and the vectors k, l, \dots may be obtained from (6) and (7), it is clear that the latter vectors lie in the same plane. Thus, the problem is reduced to a flat system of vectors.

First of all, let us consider the simplest example, analysed in detail by R. Karplus, C. M. Sommerfield and E. H. Wichman³, as shown in Fig. 4. It can easily be seen that the relation between vectors is represented by the scheme of Fig. 5. The condition (7) obviously requires the point O to lie inside the triangle. It is implied that all vectors possess the properties of euclidean (not pseudo-euclidean) vectors, which can easily be proved for the nearest singularities.

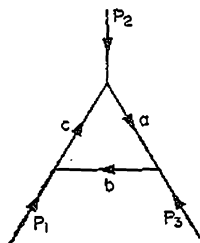


FIG. 4.

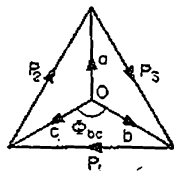


FIG. 5.

Preliminarily, let us show how to express analytically the corresponding relations. First consider the scheme of Fig. 5 which corresponds to the graph of Fig. 4. If unit vectors $n_a = q_a/m_a$, $n_b = q_b/m_b$, $n_c = q_c/m_c$ are introduced, the condition (7) may be written as

$$\beta_a n_a + \beta_b n_b + \beta_c n_c = 0, \quad (8)$$

where $\beta_i = \alpha_i m_i$ are obviously positive quantities also. Projecting these equations on the vectors n_a , n_b , n_c consecutively and introducing the notations $(n_a n_b) = \mu_c = \cos \varphi_{ab}$ etc., we obtain three equations:

$$\begin{aligned} \beta_a + \beta_b \mu_c + \beta_c \mu_b &= 0, \\ \beta_a \mu_c + \beta_b + \beta_c \mu_a &= 0, \\ \beta_a \mu_b + \beta_b \mu_a + \beta_c &= 0. \end{aligned} \quad (9)$$



FIG. 6.

Making the determinant of this system equal to zero, we obtain the equation

$$1 + 2\mu_a \mu_b \mu_c = \mu_a^2 + \mu_b^2 + \mu_c^2, \quad (10)$$

which determines the situation of a singularity (if all β are simultaneously positive); μ_a , μ_b , μ_c are connected with p_1^2 , p_2^2 , p_3^2 by the formulae

$$\mu_a = \frac{m_b^2 + m_c^2 - p_1^2}{2m_b m_c}, \quad \mu_b = \frac{m_a^2 + m_c^2 - p_2^2}{2m_a m_c}, \quad \mu_c = \frac{m_a^2 + m_b^2 - p_3^2}{2m_a m_b}. \quad (11)$$

To return to the problem of the properties of vectors for the nearest singularities. Consider any angle, say φ_{bc} . According to (11) μ_a is a real quantity. It is greater than -1 , since otherwise $p^2 > (m_b + m_c)^2$, i.e. we would transcend the singularity determined by the graph in Fig. 6, which is obtained from the graph under consideration through the elimination of one of the lines. At the same time we can see from (9) that no less than two out of three cosines of the angles should be negative. Now, a real angle corresponds to a negative cosine greater than -1 . At least two of the three angles, therefore, are real. Since the sum of them equals 2π , the third angle is real as well. Thus, Fig. 5, which was mentioned in ref. 2, solves the problem completely.

3.3. More complex graphs with three external ends

Let us turn to more complex graphs. As an example, consider the graph shown in Fig. 7. The scheme in Fig. 8 corresponds to this graph. The construction of this scheme is simple enough since every point in the Feynman graph

corresponds to a polygon in the scheme with a number of sides equal to that of the rays converging at this point, while every polygon in the Feynman graph corresponds to a point of the scheme. The condition (7) calls for a certain disposition of the vectors a , b and d on the one hand, and b , c and e , on the other. It is not difficult to write an analytic expression of this figure. The triangle bde , as all other parts of the figure, is real, since the graph comprises only stable particles, and the mass of every particle in b , d and e is, therefore, less than the sum of the other two.

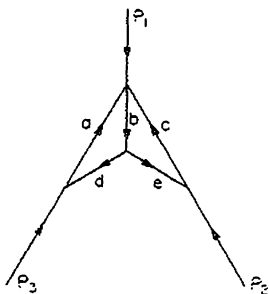


FIG. 7.

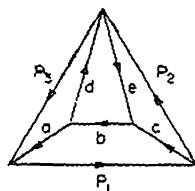


FIG. 8.

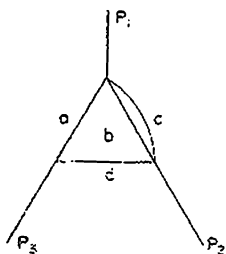


FIG. 9.

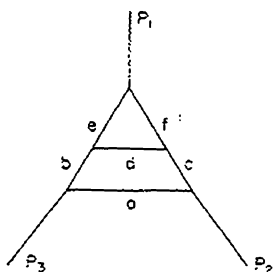


FIG. 10.

If one or several coefficients $\alpha_i = 0$ the graph is simplified. If, for example, $\alpha_c = 0$ the graph of Fig. 7 will be reduced to the graph of Fig. 9, similar to Fig. 4.

As we proceed to more complex Feynman graphs, it becomes apparent that often the case when all $\alpha_i = 0$ is impossible. Let us consider, for example, the graph in Fig. 10. The number of vector equations for α_i equals two in this case, i.e. we have four equations because the vectors lie in the same plane, while the number of the quantities α_i is six and therefore one of them may be assumed to equal zero. The same applies to the graph of the type shown in Fig. 11.

It can easily be seen how the above results change when what is meant is not the nearest singularities, that is, the emergence of complex values, but singularities of quantities that are already complex. If the singularity is not the nearest, we cannot, generally speaking, regard all angles as real, as was the case for the nearest singularities considered above.

First of all, consider the scheme of Fig. 5 which corresponds to the graph of Fig. 4. In the "Euclidean" case with all the angles real, all μ_i lie between -1 and $+1$. In the general case a study of equation (10) shows that the corresponding surface in the space μ_a, μ_b, μ_c consists of four parts intersecting each other at three points: $\mu_a = 1, \mu_b = -1, \mu_c = -1$, and the points obtained through the permutations of the indices a, b and c . One of these surfaces is limited by the triangle whose apices are the three points mentioned above. This triangle corresponds to the "Euclidean" case. The three other surfaces extend to infinity' each starting from the corresponding point of intersection. On each of these surfaces one of the μ 's is positive and exceeds 1, while the other

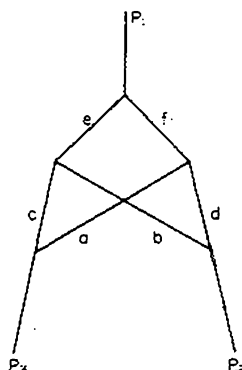


FIG. 11.

two are negative and also exceed 1 in absolute value. As was pointed out, these surfaces always correspond to the non-nearest singular points.

It can easily be seen that the non-euclidean case is possible exclusively in the graph of Fig. 4 out of all graphs considered above. In the scheme of Fig. 8, for example, which corresponds to the graph of Fig. 7, the vectors a, b and c on the one hand and a, b and d on the other, play the role of the vectors b, c and e . But, as was pointed out, the angles of the triangle formed by the vectors b, d and e are always real. Otherwise, one of the particles b, d and e would not be stable. Yet, as was just mentioned, all cosines in the non-euclidean case should exceed unity in absolute value and consequently all angles should be complex.

3.4. Graphs with four external ends

Let us proceed to the Feynman graphs with four external lines. In this case it is obvious that we have to consider schemes which are located not in a plane but in space, which, naturally, complicates the problem a great deal. The most important of such graphs are the graphs with physical ends, i.e. those in which the squares of the corresponding momenta equal the squares of the particle masses. The simplest singularities are connected with the graphs of the type shown in Fig. 12. The middle line may here correspond to a single

particle or to several of them. The singularities of such a graph obviously correspond to

$$(p_1 + p_2)^2 = m_a^2,$$

where m_a is the sum of the masses of the corresponding particles. The case of one particle yields an isolated pole, the case of two particles gives a singular line in the complex region.

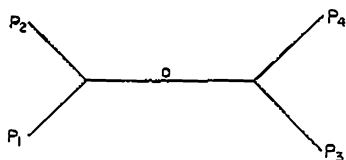


FIG. 12.

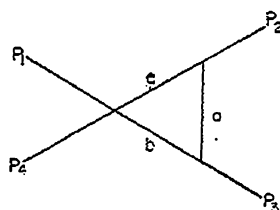


FIG. 13.

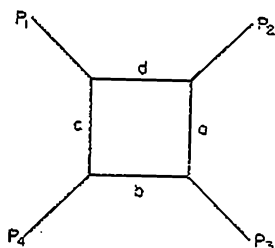


FIG. 14.

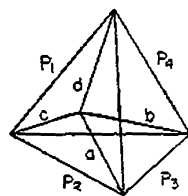


FIG. 15.

The graph in Fig. 13 fully corresponds to the graph in Fig. 4 and therefore no additional discussion is necessary.

The graph in Fig. 14 corresponds to the scheme in Fig. 15 where four of the six edges of the tetrahedron are determined by the length of free ends, while the two others equal $p_1 + p_2$ and $p_2 + p_3$ respectively. (It should be noted that the graphic construction of this graph can be found in the paper of Karplus *et al.*³⁾ A detailed analysis shows that there as well, if the nearest singularity is discussed, all angles are real and the condition that the central point should be located within the tetrahedron corresponds to the condition of the positiveness of the coefficients α . A more complex situation, which will not be discussed in the present paper, corresponds to the singularities in the complex region. Just as was done in the analysis of the three-end graphs, it is convenient to introduce four unit vectors in the directions of the vectors a , b , c and d and find an equation connecting the cosines of the angles between them. An equation of the fourth degree arises instead of equation (10). The following fact is worth mentioning. If graphs with four physical ends are considered the following condition should be fulfilled for the graph of Fig. 13, as is clear from the scheme of Fig. 5: the sum of the angle of the triangle $a b p_3$ between the sides a and b , and the angle of the triangle $a c p_2$ between the sides a and c ,

both of which are fixed by the masses of the corresponding particles, should exceed π . An analogous condition for the graph of Fig. 14 evidently is that the sum of the four angles in the triangles, corresponding to the four vertices, should exceed 2π . Hence, it directly follows that a necessary, though certainly not sufficient condition, for the existence of the "non-trivial" nearest singularities is the presence of an obtuse angle in at least one of the triangles, which, in its turn, requires that the virtual decay of at least one particle should be described by an inequality of the type

$$m_2^2 > m_a^2 + m_c^2.$$

This relation obviously cannot be fulfilled either for π -mesons or for nucleons. The nearest singularities in the scattering of these particles on each other, therefore, correspond to the graph of Fig. 12. It should be emphasised, however, that this does not apply to the singularities in the complex region.

4. NATURE OF SINGULARITIES

Now we shall consider the nature of the singularities thus obtained. Let us return to the basic formula (1), writing it as

$$\int (\varphi + K)^{-n} B d^4 k' d^4 l' \cdots d\alpha_1 d\alpha_2 \cdots \delta(\alpha_1 + \cdots + \alpha_n - 1).$$

Expanding φ with respect to the powers $\alpha'_i = \alpha_i - \alpha_{0i}$ where α_{0i} corresponds to the minimum value of φ , we can write this integral

$$\int (\varphi_0 + Q)^{-n} B d^4 k' d^4 l' \cdots d\alpha'_1 d\alpha'_2 \cdots \delta(\alpha'_1 + \cdots + \alpha'_n), \quad (12)$$

where φ_0 is the minimum value of φ equal to zero in the singularity itself (the values of the vectors at the external ends being given), while Q is a quadratic function of the variables k', l', \dots and the variables α'_i . To elucidate the nature of the singularity, it is sufficient to consider the values B at $k' = l' = \cdots = 0$. If the power of the numerator with respect to the integration variables is lower than the power of the expression Q^n , the integral (12) converges at large values of these variables; in other words, its value is determined by the values of the variables corresponding to $Q \approx \varphi_0$, i.e. integration is carried out at small values of the variables, for which (12) has sufficient accuracy. It is evident that in this case (12) may be written as

$$\text{const} \cdot \varphi_0^{(m/2) - n} \quad (13)$$

where m is a number of integrations. If $m \geq 2n$, the integral (12) does not converge and these arguments do not apply. To determine the nature of a singularity in this case the simplest way is to differentiate (12) with respect to φ_0 as many times as is necessary to make the power of the denominator greater than that of the numerator. After that we can make use of the formula obtained which should be integrated as many times. The integration constants obviously yield integer powers φ_0 which have no singularity at $\varphi_0 = 0$. Accordingly, we again obtain formula (13), with the exception of the case

when $\frac{1}{2}m - n$ is zero or a positive integer. In this case, instead of (12) we obviously obtain

$$\text{const} \cdot \varphi_0^{(m/2) - n} \cdot \ln \varphi_0.$$

It should be noted that though the minimum is now discussed, these results may be applied to any extremum φ as well.

The quantity n equals the number of internal lines in the Feynman graph; the number of vectors k with respect to which integration is carried out, equals the number ν of independent contours which make up the graph under consideration. Accordingly,

$$m = 4\nu + n - 1.$$

Hence it follows that the nature of a singularity is determined by the expression $\varphi_0^{2\nu - 1/2(n+1)}$ and, if $2\nu - \frac{1}{2}(n+1)$ is zero or a positive integer, by

$$\varphi_0^{2\nu - 1/2(n+1)} \ln \varphi_0.$$

The quantity φ_0 is obviously proportional to the distance between the point under consideration and the hypersurface in the space p_i^2 , $(p_i + p_k)^2$, in which singular points are situated. The number of contours should also include "biangles" which emerge when not one but several particles are conveyed along one line. For instance, $\nu = 2$ in the graph of Fig. 9, and since $n = 4$, the singularity is of the nature of $\varphi_0^{3/2}$.

It will be noted that a more convenient number of vertices may be used than the number of independent contours. The number of independent contours, i.e. the number of independent integrations, equals the number of lines minus the number of additional conditions. The latter equals the number of vertices less one, since one δ -function goes over into a final answer. Thus,

$$\nu = n - v + 1$$

(where v is a number of vertices)[†] and the power in the singularity may be written as $(3/2)(n+1) - 2v$.

5. SUMMARY

Let us formulate in brief the general rules for finding singularities. Various graphs with given external ends are considered. An arbitrary number of lines may converge at each vertex of such graphs, in agreement, of course, with the conservation laws (an odd number of π -meson lines cannot pass through one point, for example). All particles stable in strong interactions may figure as lines. After that, a study is made of the scheme built on the principle of replacing the polygons in the graph by a scheme of vertices. The lengths of all internal lines in the scheme are equal to the corresponding masses. The essential intersections in the schemes (such as are obtained from the polygons in the graph) should satisfy the relation $\sum \alpha_i q_i = 0$, where q_i are the vectors proceeding from this intersection, and all $\alpha_i > 0$. The nearest singularities correspond to a scheme in euclidean space.

[†] This was suggested to me by L. Okun and A. Rudik.

6. APPLICATION TO SCATTERING AMPLITUDE

A number of facts should be borne in mind in applying the results thus obtained to the scattering amplitude. Let us consider the scattering amplitude as a function of one variable x (which may be, for example, total energy or transferred momentum), regarding all other variables as given. The integrals considered above determine the functions whose values in the upper and lower semi-planes are connected by the relation $f(x^*) = f^*(x)$; in other words, the functions which are considered above and below the axis are actually different functions which are by no means analytic continuations of each other, and, generally speaking, we have a discontinuity if the values x are real. The scattering amplitude which is obtained when the Feynman rules of direction are observed, is in general of the form $a(x + i\delta) + b(x - i\delta)$, where δ is infinitesimal. The analytic properties considered above refer to the function $a(x + i\delta)$, extended by the variable x into the upper semi-plane, and to the function $b(x - i\delta)$, extended into the lower semi-plane. The behaviour of the analytic continuation of the function $a(x + i\delta)$ into the lower semi-plane and of the function $b(x - i\delta)$ into the upper semi-plane by no means follows from the above.

In an "alien" semi-plane the functions a and b may have any singularities located in any manner. Nor can they be determined from any general considerations. As is well known, the proton-neutron scattering amplitude has, for example, apart from the singularity connected with the formation of a deuteron, another singularity connected with the virtual state of this system, which does not correspond to any real particle and lies precisely in an "alien" semi-plane, with respect to the total energy of the system. Another example is the well-known resonance in the scattering of π -mesons on nucleons, which also corresponds to a singularity in an "alien" semi-plane, the total energy of the system being obviously complex. It is clear that such singularities cannot in principle be anticipated from general considerations, but may be obtained only from a theory yielding specific expressions for the scattering amplitude.

The problem is very much simplified in the case when there exists a region of values of x , for which the amplitude in question is real. Then we have two cuts through the real axis in the complex plane, and it is easy to see that the function $a(x + i\delta)$ has only the right cut, while the function $b(x - i\delta)$ has only the left one. If the quantity $a + b^*$ is considered instead of the scattering amplitude, for which it is sufficient to change the sign of the imaginary part of the amplitude in the left cut, we shall obtain a function which has no singularities in the upper semi-plane, which leads to the conventional dispersion relations.

The author is indebted to L. B. Okun, A. P. Rudik and Ya. A. Smorodinskii for numerous valuable comments.

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99. SMALL BINDING ENERGIES IN QUANTUM FIELD THEORY

THE recent trend in quantum field theory has been to consider not Hamiltonians but only the unitarity relation and analyticity properties of diagrams, expressing the local nature of the interaction. It is true that one has not succeeded as yet in writing down the complete equations for such a theory; however, one might hope that such equations would represent, in essence, a new theory, basically different from a discussion of interaction processes.

Occasionally the thought has been expressed that such a theory would only constitute a "framework," i.e. that it would allow the introduction of all kinds of particles with all kinds of relations among them. The purpose of this note is to show that for loosely bound (but strongly interacting) particles there arises in such a theory a direct relation between the coupling constant and the binding energy of the particles.

Let us denote by the letter c a particle which, when given a comparatively small amount of energy, can decay into two particles a and b . Obviously

$$M_c = M_a + M_b - \varepsilon, \quad (1)$$

where ε is the binding energy. These conditions, as is well known, are satisfied by the deuteron. It is true that it is customary to treat the deuteron as a composite particle, "consisting" of a proton and neutron. In reality, however, the concepts of "elementary" and "composite" lose their meaning in relativistic quantum field theory and the assertion that the deuteron is a "composite" particle differs only quantitatively from the assertion that mesons "consist" of nucleons and antinucleons. Only the assumption that all particles under consideration interact strongly is relevant.

For simplicity let us suppose that all particles have zero spin. It is understood that the particle c has even parity relative to the particles a and b (s -state!). Let us denote by the letter g the full vertex part for the transition $a + b \rightarrow c$. We look now at the amplitude for the mutual scattering of particles a and b . This amplitude has a pole term, associated with the virtual production of the particle c , equal to

$$(g^2/8\pi)/(M_a + M_b) [(p_a + p_b)^2 - M_c^2]. \quad (2)$$

Replacing $(p_a + p_b)^2$ by $(M_a + M_b + E)^2$, where E is the energy in the barycentric frame after subtraction of the rest energy, expressing M_c in terms of

Л. Д. Ландау, Малые энергии связи в квантовой теории поля, *Журнал Экспериментальной и Теоретической Физики* 39, 1856 (1960).

L. D. Landau, Small binding energies in quantum field theory, *Soviet Phys.-JETP* 12, 1294 (1961).

equation (1), and assuming that E and ε are small, we find

$$(g^2/16\pi)/(M_a + M_b)^2(E + \varepsilon). \quad (3)$$

Let us compare this expression with the well-known expression for the resonant scattering amplitude for not too high energies. The latter is given by

$$\sqrt{1/2}(1/M_a + 1/M_b)/[\sqrt{\varepsilon} + i\sqrt{E}]. \quad (4)$$

It is obvious that the expression $1/(\sqrt{\varepsilon} + i\sqrt{E})$ evaluated at the point $E = -\varepsilon$ is equal to $2\sqrt{\varepsilon}$. One may therefore assert that

$$g^2/16\pi(M_a + M_b)^2 = \sqrt{2(1/M_a + 1/M_b)} \varepsilon.$$

Finally we obtain

$$\varepsilon = (g^4/512\pi^2) M_a M_b / (M_a + M_b)^5. \quad (5)$$

This is the desired formula which expresses the binding energy, assuming it is small, in terms of the corresponding coupling constant.

It should be noted that analogous relations were already encountered by Fainberg and Fradkin¹ in writing dispersion relations for nucleon scattering.

It seems to me that it would be of interest to derive this formula directly from unitarity and analyticity properties of the amplitudes.

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100. FUNDAMENTAL PROBLEMS

It is with the deepest sorrow that I send this article written in the honour of the sixtieth birthday of Wolfgang Pauli to a volume dedicated to his memory, which will always be cherished by those who had the good luck of knowing him personally.

It will be impossible now to know his opinion about the ideas expressed in this article but I am still encouraged by the thought that his views on the subject would not be very different.

It is well known that theoretical physics is at present almost helpless in dealing with the problem of strong interactions. For this reason, any remarks on the subject must necessarily be of the nature of forecasts, and their authors are peculiarly apt to find themselves barking up the wrong tree.

It was long thought that the main difficulty of the theory lies in the occurrence of infinities which can be avoided only by the use of perturbation theory. The habit of using the device of renormalisation, which had achieved brilliant success in perturbation theory, was carried so far that the concept of re-normalisation acquired a certain mystical aura. The situation, however, becomes clear if, as is usual in theoretical physics, point interaction is regarded as the limit of some "distributed" interaction. This treatment, although it assumes weak interaction, goes considerably beyond the scope of perturbation theory, and makes possible the derivation of asymptotic expressions for the energy dependence of the basic physical quantities¹. These expressions show that the effective interaction always diminishes with decreasing energy, so that the physical interaction at finite energies is always less than the interaction at energies of the order of the cut off limit which is given by the bare coupling constant appearing in the Hamiltonian.

Since the magnitude of the re-normalisation increases indefinitely with the cut-off limit, it follows that even an extremely weak interaction implies a large bare coupling constant when the cut-off radius is sufficiently small. Thus it was supposed that the main problem is to devise a theory of very strong interactions.

Further investigation showed, however, that the matter was by no means so easily dealt with. It was demonstrated by Pomeranchuk in a series of papers² that, as the cut-off limit is increased, the physical interaction tends to zero, no matter how large the bare coupling constant is. At about the same time Pauli and Källén³ obtained the same result for the so-called Lee model.

L. D. Landau, Fundamental problems, *Theoretical Physics in the Twentieth Century, a Memorial Volume to Wolfgang Pauli*, ed. by M. Fierz and V. F. Weisskopf (Interscience, New York 1960), p. 245.

L. D. Landau, On analytical properties of vertex parts in quantum field theory, *Proc. Ninth. Ann. Conf. on High Energy Physics, Kiev, July 1959 (Moscow 1962)*, p. 95.

The correctness of "nullifying" the theory has often been called in question. The Lee model is a very special one, considerably differing in several respects from physical interactions; and the validity of Pomeranchuk's proofs has been doubted. In my opinion such doubts are unfounded. For example, Källén has several times put forward the view that unusual properties of the series to be summed are involved, but he has never given reasons to support this view. By now, the "nullification" of the theory is tacitly accepted even by theoretical physicists who profess to dispute it. This is evident from the almost complete disappearance of papers on meson theory, and particularly from Dyson's assertion⁴ that the correct theory will not be found in the next hundred years—a piece of pessimism which would be impossible to understand if one supposed that the present meson theory leads to finite results which we are yet unable to derive from it. It therefore seems to me inopportune to attempt an improvement in the rigour of Pomeranchuk's proofs, especially as the brevity of life does not allow us the luxury of spending time on problems which will lead to no new results.

The vanishing of the point interaction in the present-day theory leads to the idea that it is necessary to consider "distributed", non-local, interactions. Unfortunately, the non-local nature of the interaction renders completely useless the technique of the present existing theory. Of course the undesirability of this occurrence is a poor argument against the non-local nature of the interaction; but there are stronger arguments against it. All the conclusions derived by means of the quantum theory of fields without the use of particular Hamiltonians seem to get confirmed experimentally. They include, in the first place, dispersion relations. Moreover, the number of mesons formed in high-energy collisions is in agreement with Fermi's formula,⁶ which involves the use of the ideas of statistical thermodynamics for dimensions very much less than any possible radius of interaction.

The idea of the possibility of a marked modification of the existing theory without abandoning local interaction, was first suggested by Heisenberg⁷. Besides this general idea, Heisenberg has also suggested a number of further assumptions, which to me appear dubious. I shall therefore attempt to describe the general situation in what seems to me the most convincing form.

Almost 30 years ago Peierls and I had noticed that in the region of relativistic quantum theory no quantities concerning interacting particles can be measured and the only observable quantities are the momenta and polarisations of freely moving particles. Therefore if we do not want to introduce unobservables we may introduce in the theory as fundamental quantities only the scattering amplitudes.

The ψ operators which contain unobservable information must disappear from the theory and, since a Hamiltonian can be built only from ψ operators, we are driven to the conclusion that the Hamiltonian method for strong interaction is dead and must be buried, although of course with deserved honour.

The foundation of the new theory must consist of a new diagrammatic technique which must deal only with diagrams with "free" ends i.e. with scattering amplitudes and their analytic continuation. The physical basis of this

technique is the unitarity conditions and the principle of locality of interaction which expresses itself in the analytic properties of the fundamental quantities of the theory, such as the different kinds of dispersion relations.

As such a new diagrammatic theory is not as yet constructed, we are obliged to derive the analytical properties of the vertex parts from a Hamiltonian formalism, but it requires much naivety to try to make such derivations "rigorous", forgetting that we derive existing equations from Hamiltonians which do not really exist.

One of the conclusions of such an approach to the theory is that the old problem of the elementarity of particles finally loses its meaning, as it cannot be formulated without considering interactions between particles.

I think that the development of a theory on such lines has progressed very much in recent times and the time is not far away when the equations of the new theory will be finally written down.

But one must remember that in this case, contrary to what has happened in all the previous stages of theoretical physics, the writing down of the equations will mark not the end but only the beginning of the construction of the theory. The equations of the theory will be an infinite system of integral equations, each of which has the form of an infinite series, and it will be a hard task to learn how to work with such equations.

It is, of course, impossible to predict now how many constants in the theory may be chosen arbitrarily. We cannot even exclude the possibility that the equations will have no solutions at all, i.e. that the theory will also be nullified. This might be regarded as the rigorous proof of the non-locality of nature, but it might also mean that a theory of strong interactions alone cannot exist by itself and that weak interactions, and especially electrodynamics, must also be included in the picture. The infrared "catastrophe" would then make the situation infinitely more complicated.

But even in the best case we have still a great struggle before us, a struggle which has now become much more difficult without the brilliant unerring light of the mind of Wolfgang Pauli.

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APPENDIX A

APART from his many research papers Landau has also been the co-author of many books. Most of these have been or are being translated into English or have been incorporated in the well-known *Landau-Lifshitz Course of Theoretical Physics*. Of this course the following volumes have appeared: Mechanics (the first edition was written together with Pyatigorskii; this volume incorporates a problem book written together with Lifshitz and Rozenkevich), Statistical Physics, Fluid Mechanics, Classical Theory of Fields, Non-relativistic Quantum Theory, Theory of Elasticity, and Electrodynamics of Continuous Media. Translations have also appeared or are being prepared of a semi-popular booklet on Relativity (together with Ya. B. Rumer), a Lecture Course on Nuclear Theory (together with Smorodinskii), and a popular book *Physics for Everybody* (together with Kitaigoridskii). There is also an early booklet on the conductivity of metals and as it is relatively unobtainable, we give in this appendix a translation of it. It was published in 1935 in Kharkov and was written together with A. Kompaneets.

THE ELECTRICAL CONDUCTIVITY OF METALS

1. One can divide all conductors of electricity into two classes depending on the temperature-dependence of the resistance. The resistance of a conductor may at absolute zero either become infinite, or stay finite. The resistivity of electrolytes and electronic semiconductors becomes infinite. On the other hand, the resistivity of metals remains at absolute zero at least finite (sometimes it vanishes). Since the topic of this survey is metallic conductivity, it is plausible to base the physical definition of a metal on its finite conductivity at absolute zero.

One must try to construct a theoretical model which will possess properties which are defined purely physically. It is essential that an arbitrarily weak and constant electromagnetic field produces a non-vanishing current in a metal. Moreover, this is true right down to the absolute zero. However, at the absolute zero the electrons in the metal are in the ground state. It is thus clear that the electronic ground state in a metal is perturbed by an arbitrarily weak field of an *arbitrarily low frequency* (that is, a constant field). However, it follows from this necessarily that the electronic ground state of a metal is part of a continuum. In other words, there exists a whole set of electronic levels which lie infinitesimally close, immediately adjoining the ground level. Indeed, the change in energy for any electronic transition in a sufficiently weak field is $\hbar \omega$, where ω is the frequency of the field causing the transition. If ω is infinitesimally small, the energy changes arbitrarily little, that is, the electron makes a transition to an infinitesimally close level.

It follows from this only that continuous electronic levels must necessarily occur in a metal. It is, however, by no means possible to show that it is also a sufficient condition which unambiguously defines a metal, although it is very probable.

In general, this model has not been considered by anybody. In fact, there is a built-in limitation which is very important for calculations, but which is completely arbitrary from a physical point of view.

The electrons in a metal interact both with one another and with the ions in the crystal lattice. There is no actually existing metal for which there is any basis to assume that these two interactions are of a different order of magnitude. All the same, so far it has not been possible to construct a theoretical model of the metallic state which takes the electron-electron interaction into account. In calculations one considers the electrons to be completely independent and to be present only in the given field of the ions in the crystal lattice.

It is difficult to establish how valid such a method of attack is. Even if the electron-electron interaction is small compared with the electron-lattice interaction, it does not necessarily follow that it is unimportant for the phenomenon of metallic conduction. It is well known that in transport processes also very small quantities are usually important which do not play a role for statistical equilibrium. An example of this is the interaction between the molecules of a perfect gas. This interaction does not enter at all into the thermodynamic quantities of a perfect gas. On the other hand, transport quantities such as the thermal conductivity, viscosity, and so on, must necessarily take the interaction between the molecules into account. In exactly the same way could the electrical conductivity of a metal depend on the relatively weak electron-electron interaction. Afterwards we shall show that, indeed, the electronic interactions influence the metallic conduction but little. We must again state that there is no ground whatever to assume the electronic interactions in a metal to be small.

On the other hand, a simplified model is already unable qualitatively to elucidate such a phenomenon as the superconductivity of a metal.

2. Let us first of all consider liquid metals and also such metals which are constructed from arbitrarily distributed atoms.

One sees easily that the motion of an electron in such a solid can not appreciably depend on the thermal motion of the atoms which is slow compared to the motion of the electrons. One must thus expect that the resistivity of a liquid metal and of a specially disordered alloy must essentially be independent of temperature. Only a change in the configuration of the nuclei may influence it.†

† It is sometimes emphasised that the temperature dependence of the resistivity of liquid metals is similar to that of solid metals. One uses then the data in tables referring to constant pressure, while one should use data referring to constant volume.

It is of special interest to study the resistivity of a liquid metal near the critical point (cf. Birch, *Phys. Rev.* **41** (1933), 641) but this has up to now only been done for mercury. There is then a continuous transition from the metallic state to the gaseous one which has an infinitely large resistivity. The way this transition takes place could be a very helpful hint for a future theory of the metallic state—more complete than the one existing at present.

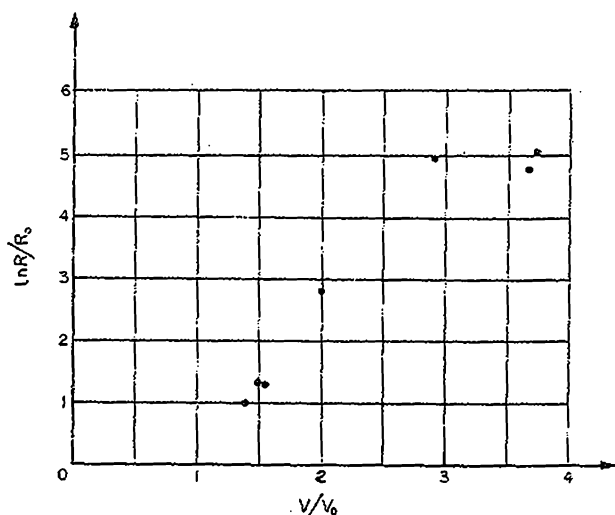


FIG. 1.

Unfortunately, Birch's data on the resistivity of mercury do not refer to constant volume and there are no data on the compressibility of mercury near the critical point. To reduce the data to constant volume one must thus apply the Van der Waals' equation, using the critical constants of mercury, obtained by Birch: $T_c = 1460^\circ\text{C}$ and $p_c = 1640$ atm. The result will now, of course, only be qualitative. The measurements of Birch, referred to constant volume, are shown in Fig. 1.

3. Let us now turn to solid metals. A solid metal has a crystalline structure, that is, it consists of regularly arranged atoms, in the field of which the conduction electrons move. The theory of metallic conduction starts in the following from the same approximation as the theory of molecules. Since the nuclei

Continuation from p. 804

One must consider the temperature coefficient of the resistivity in the form

$$\frac{1}{R} \left(\frac{\partial R}{\partial T} \right)_p = \frac{1}{R} \left(\frac{\partial R}{\partial T} \right)_p - \frac{\frac{1}{R} \left(\frac{\partial R}{\partial p} \right)_p \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p}{\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_p}.$$

Of all liquid metals, one knows only all of the relevant quantities for mercury, and from these data it follows that the required temperature coefficient at constant volume is equal to -0.57×10^{-3} : the resistivity of liquid mercury decreases thus with increasing temperature.

in the crystalline lattice are very heavy compared with the electrons, one does not study the general quantum-mechanical system, consisting of nuclei and electrons, but only the motion of the electrons in a given configuration of the nuclei in the form of a crystalline lattice. One can then study separately the electronic and the vibrational levels of the lattice just as this is done for molecules—and this procedure is equally valid in both cases.

A perfect crystalline lattice has in space a three-fold periodic force field. We shall determine the form of the electron function ψ in such a field, assuming it to continue indefinitely in all three directions. Let the symmetry of the lattice be arbitrary. We can, however, choose different unit lengths for the axes in such a way that the basic period in all three directions is equal to unity. It is, moreover, convenient to change to a skew system of co-ordinates, the axes of which are parallel to the three edges of the elementary parallelepiped. The periodicity of the crystalline lattice is most simply described in such a system of co-ordinates. The potential energy of the electron $U(x, y, z)$ should not change under the substitution $x \rightarrow x + 1$, $y \rightarrow y + 1$, or $z \rightarrow z + 1$, or, in general

$$U(\mathbf{r} + \mathbf{n}) = U(\mathbf{r}).$$

Here \mathbf{r} is the radius vector of an arbitrary point $\mathbf{r}(x, y, z)$ and \mathbf{n} is an arbitrary "integer" vector, that is, a vector such that its components are integral multiples of the basic periods along the corresponding axes.

This enables us to reach several conclusions about the form of the electron function ψ in the lattice. To simplify the discussion it is sufficient to consider the one-dimensional problem as one can immediately generalise the results to the three-dimensional case. Let, therefore, the electron move in a field of force with potential energy $U(x)$ which is invariant under the transformation $x \rightarrow x + 1$. The substitution $x \rightarrow x + 1$ will then not change the wave equation. Hence, under such a substitution, ψ can only be multiplied by a constant, which we denote by $e^{i\xi}$. Thus

$$\psi(x + 1) = e^{i\xi} \psi(x).$$

This finite difference equation can be solved by the function

$$\psi(x) = e^{e^{i\xi}x} \varphi(\xi, x),$$

where $\varphi(\xi, x)$ has the periodicity of the lattice:

$$\varphi(\xi, x + 1) = \varphi(\xi, x).$$

One sees, moreover, easily that ξ is purely real. Indeed, the wave function ψ must remain finite as $x \rightarrow \pm \infty$. If, however, ξ possessed a positive or negative, imaginary part, the factor $e^{i\xi x}$ would tend to infinity as x tended to $-\infty$ or $+\infty$.

One can generalise these considerations easily to cover the three-dimensional case. Then ξ becomes a vector and the wave function has the form

$$\psi(\mathbf{r}) = e^{i(\xi \cdot \mathbf{r})} \varphi(\xi, \mathbf{r}),$$

where

$$\varphi(\xi, \mathbf{r} + \mathbf{n}) = \varphi(\xi, \mathbf{r}).$$

It is important that the components of ξ by definition have a meaning only when given modulo 2π . The electron energy which is a function of ξ , $\varepsilon(\xi)$, must be periodic in the components of ξ with period 2π :

$$\varepsilon(\xi + 2\pi n) = \varepsilon(\xi).$$

Here, n is an "integer" vector in ξ -space.

Finally, the solution of the wave equation has the following form:

$$\psi(r, t) = e^{-i\varepsilon(\xi)t/\hbar} e^{i(\xi \cdot r)} \varphi(\xi, r).$$

In this form, the wave function of an electron in a lattice must be very similar to a free electron wave function:

$$\psi(r, t) = e^{-i\varepsilon(p)t/\hbar} e^{i(p \cdot r)/\hbar}$$

The quantity $\hbar \xi$ takes here the place of the momentum and we shall simply call ξ in the following the quasi-momentum of the electron in the lattice. Instead of $\varepsilon(p)$ we get for an electron in a lattice $\varepsilon(\xi)$ which adds to the similarity. The only difference lies in the modulating factor $\varphi(\xi, r)$, and in the fact the quasi-momentum is not defined exactly but only modulo 2π and that $\varepsilon \neq \hbar^2 \xi^2/2m$.

Knowing the expression for the electron energy we can also simply construct an expression for its velocity. To do this we must construct a wave packet from the wave functions and find its group velocity,

We can write for this wave packet

$$\psi(x) = \int a(\xi) e^{i\xi x} \varphi(\xi, x) d\xi,$$

where the integration is over a very small ξ -interval. This can also be expressed in the form that the function $a(\xi)$ is appreciably different from zero only in a small ξ -interval. However, in a sufficiently small interval we may consider the function $\varphi(\xi, x)$ which is periodic in x to be practically independent of ξ for arbitrary x . On the other hand, $e^{i\xi x}$ can for large x certainly not be considered to be constant in however small a ξ -interval as for sufficiently large x it changes arbitrarily rapidly as function of ξ . Therefore, for the given shape of $a(\xi)$ we may validly take $\varphi(\xi, x)$ outside the integral sign, but must, on the contrary, retain $e^{i\xi x}$. In that case, $\varphi(\xi, x)$ has the character of some modulating factor while the integral

$$\int a(\xi) e^{i\xi x} d\xi$$

has exactly the same form as the corresponding integral in the expression for the wave packet for a free electron.

If the wave packet is placed in an almost uniform force field it will behave as a classical electron with the velocity v of the given wave packet. For this it is necessary that the potential energy in such a field changes little over distances of the order of the de Broglie wavelength \hbar/mv . For electrons in a crystalline

lattice, the order of magnitude of the de Broglie wavelength is of the order of the lattice constant, and this gives us the measure for the uniformity of the field. The quantity ξ replaces the momentum of the free electron. The Hamiltonian of the electron function in the lattice has the form $\varepsilon(\xi) - e(\mathbf{F} \cdot \mathbf{r})$. Here $\varepsilon(\xi)$ is the energy without an external field, and $-e(\mathbf{F} \cdot \mathbf{r})$ the potential energy in a weak external, almost uniform field \mathbf{F} . After what we have said, it is valid to apply the Hamiltonian equation to the electron in a lattice. We then get for its velocity

$$\mathbf{v} = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial \xi},$$

and for the change in momentum

$$\dot{\xi} = -\frac{1}{\hbar} \frac{\partial [\varepsilon - e(\mathbf{F} \cdot \mathbf{r})]}{\partial \mathbf{r}} = \frac{e \mathbf{F}}{\hbar}.$$

The momentum of an electron in a perfect crystalline lattice in an external field \mathbf{F} increases continuously. Such a lattice has therefore no electrical resistivity. In particular, one can not speak about any proportionality between current and field—an arbitrarily weak field can produce an arbitrarily large current, if we wait sufficiently long. Indeed, the resistivity is caused by a violation of the strict periodicity of the lattice by the thermal vibrations of the nuclei.

After we know on what variables the energy of the electrons in a lattice depends, we can apply statistics to them.

They form at absolute zero a completely degenerate fermion gas or, to put it differently, they fill all quantum states inside the $\varepsilon = \varepsilon_0$ -surface. It is well-known that a fermion gas is strongly degenerate up to a temperature T for which kT is comparable to the Fermi energy ε_0 . This follows from the fact that at on-vanishing temperatures the electrons do not occupy the Fermi sphere completely—some of them go out over the boundary ε_0 . Near this boundary there is a small “smeared out” region with a width of the order of kT . As long as the dimension of that region remains much smaller than the whole volume of the Fermi sphere the electron gas is strongly degenerate. On the other hand, if the temperature is such that $kT \approx \varepsilon_0$ the degeneracy is lifted. One can estimate for which temperatures the electron gas remains degenerate. For such an estimate it is sufficient to consider the electrons in the lattice to be free and to put $\varepsilon \sim \mathbf{P}^2/2m$. The total phase volume of the sphere $\varepsilon = \varepsilon_0$ is then equal to

$$\frac{4\pi}{3} P_0^3 V = \frac{4\pi}{3} (2m \varepsilon_0)^{3/2} V.$$

Here V is the volume occupied by the electrons in configuration space.

On the other hand, one can easily estimate the phase volume directly. Since there can be at most two electrons with opposite spins (according to the Pauli principle) in each phase cell of volume $(2\pi \hbar)^3$, the total phase volume of the

system of N electrons must be equal to

$$\frac{1}{2} N (2\pi \hbar)^3.$$

Comparing this with the earlier obtained expression of the phase volume in terms of ε_0 we find for ε_0 easily

$$\varepsilon_0 = \left(\frac{3N}{8\pi V} \right)^{2/3} \frac{4\pi^2 \hbar^2}{2m}.$$

We put the criterion for degeneracy as $kT \leq \varepsilon_0$. Hence, choosing the lattice constant to be about 3×10^{-8} cm and assuming one electron to fit in an elementary cube, we get

$$T \leq 3 \times 10^4 \text{ K}.$$

For all experimentally accessible temperatures the electron gas is thus degenerate to a very large degree, the more so as at temperature of the order of 10^3 °K the metal melts.

Since we have not considered the electron-electron interaction in the present, theory, we must statistically describe them as a perfect gas in the force field of the atoms in the lattice. We must then take for them the usual Fermi distribution function where we put instead of the energy the known function $\varepsilon(\xi)$. We must still determine the weight of the states corresponding to a given value of ξ .

We find it easily from the same considerations which are applied to free electrons. To do this we must consider the electron gas as a superposition of plane ψ -waves on which we impose the periodicity condition

$$\psi(\mathbf{r} + \mathbf{n}) = \psi(\mathbf{r}).$$

Here \mathbf{n} is a constant "integer" vector, that is, the substitution is nothing but a translation over an integral number of lattice periods.

The modulating factor $\psi(\xi, \mathbf{r})$ satisfies the periodicity condition automatically. We need thus consider only the function $e^{i(\xi \cdot \mathbf{r})}$, as usual.

It is however, well known that the weight of the states corresponding to a volume element $d\tau$ of ξ -space and dV of configuration space is equal to

$$d\Gamma = 2 \frac{dV d\tau}{(2\pi)^3}.$$

The factor 2 derives from the fact that the same state in phase space can have electrons with opposite spins. The electron distribution finally takes the form

$$n d\Gamma = 2 \frac{dV d\tau}{(2\pi)^3} \frac{1}{e^{(\varepsilon - \mu)/kT} + 1},$$

where μ is the chemical potential which is independent of ξ .

4. We explained that a perfect crystalline lattice does not have an electrical resistivity. It is thus natural to attempt to explain the resistivity of a metal by the violation of the perfectness of the lattice properties. This violation can be two-fold—either static through some lattice defects or time-dependent through thermal vibrations. In the present section we shall consider only the latter.

If the strict periodicity of the lattice is violated, the electron wave function is not of the form found in section 3. However, if we assume that the thermal displacements of the nuclei are small, we can consider them as small perturbations imposed upon the perfect lattice. The known wave functions are then the zero-order approximations. Using them we can evaluate the probability for an electronic transition $\xi \rightarrow \xi'$, $\varepsilon(\xi) \rightarrow \varepsilon(\xi')$ per unit time under the action of some perturbing potential.

We shall elucidate what the form must be of this perturbing potential. If we assume the thermal displacements of the nuclei to be small, one can validly expand the potential energy of the electron in a power series in terms of them and restrict oneself to the first powers.

The term in the expansion which does not contain at all the displacements is the already known potential energy $U(r)$. The wave equation with such a U determines the zero-order function. The terms containing the first powers of the displacements enter as a perturbing energy.

It is well known that one can represent the vibrations of the atoms in the lattice as a superposition of travelling plane waves differing in their wave vector and polarisation.

We see now what form the perturbing energy will have under the given assumptions. Since the thermal displacements of the nuclei are small, the perturbing energy will be a superposition of the displacements produced by all the nuclei separately

$$V = \sum_n (\Phi(r - n) \cdot u_n).$$

Here u_n is the displacement of the n th nucleus. We can expand it in travelling plane waves, as follows,

$$2u_n = \sum_f [a_f e^{i(f \cdot n)} + a_f^* e^{-i(f \cdot n)}],$$

where f is the wave vector of the f th vibration (that is, the vibration with a given triple of numbers f_1, f_2, f_3) and a_f its amplitude. If we substitute this expression into the expression for V , the perturbing energy caused by the f th eigenvibration of the lattice will be of the form

$$\begin{aligned} \text{Re}(a_f \cdot \sum_n \Phi(r - n) e^{i(f \cdot n)}) &= \text{Re}(a_f \cdot e^{i(f \cdot r)} \sum_n \Phi(r - n) e^{i(f \cdot n - r)}) \\ &= \text{Re}(a_f \cdot \Pi(r) e^{i(f \cdot r)}). \end{aligned}$$

From the definition of Π we see easily that it has the period of the lattice, that is, $\Pi(r) = \Pi(r + i)$, where i is a vector of which each component is zero or unity. Indeed, the substitution $r \rightarrow r + i$ is equivalent to the substitution $n \rightarrow n - i$ under the summation sign. However, the summation is over all

nuclei of the infinite lattice and the substitution $n \rightarrow n - i$ therefore cannot change it.

Since the perturbing energy is linear in the displacements, the actions of the different eigenvibrations of the lattice upon the electron are independent of one another and can be superimposed upon one another.

Such a model is completely analogous to placing an electron in an electromagnetic field which is produced by the superposition of plane monochromatic plane vibrations. The various vibrations differ in wave vector f and polarisation. If the frequency corresponding to the wavevector f is $\omega(f)$, such a wave field can conveniently be described in terms of light quanta. Indeed, we can assign to each vibration f quanta of energy $\omega(f)$ and momentum $\hbar f$. The quanta of the electromagnetic field are called photons. By analogy, the quanta of the elastic lattice vibrations are called phonons (acoustic quanta).

It is well known that the probability for a quantum transition $\xi \rightarrow \xi'$ per unit time is proportional to the square of the absolute magnitude of the matrix element $V_{\xi\xi'}$ of the perturbing energy. This matrix element we must now determine.

First of all, we note that the perturbing energy contains the vibrational amplitudes linearly. It is well known that its matrix elements are thus non-vanishing only for transitions corresponding to the absorption or emission of one quantum by the electron. Transitions for which the energy changes by two quanta are a higher-order effect. Let the lattice before the transition be in such a quantum state that the total number of quanta with wave vector f is equal to $N(f)$. It is then well known that the probability for a transition where the lattice gives off one quantum of energy $\hbar \omega(f)$ to the electron is proportional to $N(f)$ and the transition where the electron transfers to the lattice one vibrational quantum proportional to $N(f) + 1$. The proportionality coefficient is the same in both cases. It depends only on f , but not on $N(f)$. We must now determine it.

As it is independent of $N(f)$ we can validly evaluate it for large $N(f)$. This limiting case corresponds to a transition to classical mechanics. The perturbing energy and its time-dependence is then a travelling plane wave of the form

$$V = \text{Re}(a_f \cdot e^{i[\omega(f)t - (f \cdot r)]} \Pi(r)). \quad (1)$$

Its matrix element is equal to

$$\begin{aligned} V_{\xi\xi'} &= e^{it[\epsilon(\xi') - \epsilon(\xi) + \hbar\omega(f)]/\hbar} \int e^{i(\xi - \xi' + f \cdot r)} (a_f \cdot \Pi(r)) \varphi^*(\xi', r) \varphi(\xi, r) d^3r \\ &+ e^{it[\epsilon(\xi') - \epsilon(\xi) - \hbar\omega(f)]/\hbar} \int e^{i(\xi - \xi' - f \cdot r)} (a_f^* \cdot \Pi^*(r)) \varphi^*(\xi', r) \varphi^*(\xi, r) \varphi(\xi, r) d^3r. \end{aligned}$$

To find the probability for the transition $\xi \rightarrow \xi'$ we must integrate $V_{\xi\xi'}$ over the time from zero until some value t , take the absolute square of the integral and integrate it again, over ξ' . It is well known that for the last integration only those $V_{\xi\xi'}$ are important which are independent of the time, that is, those for which the condition

$$\epsilon(\xi') - \epsilon(\xi) \pm \hbar \omega(f) = 0 \quad (2)$$

is fulfilled.

We can indicate this simply by multiplying the transition probability by a δ -function-like factor of the argument $\varepsilon(\xi') - \varepsilon(\xi) \pm \hbar \omega(f)$. Physically this describes the law of conservation of energy during a collision. The case where we have the upper sign requires that for the transition $\varepsilon(\xi') = \varepsilon(\xi) - \hbar \omega(f)$, that is, that the electron gives off one lattice vibration quantum. The case where we have the lower sign requires, on the other hand, that the electron absorbs one vibrational quantum.

We must now find out what becomes of the integrals occurring in $V_{\xi\xi'}$. To evaluate them it is convenient to make the substitution $r \rightarrow r + n$:

$$\int e^{i(\xi - \xi' \pm f \cdot r)} (a_f \cdot \Pi(r))^{(*)} \varphi^*(\xi', r) \varphi(\xi, r) d^3r \\ = \int e^{i(\xi - \xi' \pm f \cdot n)} e^{i(\xi - \xi' \pm f \cdot r)} (a_f \cdot \Pi(r))^{(*)} \varphi^*(\xi', r) \varphi(\xi, r) d^3r;$$

(*) indicates here that for the lower sign we must replace $(a_f \cdot \Pi)$ by $(a_{f^*} \cdot \Pi^*)$. The functions Π and φ have the periodicity of the lattice so that we could replace their arguments $r + n$ by r . It is clear from this that the integral over the n th elementary cell differs from the same integral over the "first" cell by a factor $e^{i(\xi - \xi' \pm f \cdot n)}$. The integration over the whole volume of the crystal can thus be replaced by an integration over one elementary cell followed by a summation over all elementary cells. In the last summation we have only the terms $e^{i(\xi - \xi' \pm f \cdot n)}$ since the integral itself does not contain n . We get thus

$$\left[\int e^{i(\xi - \xi' \pm f \cdot r)} (a_f \cdot \Pi(r))^{(*)} \varphi^*(\xi', r) \varphi(\xi, r) d^3r \right] \sum_n e^{i(\xi - \xi' \pm f \cdot n)},$$

where the integral is taken over one elementary parallelepiped. The sum is non-vanishing only when the exponent is 0 or an integral multiple of 2π . Important is only the case when it is zero, as ξ , ξ' , and f are defined only modulo 2π . For given ξ and f , ξ' is thus completely determined. Because of this, for every elementary transition not only is it necessary to conserve energy, but the law of conservation of quasi-momentum in the form

$$\xi - \xi' \pm f = 0 \quad (2a)$$

must also be satisfied. Using these conditions we can state that for electronic transitions starting from a given state not all vibrations with arbitrary f can take part, but only those which satisfy the relation

$$\varepsilon(\xi \pm f) - \varepsilon(\xi) \mp \hbar \omega(f) = 0.$$

The conservation of quasi-momentum can be expressed by multiplying the transition probability by a δ -function of argument $\xi - \xi' \pm f$.

We can thus finally write the probability for single quantum transition per unit time in the form

$$W_{\xi\xi'}^{\pm} = A(\xi, f) \delta[\varepsilon(\xi') - \varepsilon(\xi) \pm \hbar \omega(f)] \delta(\xi - \xi' \mp f) \{N(f)\}^{\pm 1}, \quad (3)$$

where the upper sign and $N(f) + 1$ refer to the absorption of one quantum by the lattice, and the lower sign and $N(f)$ to the absorption of one quantum

by the electron†. It is important to know the dependence of the transition probability on f for small f . Small f correspond to long lattice deformation waves (by definition the wavelength is $2\pi/f$). But if the deformation waves are sufficiently long, the electron again enters into almost uniform conditions, that is, into a lattice with a slightly changed constant. If we denote the elastic deformation tensor by V_{ik} , the Hamiltonian function of the electron $\varepsilon(\xi')$ in such a deformed lattice can be expanded in a power series in powers of the tensor V_{ik} and we can restrict ourselves to the first terms:

$$\varepsilon'(\xi) = \varepsilon(\xi) + \lambda_{ik} V_{ik}.$$

Here $\varepsilon(\xi)$ is the energy of the electron in the undeformed lattice and λ_{ik} a quantity depending on ξ . It is important that the first term of the expression in that case contains already not the displacements u_n of the lattice sites, but the deformation of the lattice, that is, the relative displacements, since

$$V_{ik} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right].$$

The terms $\lambda_{ik} V_{ik}$ plays the role of a perturbing energy. Its matrix element, from which we obtain the probability for the transition $\xi \rightarrow \xi'$ is equal to

$$\lambda_{ik} \int \psi^*(\xi', r) V_{ik} \psi(\xi, r) d^3r.$$

The lattice deformation V_{ik} and the displacement can easily be written as a travelling plane wave. The transition probability will then contain the δ -functions, already encountered, which multiply the square of the modulus of the elastic deformation, corresponding to a given wave vector f , and the integral over one phase cell known from the previous expression for the probability. For $f = 0$ this integral tends neither to zero nor to ∞ , and we can thus validly put in it $f = 0$ for sufficiently small f . The square of the modulus of the elastic deformation corresponding to a given f is proportional to the energy density of the corresponding deformation. We shall denote it by $\varrho(f)$. The probability for the transition for small f is thus proportional only to the well-known δ -functions and the energy density of the deformed lattice. The coefficient of proportionality depends then only on ξ , but not on f . Since $\varrho(f) = \hbar \omega(f) N(f)$, we write

$$W_{\xi\xi'} = A(\xi) \hbar \omega(f) \delta[\varepsilon(\xi') - \varepsilon(\xi) \pm \hbar \omega(f)] \delta(\xi' - \xi \mp f) \{N(f)\}^{\pm 1}. \quad (4)$$

5. If we apply to a metal an external electrical field, the equilibrium of the electron gas is violated. In fact, since in an external field F there is no stationary state of an electron, the electrons must accelerate continuously according to the law $\xi = eF/\hbar$, already encountered. On the other hand, the thermal lattice vibrations will scatter electrons both with respect to their direction and their energy. There must thus be established some stationary process where the additional momentum in the direction of the field, produced by the field

† It is sufficient to take the coefficient of proportionality $A(\xi, f)$ to depend only on ξ and f , since ξ' can be eliminated from the condition $\xi \pm \xi' + f = 0$.

per unit time, is completely scattered by the lattice and the total extra kinetic energy which the electrons obtained, being accelerated in the field, is transferred to the thermal vibrations. It is natural to expect that in that case the electron gas can no longer be described by the well-known fermion distribution function. One must expect this *a priori* because one does not obtain any current from the fermion distribution function—since all directions for the velocity in it are equally probable. One must thus find the electronic distribution function in an external electrical field.

We can write the following equation for the distribution function for the electrons in a field:

$$d\tau \frac{dn(\xi)}{dt} = n_1 - n_2, \quad (5)$$

where $dn(\xi)/dt$ is the change of $n(\xi)$ per unit time, n_1 is the number of electrons entering the given cell in ξ -space per unit time due to collisions with the lattice, and n_2 the number of electrons leaving it due to the same cause.

We can write the derivative $dn(\xi)/dt$ in more detail as follows

$$\frac{dn(\xi)}{dt} = \frac{\partial n(\xi)}{\partial t} + \left(\frac{\partial n(\xi)}{\partial \xi} \cdot \frac{d\xi}{dt} \right).$$

However, $d\xi/dt$ is equal to eF/\hbar in an electrical field. We can thus rewrite the left hand side of the equation which determines n in the form

$$\frac{\partial n}{\partial t} + \left(\frac{\partial n}{\partial \xi} \cdot \frac{eF}{\hbar} \right).$$

We must now find the difference $n_1 - n_2$, that is, the difference between the total number of electrons, entering the state ξ per unit time and the number of electrons, leaving it.

To do this we consider first of all apart from the state ξ only one more state, ξ' , and determine the balance of all transitions $\xi \rightarrow \xi'$ and their reverse $\xi' \rightarrow \xi$. We determined in the preceding section the probability $W_{\xi\xi'}^F$, for the transition of one electron from the state ξ into ξ' , without taking into account the fact that ξ' may turn out to be occupied by another electron so that the transition $\xi \rightarrow \xi'$ is forbidden by the Pauli principle. However, any theory of the metallic state must take this necessarily into account as at all experimentally accessible temperatures the electron gas is completely degenerate so that there are practically no free states inside the surface $\varepsilon = \varepsilon_0$.

Let us thus consider a state ξ . The possibility that an electron of given ξ makes a transition to a *free state* ξ' is equal to $W_{\xi\xi'}^F$ —which we have already evaluated. We must multiply it by the number of free states ξ' . The total number of states per unit volume is equal to the weight of ξ' , that is, $d\tau'$, and the number of occupied states is equal to $n(\xi') d\tau'$ and the number of unoccupied states to $[1 - n(\xi')] d\tau'$. Moreover, the transition probability is proportional to the number of occupied states ξ from which the transition can take place. This number is $n(\xi) d\tau$. Let us assume that the transition

$\xi \rightarrow \xi'$ involves the transfer of one quantum to the lattice. The probability for the transition $\xi \rightarrow \xi'$ is then clearly equal to

$$W_{\xi\xi}^- n(\xi) [1 - n(\xi')] d\tau d\tau'.$$

The transitions $\xi \rightarrow \xi'$ can, however, partly be compensated by the reverse ones: $\xi' \rightarrow \xi$. Since the original equation contains only the balance of all direct and reverse transitions, we must find the probability for the transition $\xi' \rightarrow \xi$ and subtract it from the probability for the transition $\xi \rightarrow \xi'$. To do this we must in the following first of all interchange the indices and arguments ξ and ξ' . Then, since $\xi \rightarrow \xi'$ corresponds to the transfer of one quantum by the electron, $\xi' \rightarrow \xi$ must correspond to the transfer of one quantum by the lattice. The probability obtains thus a different index. $W_{\xi\xi'}^-$, and $W_{\xi\xi}^+$ differ only by the factors $N(f) + 1$ and $N(f)$. Indeed, we must in the δ -functions occurring in $W_{\xi\xi}^+$ by definition take the lower signs, but also we must interchange the arguments ξ and ξ' . The arguments of the δ -function thus change sign under these operations, but as the δ -functions are even functions, they are not changed at all.

The total balance of all $\xi \rightarrow \xi'$ -transitions involving the transfer of one quantum by the electron can thus be written in the form

$$\{W_{\xi\xi'}^- n(\xi) [1 - n(\xi')] - W_{\xi'\xi}^+ n(\xi') [1 - n(\xi)]\} d\tau d\tau'.$$

Transitions $\xi \rightarrow \xi'$ which are such that the lattice gives up one quantum are clearly also possible. To evaluate the balance of these transitions also, it is clearly sufficient to interchange the + and - signs in the expression obtained above:

$$W_{\xi\xi}^+ n(\xi) [1 - n(\xi')] - W_{\xi'\xi}^- n(\xi') [1 - n(\xi)] d\tau d\tau'.$$

According to our earlier remarks, we can write $W_{\xi\xi}^-$ and $W_{\xi\xi}^+$ in the form

$$W_{\xi\xi'}^- = B^{(1)} [N(f) + 1], \quad W_{\xi'\xi}^+ = B^{(1)} N(f),$$

$$W_{\xi\xi}^+ = B^{(2)} N(f), \quad W_{\xi'\xi}^- = B^{(2)} [N(f) + 1],$$

where $B^{(1)}$ and $B^{(2)}$ differ in the signs in the arguments of the δ -functions. In fact, according to our definitions, in $B^{(1)}$ the upper and in $B^{(2)}$ the lower sign occurs.

We must now evaluate all states ξ' and also all possible f of the phonon which takes part in the transition. To do this we must integrate the expression for the balance of the transitions $\xi \rightleftharpoons \xi'$ over all ξ' , that is, over $d\tau'$ and over all f , that is, over $d\tau_f$. The expression obtained must be put equal to $(dn(\xi)/dt) d\tau$ with the opposite sign since we evaluated in fact the difference between the numbers of electrons leaving and entering the state, while $(dn/dt) d\tau$ is equal to the opposite expression.

Because of the δ -functions occurring in $B^{(1)}$ and $B^{(2)}$, four of the six integrations over $d\tau'$ and $d\tau_f$ can be performed at once, once we know the form of the functions $\varepsilon(\xi)$ and $\omega(f)$. Three of them, viz. those where we use the δ -functions of $\xi - \xi' \pm f$ can be performed explicitly—by simply substituting $\xi' = \xi \pm f$ in $n(\xi')$. As to the fourth one—the one taking conservation of

energy into account—we can say that it everywhere leads to a replacement of $\varepsilon(\xi')$ by $\varepsilon(\xi) \pm \hbar \omega(f)$, and, moreover, to a multiplication of the integrand by an unknown coefficient $a_1(a_2)$ depending on ξ and f .

We find finally for the unknown electron distribution function in a field the integro-differential equation

$$\begin{aligned} \frac{\partial n(\xi)}{\partial t} + \left(\frac{\partial n(\xi)}{\partial \xi} \cdot \frac{e \mathbf{F}(t)}{\hbar} \right) = & \int a_1 B^{(1)} \{n(\xi) [1 - n(\xi')] [N(f) + 1] \\ & - n(\xi') [1 - n(\xi)] N(f)\} d^3 f + \int a_2 B^{(2)} \{n(\xi) [1 - n(\xi')] N(f) \\ & - n(\xi') [1 - n(\xi)] [N(f) + 1]\} d^3 f, \end{aligned} \quad (5a)$$

where we must put in the first integral $\xi' = \xi + f$ and $\varepsilon' = \varepsilon - \hbar \omega$, and in the second integral $\xi' = \xi - f$ and $\varepsilon' = \varepsilon + \hbar \omega$. The integration is over all those f which can take part in transitions from the given state ξ , that is, which satisfy the equation $\varepsilon(\xi \pm f) - \varepsilon(\xi) \mp \hbar \omega(f) = 0$.

If we assume that the gas is in a state of statistical equilibrium $\partial n / \partial t = 0$, when $\mathbf{F} = 0$, equation (5) is identically satisfied by the fermion distribution for the electrons and the Planck distribution $N(f) = [e^{\hbar \omega(f)/kT} - 1]^{-1}$ for the phonons, as can easily be checked by simple algebra. The equilibrium between the electrons and the phonons requires the fermion form for the electron distribution function and the Planck form for the phonon distribution function.

In statistical equilibrium the total current vanishes, since the direct and the inverse directions of ξ are equally probable. If the metal is in an external electrical field, it is plausible to expect that not only the electron distribution function, but also the phonon distribution function will change due to collisions with the electrons with a distribution function which is no longer the fermion one. The integro-differential equation contains the $N(f)$ which is also unknown. We must construct another equation to determine $N(f)$. To do this we must require that the total number of phonons of given f must not change in time: $dN(f)/dt = 0$.

In an electrical field there must become established some well-defined equilibrium for the phonons such that the number of phonons of any state f which are absorbed per unit time by the lattice is exactly compensated by the number of phonons absorbed by the electrons per unit time. However, one must mention that the number of phonons of a given f can also change because of the interaction between different eigenvibrations without any participation of the electrons. However, estimating both effects, one can verify that the second source for phonon transitions is much less important. We shall neglect it in the following.

We now consider the total change in the number $N(f)$ due to collisions with the electrons. Let the absorption of one phonon by the lattice be accompanied by an electron transition $\xi \rightarrow \xi'$, $\varepsilon(\xi) \rightarrow \varepsilon(\xi')$. We have shown that the probability for such a process is equal to

$$W_{\xi\xi'}^- n(\xi) [1 - n(\xi')] d\tau d\tau',$$

or, in a more convenient notation,

$$B^{(1)} [N(\mathbf{f}) + 1] n(\xi) [1 - n(\xi')] d\tau d\tau'.$$

To find the final balance of all phonon transitions we must find the difference of the expressions obtained and integrate it over all initial and final states of the electron—over $d\tau$ and $d\tau'$. The integral must vanish:

$$B^{(1)} \{[N(\mathbf{f}) + 1] n(\xi) [1 - n(\xi')] - N(\mathbf{f}) n(\xi') [1 - n(\xi)] d\tau d\tau' = 0. \quad (5b)$$

Together with the earlier integral equation, it determines the two unknown distribution functions $N(\mathbf{f})$ and $n(\xi)$. We must again emphasise that for statistical equilibrium both functions must have the well-known form

$$n(\xi) = [e^{\{\varepsilon(\xi) - \mu\}/kT} + 1]^{-1}, \quad N(\mathbf{f}) = [e^{\hbar\omega(\mathbf{f})/kT} - 1]^{-1}.$$

6. The integral equation contains functions $\alpha(\mathbf{f}_1, \xi)$, $\omega(\mathbf{f})$ and $\varepsilon(\xi)$ which are different for different cases, and there is no physical sense in solving it exactly.

Therefore it is not impossible to evaluate the magnitude of the resistivity of the model of a metal which we used in a general form, but also to determine its temperature dependence in the whole temperature range where the crystalline lattice exists, that is, from the absolute zero until the melting temperature.

However, it is possible to find exactly the asymptotic behaviour of the temperature dependence in two limiting cases: we can determine the behaviour of the resistivity of a metal for $T \gg \Theta$ and $T \ll \Theta$, where Θ is the "characteristic" Debye temperature of the metal. The first case will be called the high-temperature case and the second the low-temperature case. Moreover, we can construct a qualitative, interpolating formula which describes the resistivity of a metal satisfactorily at $T \approx \Theta$.

Let us first consider the high-temperature case.

The condition $T \gg \Theta$ means that the energy of a lattice phonon is much less than kT . In fact, the maximum frequency ω_0 is of the order of magnitude $k\Theta/\hbar$. Since $kT/\hbar \gg k\Theta/\hbar$ by definition, we see that $kT/\hbar \gg \omega_0$, or $kT \gg \hbar\omega_0$, as we stated.

If we take for the phonon distribution function $N(\mathbf{f})$ the Planck form, we can expand it in a power series in $\hbar\omega(\mathbf{f})/kT$. If the temperature is sufficiently high, we can cut off the expansion after the first, classical, term $kT/\hbar\omega$. It follows that $N(\mathbf{f}) \gg 1$. The unknown distribution function $N(\mathbf{f})$ which is not of the Planck form occurs in the integral equation and there is no reason whatever to expect that in a weak electrical field the distribution function will change its order of magnitude. Therefore for $T \gg \Theta$ we have $N(\mathbf{f}) \gg 1$.

We can then, however, in the expression for the transition probability $W_{\xi\xi'}^-$, validly neglect 1 as compared to $N(\mathbf{f})$ in the factors $N(\mathbf{f}) + 1$ and put

$$W_{\xi\xi'}^\mp = W_{\xi\xi'}^\pm.$$

This simplification essentially changes the form of the integral equation. Indeed, putting $N(f) \approx N(f) + 1$, we can take it out from under the braces. Inside the braces remain two terms $n(\xi) n(\xi')$ with opposite sign which cancel one another and there only remains the difference $n(\xi) - n(\xi')$. The factors $1 - n(\xi)$ and $1 - n(\xi')$ which give the probability density for free states and which are typical for the Fermi statistics have thus disappeared. The total balance of electronic transitions $\xi \rightleftharpoons \xi'$ is now thus of the same form as if the probability for the direct and reverse transitions depended only on the number of electrons of the state from which the transition takes place and did not depend at all on the probability that the final state is already occupied.

The probability for the transition $\xi \rightarrow \xi'$ of one electron is thus simply equal to $W_{\xi\xi'}^\pm$, where we have already put $N(f) \approx N(f) + 1$.

The total energy interval where transitions are possible is the "smeared out" zone of the fermion distribution. Its order of magnitude is well known to be kT . However, at each collision the energy of the electron is changed by an amount $\hbar\omega$ which is appreciably less than kT (in the high-temperature case). The energy changes thus slowly even in that small region where it can change in practice. Thus, at high temperatures the energy of the electron is essentially conserved.

Let $\varepsilon(\xi)$ be its energy when there is no external field and $e\varphi$ its energy in the field (φ : the field potential). If we denote the total energy by E , we can then write $E = \varepsilon(\xi) + e\varphi$. Collisions with lattice phonons lead only to a redistribution of the electrons over their "momenta", but not over their energies.

This redistribution leads to a slow exchange between the electrons of the given value of the total energy E . This exchange will take place in configuration space. Indeed, if the electron changes its "momentum" ξ , its "kinetic" energy $\varepsilon(\xi)$ also changes. However, since the total energy remains equal to E , its potential energy $e\varphi(x, y, z)$ must change, that is, its position in configuration space.

One can express this in a different way. Let $n(\varepsilon)$ be the distribution function of the electrons. Since $\varepsilon = E - e\varphi$, this distribution function depends on the spatial co-ordinates: $n(\varepsilon) = n(E - e\varphi)$. The spatial density which is inhomogeneous with respect to φ must lead to a diffusion of the electrons in the direction of the lower spatial concentration.

To simplify the further discussions we assume the crystal to be cubic. Of course, this can not influence the temperature dependence and also makes the consideration of the diffusion current easier, because the diffusion coefficient becomes then a scalar while in general it is a second-order symmetrical tensor.

Furthermore, we are interested only in the exchange of electrons with a given value of E and the diffusion current must thus be averaged over all directions of ξ corresponding to the same value of $\varepsilon(\xi)$. After this operation the diffusion coefficient depends now only on ε . We denote it by $D(\varepsilon)$.

One can thus easily write down the diffusion current, pertaining to electrons with a given value of $\varepsilon(\xi)$. We must choose a phase cell $d\tau_\varepsilon$ including all states with a given ε . We multiply its volume by two since in each elementary cell there are still two states possible with opposite spin. Moreover, we must

multiply the volume of this phase cell by the geometrical gradient of $n(\varepsilon) = n(E - e\varphi)$ evaluated for constant energy (we have assumed that E is conserved) and by the diffusion coefficient $D(\varepsilon)$. The total current of electrons of given E and ε is thus equal to

$$-2d\tau_e D(\varepsilon) \text{grad } n(\varepsilon).$$

However, we can very easily evaluate the geometrical gradient of $n(\varepsilon)$; it is equal to

$$\frac{\partial n}{\partial \varepsilon} \nabla \varepsilon = \left[\frac{\partial n}{\partial \varepsilon} \nabla (E - e\varphi) \right]_{E=\text{const}} = -\frac{\partial n}{\partial \varepsilon} e \nabla \varphi = \frac{\partial n}{\partial \varepsilon} e F,$$

where F is the electrical field strength.

We obtain the density of the electrical current of the electron of given ε from the diffusion current, simply by multiplying by e . Moreover, the total current density j is obtained from the same expression by integrating over ε :

$$j = -2e^2 F \int D(\varepsilon) \frac{d\tau_e}{d\varepsilon} \frac{\partial n}{\partial \varepsilon} d\varepsilon.$$

At temperatures $T \gg \Theta$ the electron gas is still highly degenerate. Therefore $n(\varepsilon)$ has a steep discontinuity at $\varepsilon = \varepsilon_0$. Since the other functions such as $D(\varepsilon)$ and $d\tau_e/d\varepsilon$ change smoothly, we can take for $n(\varepsilon)$ the same form as at absolute zero; that is, $n(\varepsilon) = 1$ for $\varepsilon < \varepsilon_0$ and $n(\varepsilon) = 0$ for $\varepsilon > \varepsilon_0$. The smearing out of the distribution can be neglected since $kT \ll \varepsilon_0$. Therefore $\partial n/\partial \varepsilon$ is now δ -function-like function of $\varepsilon - \varepsilon_0$. Indeed, it satisfies the definition of a δ -function—with the opposite sign: $\partial n/\partial \varepsilon = 0$ for $\varepsilon \neq \varepsilon_0$, and

$$\begin{aligned} \int \frac{\partial n}{\partial \varepsilon} d\varepsilon &= n(\varepsilon) \Big|_0^\infty = -1, \quad \int f \frac{\partial n}{\partial \varepsilon} d\varepsilon = f(n) \Big|_0^\infty - \int n \frac{\partial f}{\partial \varepsilon} d\varepsilon \\ &= -f(0) - \int_0^{\varepsilon_0} \frac{\partial f}{\partial \varepsilon} d\varepsilon = -f(\varepsilon_0). \end{aligned}$$

In the last integral we took n out from under the integral sign and at the same time replaced the upper limit of the interval over which we integrate by ε_0 .

The total current density is thus equal to

$$j = 2e^2 F \left(\frac{d\tau_e}{d\varepsilon} \right)_{\varepsilon_0} D(\varepsilon_0).$$

Let us determine the temperature dependence of σ . As it is a purely mechanical quantity, $(d\tau_e/d\varepsilon)_\varepsilon$ does not depend on the temperature. We know from general considerations about diffusion that $D(\varepsilon)$ is of the order of $\frac{1}{2}v\lambda$, where v is the electron velocity averaged in some way or other and λ is the electron mean free path; $v(\varepsilon_0)$ is temperature independent.

The electron mean free path λ is by definition inversely proportional to the probability W of a collision with phonons per unit time. On the other hand, the probability for a collision is at high temperatures proportional to $N(f)$ and this, in turn, is proportional to T . The terms containing f can in no way change the temperature dependence of the transition probability. Indeed, in order to obtain the mean free path, one must average the transition probability over all f from zero to f_0 , defined by the condition $\hbar\omega(f_0) \approx k\Theta$. Such an operation cannot lead to an additional temperature dependence of W .

We find thus that λ is inversely proportional to the temperature. The conductivity of the metal, σ , which is by definition equal to j/E is thus also inversely proportional to the temperature and the resistivity R is directly proportional to T .

This law is exact, independent of the order of magnitude estimates made. The latter refer only to the numerical proportionality coefficients which are temperature-independent at sufficiently high temperatures. The law $R \sim T$ itself asymptotically exact provided only that $T \gg \Theta$.

We shall now consider in how far this law agrees with experimental data on metals. It is clear that we must take such metals that their conductivity can be studied in a broad temperature interval $T \gg \Theta$. Such metals with a high melting point are tungsten, tantalum, and molybdenum.

For tungsten we have ($\Theta \approx 380^\circ\text{K}$)†

$$T = 2500^\circ\text{K}, \quad R = 94.31, \quad \frac{R}{T} = 376.5;$$

$$T = 3500^\circ\text{K}, \quad R = 141.42, \quad \frac{R}{T} = 400.4.$$

For molybdenum ($\Theta = ?$)

$$T = 1800^\circ\text{K}, \quad R = 47, \quad \frac{R}{T} = 261;$$

$$T = 2800^\circ\text{K}, \quad R = 78.2, \quad \frac{R}{T} = 280.$$

For tantalum ($\Theta \approx 288^\circ\text{K}$)

$$T = 1800^\circ\text{K}, \quad R = 67.6, \quad \frac{R}{T} = 375;$$

$$T = 2800^\circ\text{K}, \quad R = 105.0, \quad \frac{R}{T} = 375.$$

The agreement for tantalum is thus already very good, even though T is only eight times higher than Θ .

It is of interest to estimate from the experimental data the number λ/a , that is, to find out how many atomic distance a are included in one free path

† The resistivity is expressed in arbitrary units.

of the electron. To do this, we first of all estimate σ . In it there occurs $d\tau_\epsilon/d\epsilon$. We can write the latter in the form $(d\tau_\epsilon/d\xi)(d\xi/d\epsilon)$. However, $d\epsilon/d\xi$ is nothing but $\hbar v$ where v is the electronic velocity (see section 3). We must now find the order of magnitude of the quantity $d\tau_\epsilon/d\xi$. Since our considerations have purely an order-of-magnitude character, we may assume that ϵ depends only on the magnitude, but not on the direction of ξ . In that case $d\tau_\epsilon$ is the volume between two infinitely close spheres in the ξ -space:

$$d\tau_\epsilon \approx \frac{4\pi \xi^2 d\xi}{(2\pi)^3},$$

and we have for the conductivity

$$\sigma \approx \frac{1}{3\pi^2} \frac{e^2}{\hbar} \xi^2 \lambda.$$

One can very easily estimate the magnitude of ξ from the condition that the electron gas is highly degenerate at all temperatures so that the phase volume occupied by it $\approx (4\pi/3) (\xi/2\pi)^3 \Delta V$ must be of the order of magnitude of the total number of electrons in the volume ΔV divided by two because of the two possible spin orientations. Assuming that the spatial density of the electrons $n = N/\Delta V$ is equal to unity divided by the elementary cube of volume a^3 , we find

$$\frac{\xi}{2\pi} \approx \left(\frac{3n}{8\pi} \right)^{1/3} \approx \frac{1}{a} \left(\frac{3}{8\pi} \right)^{1/3},$$

and for the conductivity σ :

$$\sigma = \frac{1}{3} \frac{e^2}{\hbar a} \left(\frac{\pi}{3} \right)^{2/3} \frac{\lambda}{a}$$

For most metals the conductivity is of the order of $10^6 \Omega^{-1} \text{ cm}^{-1}$ or 10^{17} sec and $a \approx 3 \times 10^{-8} \text{ cm}$; the free path comes thus out to be of the order of some hundreds of atomic distances.

It is also of interest to follow the electrical conductivity of metals up to the melting point to compare the properties of the liquid and the solid phases at the same temperature. This is therefore interesting, because the solid phase is much more ordered than the liquid one and a comparison enables us to elucidate the discontinuity in the conductivity when the ordering undergoes a discontinuity. The usual experimental data about the discontinuity of the electrical conductivity at the melting point are not altogether useful since the specific volume undergoes a discontinuous change together with the ordering. However, by extrapolation one can refer the resistivity of a liquid metal to the equivalent isotropic dilatation and thus compare the data referring to the same volume and amount of material which are in different phases. We obtained the following numbers:

Metal	Ag	Al	Au	Be	Cd	Cu	K	Li	Na	Sb	Sn	Zn
$R_{\text{liq}}/R_{\text{sol}}$	1.66	1.7	1.59	0.352	1.39	1.82	1.64	1.92	0.875(?)	0.75	1.78	1.48

The resistivities of Bi, Sb, and Na(?) decrease on melting. One must ascribe this to the fact that the motion of the electron is influenced not only by the order, but also the conditions for motion in the metal itself.

We now turn to the low-temperature case, $T \ll \Theta$. We first of all elucidate which of the thermal lattice vibrations can play an important role in this case. Although formally the whole acoustic spectrum up to $\omega_0 \approx k\Theta/\hbar$ takes part in the phonon transitions, the "maximum frequencies", that is, those vibrations for which $\hbar\omega \gg kT$ will practically not at all be excited. This follows from the shape of the Planck distribution, $N(f) = [e^{\hbar\omega(f)/kT} - 1]^{-1}$: when $\hbar\omega(f) \gg kT$, $N(f)$ is practically equal to zero.

Therefore only those phonons for which $\hbar\omega(f) \approx kT$ will turn out to be important for the transitions. Since, on the other hand, kT is the width of the smearing out of the fermion distribution, each collision will thus change the energy of the electrons over the whole possible range of its change.

One can, on the other hand, easily show that the wave number changes relatively little—the less, the lower the temperature. Indeed, at low temperatures—as we just mentioned—phonons of low frequency, that is, long wavelengths, are excited. However, long deformation waves of the lattice correspond to a limiting transition to an elastic continuum as the discrete structure does no longer play a role. It is well known that we can then put the frequency $\omega(f)$ proportional to the wave vector f with a coefficient of proportionality which depends only weakly on f . At low temperatures only phonons of small f are thus excited. On the other hand, the electron wave number ξ remains by definition of the order unity (inverse lattice constant). Since each collision changes ξ by $\pm f$, ξ must change little both in magnitude and in direction.

It is thus valid to speak about a slow redistribution of the electrons among wave numbers. One can describe this as follows: ξ varies little in magnitude remaining in the vicinity of the Fermi surface. (Electrons with smaller ξ can not undergo transitions because of the Pauli principle.) Each collision changes ξ inappreciably. There occurs thus something like a diffusion of phase points over the Fermi surface.

If we apply an external electrical field to the metal, there will be a preferred direction of motion for the electrons. Phase points collect more densely near one part of the Fermi surface than would normally happen without a field. For a stationary process electron-phonon collisions per unit time will unbalance the total electron momentum by the same amount as the field is able to balance it during the same interval. In other words, the "diffusion current" will at any point of the energy surface exactly compensate the current of phase points under the action of the field. For a given field, the total current will reach a value which is the larger the slower the diffusion on the energy surface dissipates any accumulation of points, equivalent to a current. The velocity of the dissipation of electrons is thus a measure for the electrical resistivity of a metal. The first, in turn, is measured by the diffusion of the electrons on the energy surface, and the total diffusion current is proportional to the diffusion constant D which is multiplied by the gradient of the "surface concentration" of the electrons. In a weak external field the current of the

phase points under the action of a field can be put proportional to the field—just as any change in momentum. The gradient of the “surface” concentration is nearly exactly the same as the current density in the given direction. It measures in fact the difference in the density of phase points in the given direction in a field and without a field. We find thus that the phase point current at a given point of the energy surface which is proportional to the external field is because of the stationary condition equal to the diffusion current flowing in the opposite direction which is equal to the known diffusion coefficient multiplied by the gradient of the surface concentration of the phase points; it is thus proportional to the total current. The diffusion coefficient D connects thus indeed the field and the current and one may consider it to be a measure of the electrical resistivity of the metal. The proportionality coefficient for the relation between the field and the phase point current in the given direction, on the one hand, and the proportionality coefficient for the relation between the current density and the surface concentration gradient, on the other hand, are purely mechanical quantities, and the temperature dependence of the resistivity which interests us is thus determined by the diffusion coefficient we are looking for.

The general expression for the diffusion coefficient is $D \approx v \lambda$ where v is an average velocity of the points and λ a mean free path. We must understand here by v the average velocity of the phase points on the energy surface and by λ their mean free path on the energy surface. As the position of a phase point of an electron changes due to collisions with phonons we must understand by mean free path its change of momentum in one collision, that is, $\Delta \xi$. In this form it corresponds exactly to the usual definition of a free path as the distance travelled between two collisions. We must understand by the average velocity of the phase points the distance travelled by them per unit time on the energy surface. This distance is clearly equal to the same mean free path multiplied by the number of collisions per unit time. We have already determined the number of collisions and denoted it by W . We have thus $D \approx W (\Delta \xi)^2$.

We have mentioned that at low temperatures phonons are excited with small f . However, for small f the probability W is proportional to the energy density of acoustic vibrations of a given f . Furthermore, this energy density occurs in the transition probability and the δ -function-like dependence on $\varepsilon(\xi') - \varepsilon(\xi) \pm \hbar \omega(f)$ which expresses the law of conservation of energy in a collision. We must substitute here for ξ' , $\xi \pm f$ and obtain then the result that for electrons of a given ξ phonons of arbitrary f can not take part but only those for which f lies close to the surface $\varepsilon(\xi \mp f) - \varepsilon(\xi) \pm \hbar \omega(f) = 0$. However, near this surface not all vibrations are excited, but practically only those whose frequencies are of the order of or less than kT/\hbar . Larger frequencies are according to the Planck distribution not excited at all. We must thus take into account only those phonons which lie close to the energy surface and within a region determined by the condition $\hbar \omega(f) \approx kT$. Their number is thus proportional to the total of closed curves $\hbar \omega(f) \approx kT$ on the surfaces $\varepsilon(\xi \mp f) - \varepsilon(\xi) \pm \hbar \omega(f) = 0$. However, since for sufficiently low temperatures

only low frequencies and small f are excited, one may consider these curves to be plane circles with an area proportional to f^2 . The energy density of all phonons which can take part in the transitions is thus proportional to $\hbar \omega(f) f^2 \sim f^3$. We find thus that $W \sim f^3$. Moreover, D still contains a factor $(\Delta \xi)^2$ which is proportional to f^2 since ξ changes by f in a collision. We find thus $D \sim f^5$. Since, however, $\hbar \omega(f) \approx kT$ and $\omega \approx f$, we see that the resistivity of a metal must be proportional to T^5 . As in the case of the high-temperature law $R \sim T$, this law is true independent of the estimates we made of the order of magnitude of the various quantities which only refer to the coefficients of proportionality.

7. The $R \sim T^5$ -law is asymptotically exact for the chosen model of a metal for $T \ll \Theta$. For intermediate temperatures one cannot find a general law since it has no sense to solve the transport equation in its general form. All the same it is possible to construct a qualitative interpolation formula from which we obtain both the $R \sim T^5$ -law for low temperatures and the $R \sim T$ -law for high temperatures. It describes the intermediate region $T \approx \Theta$ in satisfactory agreement with experiments. We shall derive this formula in the present section.

We have already mentioned that we must use as a measure for the electrical resistivity the change per unit time in the number of electrons of a given direction under the action of the field which in a stationary state is exactly compensated by the phonon collisions. We have already mentioned that the change in the number of electron of given ξ under the action of the field is $(e/\hbar)(F[\partial n/\partial \xi])$. To evaluate the change in the number of electrons of only the given direction of ξ , we must multiply by $d\varepsilon$ and integrate over all energies corresponding to this direction. The change in this number n under the action of the field is obtained by integrating over the energies of both intervals giving the change in the number $n(\xi)$ of given *magnitude and direction*. It is in general not possible to perform this integration since in the equation for the unknown functions n and N there occur $A(\xi, f)$, $\varepsilon(\xi)$, and $\omega(f)$ which are different for different cases. However, we can at least make an estimate of the temperature dependence of the resistivity by making one simplifying, qualitative assumption.

This assumption is the following one. For $T \approx \Theta$ the energy changes are still all by an amount of the order of kT , that is, over the whole of the possible range of change in ε . As to the changes in momentum, they proceed still rather slowly. In an electrical field there will therefore be established an approximate statistical equilibrium with a well defined chemical potential μ for electrons with the same direction of their momentum. Electrons with different directions of ξ can thus be characterised by their own Fermion distribution functions where only μ depends on the direction.

We impose on this change the condition $dN(f)/dt = 0$, that is, that the lattice phonon must remain in a stationary state. This condition is expressed analytically by equation (5b) and we only need to put definite values into it for the present case.

In equation (5b) we have integrations over $d\tau$ and $d\tau'$ for given f . The second integration we can perform immediately using the fact that $\xi' = \xi \pm f$. However, the transition must also satisfy the energy conservation law so that not all electrons with arbitrary ξ can take part, but only those for which ξ satisfies the relation $\varepsilon(\xi \pm f) = \varepsilon(\xi) \mp \hbar \omega(f)$. We can express this geometrically as follows: the vectors ξ and $\xi \pm f$ are connected with the same energy surface, but as f is given, ξ can lie only on a certain well-defined curve satisfying the condition that $\xi \pm f$ lies on the same surface.

Finally, the integration in equation (5b) leads to an averaging over all ξ corresponding to the points of this curve. We must also integrate this result over the absolute value of ξ , or, more conveniently, over $d\varepsilon$. One shows, however, easily that one can validly interchange these two operations; the energy changes are, indeed, not taking place over the whole range, but only in the vicinity of the Fermi surface, that is, in a small section of order kT .

Indeed, electrons inside the filled band can in general not undergo any transitions as soon as they are further removed from the Fermi surface than kT . In fact, each electron transition is accompanied by a change in energy of about $\hbar \omega \sim kT$ so that because of the Pauli principle it is simply impossible to make a transition from a phase cell which is further from the smeared out zone than kT . In general, there are no electrons outside the smeared out zone as follows from the shape of the fermion distribution

$$n = [e^{(\varepsilon - \mu)/kT} + 1]^{-1}.$$

Therefore, although the integration over $d\varepsilon$ formally is from 0 to ∞ , it gives a non-vanishing result only near ε_0 . In that case, the averaging over the curve on which ξ lies can be performed over the Fermi surface bearing in mind that for the subsequent integration over $d\varepsilon$ only the regions in its immediate vicinity are important. As ξ then changes in this integration practically very little, the coefficient $A(\xi)$ occurring in the transition probability, can validly be assumed to be constant: $A(\xi) = A(\xi)_0 (\varepsilon(\xi_0) = \varepsilon_0)$. The required equation (5b) can thus finally be written in the form

$$\overline{\int B^{(2)} \{n(\xi) [1 - n(\xi')] N(f) - n(\xi') [1 - n(\xi)] [N(f) + 1]\} d\varepsilon = 0},$$

where the bar over the integral indicates an average over the Fermi surface. The integration is over all energies, but $n(\xi)[1 - n(\xi')]$ vanishes for states further from the Fermi surface than kT .

If the metal is not in an external electrical field, $n(\varepsilon)$ is the Fermi distribution with a constant μ which is independent of the direction. The form of all distribution functions is then known. In an external field, some kind of approximate equilibrium of the electron gas is established with a distribution function which has again the fermion form but with μ dependent on the *direction* of the momentum. One must then clearly also change the phonon distribution function $N(f)$. Its change is not arbitrary, but must satisfy the already mentioned condition $dN(f)/dt = 0$ which expresses equation (5b), only written in a different form. This equation enables us to express the change in $N(f)$ in terms of the change in $n(\xi)$.

We now make the necessary calculations. Let $(n \equiv n(\xi), n' \equiv n(\xi'))$

$$\int A \hbar \omega(f) \{N(f) n(1 - n') - [N(f) + 1] n'(1 - n)\} d\varepsilon = 0.$$

If the external field is weak, the deviations of n and N from their equilibrium values will accordingly be small. Denoting them by δn and δN , we retain the notation $n(\xi)$ and $N(f)$ for the "zero-order" equilibrium distribution functions. Bearing in mind that these equilibrium functions reduce the integrand identically to zero we can re-write the integrand up to first order in small quantities in the following form

$$\delta N(n - n') + \delta n(N + n') + \delta n'(n - N + 1).$$

Here occur simultaneously δn and $\delta n'$, that is, the deviations from the equilibrium values of the electron distribution for two values of ξ . We have already mentioned that this is reflected by the fact that each direction has its own μ . However, the quantity μ itself is in this case not important—we are only interested in the relative change in μ when we change from the direction of ξ to that of ξ' . It is thus unnecessary because of this to introduce simultaneously into our considerations δn and $\delta n'$ —assuming n' to be given we can put $\delta n' = 0$ and introduce only the relative change in n which is equal to δn . We denote the difference

$$\mu(\xi) - \mu(\xi') \text{ by } \delta\mu.$$

We have thus only left in the integral

$$\delta N(n - n') + \delta n(N + n');$$

we can easily express n' in terms of n , using the fact that for equilibrium dN/dt must vanish. We get thus

$$n' = \frac{Nn}{N + 1 - n}.$$

Moreover, for small $\delta\mu$, $\delta n = (\partial n / \partial \mu) \delta\mu$. Since n depends only on the difference $\varepsilon - \mu$, we have $\partial n / \partial \mu = -\partial n / \partial \varepsilon$. We get thus for the integral

$$\int A \hbar \omega(f) \left[\delta N(n - n') - \frac{\partial n}{\partial \varepsilon} \delta\mu (N + n') \right] d\varepsilon = \frac{dN}{dt} = 0.$$

It is convenient to take n instead of ε as integration variable and integrate clearly, from 0 to 1. Moreover, we have already mentioned that we shall neglect the dependence of A on ξ . In the second term we have $(\partial n / \partial \varepsilon) d\varepsilon = dn$, and in the first term we put $d\varepsilon = kT dn/n(1 - n)$ as follows easily from the known form of n .

The integral thus reduces to the form

$$\int_0^1 A \hbar \omega(f) \left[\delta N \frac{(n - n') kT}{n(n - n')} - \delta\mu (N + n') \right] dn = 0.$$

However,

$$\frac{n - n'}{n(1 - n)} = \frac{1}{N + 1 - n},$$

$$N + n' = \frac{N(N + 1)}{N + 1 - n}.$$

We then get by integration over dn

$$\delta N - \frac{N(N + 1)}{kT} \delta \mu = 0.$$

Now, only $\delta \mu$ depends on the direction of the transition. We must average it over the curve on the energy surface on which ξ lies. We finally get

$$\delta N = \frac{N(N + 1)}{kT} \overline{\delta \mu}.$$

Knowing δN we can now easily evaluate the change in the number of electrons with a given direction of ξ :

$$\int A \hbar \omega(f) [n(1 - n')(N + 1) - n'(1 - n)N] d^3 f d\varepsilon$$

$$+ \int A \hbar \omega(f) [n(1 - n'')N - n''(1 - n)(N + 1)] d^3 f d\varepsilon.$$

We can chose n' and n'' to correspond to the same direction but to different energies: ξ' corresponds to an energy $\varepsilon' = \varepsilon - \hbar \omega$, and ξ'' to an energy $\varepsilon'' = \varepsilon + \hbar \omega$. The integration over $d f$ takes into consideration only those f satisfying the equation $\varepsilon(\xi \pm f) - \varepsilon(\xi) \mp \hbar \omega(f) = 0$.

It is clear that with such a choice of ξ' and ξ'' we have $\delta n' = \delta n''$. Using the same arguments as before we can put them equal to zero since they must have the same μ . After that, the integrand is of the form

$$A \hbar \omega [\delta n (2N + 1 - n' - n'') + \delta N (2n - n' - n'')].$$

Here n , n' , n'' , and N are the known equilibrium distribution functions, and thus

$$n' = \frac{n(N + 1)}{n + N}, \quad n'' = \frac{Nn}{N + 1 - n}.$$

As before, we replace δn by $(\partial n / \partial \mu) \delta \mu = -(\partial n / \partial \varepsilon) \delta \mu$ and $\delta \varepsilon$ by $kT dn / n(1 - n)$. We then express δN in terms of $\overline{\delta \mu}$ and we get finally for the integral over $d\varepsilon$ (changing to n as integration variable):

$$A \hbar \omega \int_0^1 dn \left\{ (2N + 1 - n' - n'') \delta \mu + (N + 1) N \frac{2n - n' - n''}{n(n - 1)} \overline{\delta \mu} \right\}.$$

However,

$$\frac{2n - n' - n''}{n(n - 1)} = \frac{1}{N + 1 - n} - \frac{1}{N + n},$$

and

$$2N + 1 - n' - n'' = N(N + 1) \left[\frac{1}{N + n} - \frac{1}{N + 1 - n} \right].$$

We thus get from the integral

$$A \hbar \omega (\delta\mu - \overline{\delta\mu}) N(N + 1) \ln \frac{N + 1}{N} = A \hbar \omega (\delta\mu - \overline{\delta\mu}) N(N + 1) \frac{\hbar \omega}{kT}.$$

We must still integrate this result over all f allowed for the transition. We must, however, first find the value of $\delta\mu - \overline{\delta\mu}$.

We mentioned that $\delta\mu$ is the difference $\mu(\xi) - \mu(\xi + f)$, that is, a quantity determined by ξ and f . For given f it can change only in such a way the ξ is displaced over the known curve on the energy surface; $\overline{\delta\mu}$ is the result of averaging $\delta\mu$ over this curve. We must expand the difference $\delta\mu - \overline{\delta\mu}$ for small f in a power series in f . The zeroth term of the expansion is then by definition of $\delta\mu = \mu(\xi) - \mu(\xi + f)$ identically equal to zero, as $\delta\mu$ vanishes for $f = 0$. Furthermore, there must be terms in the expansion terms linear in the components of f and terms quadratic in them, to which we must restrict the expansion.

Knowing $\delta\mu - \overline{\delta\mu}$ we can integrate over f . We have already mentioned that the integral is taken over a part of the energy surface which is limited by the condition $\hbar \omega(f) \approx k \Theta$. Bearing in mind the qualitative character of our calculations we may assume that part of the energy surface to be a plane. If we first integrate over the angles, the linear term in the expansion of $\delta\mu - \overline{\delta\mu}$ clearly vanishes and there remains the quadratic term, proportional to f^2 . We still had from the preceding integration over dn a factor $(\hbar \omega)^2/kT$ which is proportional to f^2 and an element of the plane surface is proportional to $f df$. If we replace $\hbar \omega(f)/kT$ by x (bearing in mind that purely qualitatively, $f \sim \omega(f)$) the temperature dependence of the integral is finally reduced to the factor

$$T^5 \int_0^{e/T} \frac{e^x x^5 dx}{(e^x - 1)^2}. \quad (6)$$

We remember that μ entered the calculations as an independent variable.

It follows from equation (5a) that the whole of the expression obtains is equal to

$$- \int \left(\frac{\partial n}{\partial \xi} \cdot \frac{eF}{\hbar} \right) d\varepsilon$$

because of the stationarity condition $dn/dt = 0$. We can easily evaluate this last expression; it is equal to

$$\int \frac{\partial n}{\partial \varepsilon} \left(\frac{\partial \varepsilon}{\partial \xi} \cdot \frac{eF}{\hbar} \right) d\varepsilon.$$

For n we can substitute its equilibrium, Fermi expression since $\delta\mu$ is small compared to μ for a weak field (on the right-hand side terms without δn cancel

identically) and $dn/d\varepsilon$ is the already well-known δ -function-like function: $-\delta(\varepsilon - \varepsilon_0)$. Since $\partial\varepsilon/\partial\xi$ changes relatively slowly, the whole integral is equal to

$$-\left(\left(\frac{\partial\varepsilon}{\partial\xi}\right)_{\varepsilon=\varepsilon_0} \cdot \frac{eF}{\hbar}\right).$$

However, $\partial\varepsilon/\partial\xi$ is nothing but $\hbar v$, where v is the electron velocity. The whole expression thus reduces finally to $v_0 eF$. This is, however, nothing but the change in the number of electrons of a given direction under the action of the field. It is proportional to the field. On the other hand, this change is compensated by the "diffusion" current of the electrons in ξ -space. Formally putting $F = jR$, where j is the current density, and substituting this into the equation, we see that the only quantity which may lead to a temperature dependence of the conductivity is again expression (6).

This formula agrees in the limits $T \ll \Theta$ and $T \gg \Theta$ with the results, obtained earlier. Indeed, at sufficiently low temperatures we can replace the upper limit Θ/T by infinity and the integral becomes temperature-independent and equal to 124.4. We obtain the already known result $R \sim T^5$. If, on the other hand, $T \gg \Theta$ we can validly expand the exponential in the integral in a power series and limit ourselves to the first term. The complete expression then becomes $\frac{1}{4}\Theta^4 T$, i.e. $R \sim T$ which we had derived also. Therefore, this formula can without any simplifications be considered to be a qualitative, interpolation formula for the temperature behaviour of the resistivity. It agrees satisfactorily with experimental data for $T \sim \Theta$.

To estimate the degree of approximation given by it we compare it with another interpolation Debye formula for the specific heat of solids. In fact, we compare the value Θ_{el} of Θ obtained from the electrical properties of a metal with the value Θ_{th} of Θ which is substituted into the Debye formula. One can most simply estimate Θ_{el} by comparing the resistivity at high and at low temperatures, R_1 and R_2 :

$$\frac{R_1}{R_2} = 500 \left(\frac{T_1}{\Theta} \right)^4 \frac{T_1}{T_2}. \quad (7)$$

Substituting the experimental data of R we get the following table:

Element	Na	K	Cu	Au	Cd	Pb	Na	Pt
Θ_{th}	159	99.5	315	190	168	88	310	225
Θ_{el}	212	212	357	176	131	129	375	193

Our model thus enables us to determine the temperature dependence of the resistivity of a metal in reasonable agreement with experiments. Apart from the temperature, the external pressure is known also to influence the resistivity. It is often stated that external pressure has a very considerable influence in this sense, according to a non-linear law at relatively very small

compressions. Caesium is an example, the resistivity of which initially decreases with increasing pressure (up to 5000 kg/cm²) and then again begins to increase; other examples are the other alkali metals Rb and K for which there is also a tendency to a bending in the $R(p)$ -curve.

Indeed, the resistivity of a metal deviates from a linear law by amounts of the same order as the compressibility—K, Rb and Cs are just the metals which have an anomalously large compressibility—so that experimentally attained pressures which compel us to think about a non-linear behaviour of $R(p)$ correspond to compressions of the order of 20–30 per cent.

8. Impurities and inclusions have a very large influence on the conditions of metals. We shall understand by impurities here not crystalline inclusions, but a small amount of a foreign element in the form of a solid solution. The lattice of the main metal must, of course, lose its strict periodicity in as far as the impurity atoms distort it. In this sense, small local “defects” of the lattice are completely analogous to impurities.

If the relative amount of impurities is very small, we can consider the electron motion in the crystal in first approximation to take place in a periodic field upon which is imposed, a small spatial periodic perturbation. The perturbing energy does not contain the time explicitly and the collisions with the impurity atoms will therefore not involve energy transfer. The impurities scatter the electrons thus only into a different direction, similar to what is done by the thermal vibrations at high temperatures. We are thus justified to take over for this case our earlier calculations with the only difference that the mean free path occurring in the final formula is now independent of the energy density of the acoustic vibrations. The latter was proportional to the absolute temperature which led to the relation $R \sim T$. The mean free path caused by the impurities, on the other hand, is temperature-independent. To a first approximation one can assume qualitatively that the scattering of the electrons by impurities is simply superimposed upon the scattering by the thermal lattice vibrations—and the temperature dependence of the resistivity of a dilute solid solution is the same as that of a pure metal. Matthiesen was the first to express this rule. This refers only to very dilute (“infinitely dilute”) solutions as one can select alloys with relative amounts of the components very close to one another for which the resistivity is practically constant over a very wide temperature range.

Impurities, and also other lattice defects (deformations, dislocations), lead to the phenomenon of the so-called residual resistivity which consists in the fact that the resistivity of all metals in practice does tend not to zero as $T \rightarrow 0$ but to a finite value—the so-called residual resistivity. The impurities also change the form of the electron distribution function (since the thermal vibrations change it arbitrarily little as $T \rightarrow 0$), and one can not split the resistivity of a metal into a part caused by the thermal vibrations and one caused by the impurities. This effect makes it extremely difficult to check experimentally the $R \sim T^5$ -law or, in other words, to compare our model of a metal with actually existing metals.

A peculiar form of a solid solution is a regular alternation of the atoms of the components of an alloy in a lattice when their relative numbers are known (for instance, $\text{Cu} : \text{Au} = 1 : 1$). This structure, or intermetallic compound, is called a "superstructure". It produces again a strict periodicity in the lattice. One must, of course, expect that the superstructure is strongly reflected in the resistivity of the alloy, which will be approximately of the same order of magnitude as the resistivity of a pure metal. This has also been observed for all clearly established cases of superstructure. However, at high temperatures the superstructure contains many irregularities which later freeze in and lead to a residual resistance of a higher order of magnitude than the one of the pure components of the alloy.

9. We mentioned in section 1 that the theory developed here has a serious defect in that it does not take into account the electron-electron interactions. We can, however, consider at least what these would lead to if their order of magnitude were such that the electron distribution function when there is no field is not appreciably changed. We have already stated that there are no reasons whatever for assuming these interactions to be small in metal, but in transport processes effects due to perturbations with energies much less than kT may also be important.

First of all, the free electron gas must have a zero resistivity, as the total momentum of the electrons is not changed in each elementary "collision" and the collision can therefore not change the total momentum of the whole of the gas. One would think that ξ would take the place of the momentum of the electrons in the crystal lattice and the electron-electron collisions could also not lead to an additional resistance. However, in actual fact ξ is for each electron defined only modulus 2π and its sum could thus change for the two electrons by 2π so that each of the relevant ξ could change by less than 2π . The velocities of the electrons which were defined in section 2 as functions of period 2π change thus also without conserving their sum. Hence, electron-electron collisions in the lattice change the total current.

We shall estimate how the probability of such an effect depends on the temperature. First of all, we note that effective collisions can not take place with electrons of all velocities, but only with those whose energy lies in the smeared-out region of the Fermi distribution. Indeed, electrons inside the fully occupied region can only interchange their positions and this means that there is no effect. Electrons from the interior and the smeared-out region can also merely interchange positions—because of the law of conservation of momentum the electron from the interior region can only go to a position of the other electron and nowhere else, since the other electron can only take up the former's position because of the Pauli principle. Outside the smeared-out region there are in general no electrons, at all, and collisions are therefore only possible only in the smeared-out region.

The collision probability is then first of all proportional to the square of the number of electrons in the smeared-out region, and thus, since that number is proportional to the width, kT , of the region, to the square of kT . Secondly,

one of the electrons must necessarily fall after the collision into the smeared-out region: it cannot fall into the interior region because of the Pauli principle and not into the exterior region because in that case its partner would—due to the energy conservation law—fall into the interior region which is again forbidden. The collision probability must thus once more be proportional to the width of the region—and the final effect caused by electron-electron collisions will thus be $\sim T^3$ up to very high temperatures.

Let us compare the resistivity caused by the lattice with that caused by the electron-electron interactions; this can, of course, merely be done very approximately since it is impossible to separate them physically. The two become of the same order of magnitude at a temperature τ where the Fermi distribution breaks down as the mean free path of the electrons in that case is of the order of magnitude of the atomic spacing. In fact when the temperature reaches the order of magnitude of electronic temperatures, it must be possible from simple dimensional considerations to construct expressions for both resistivities from the same universal constants. To compare the two effects at low temperatures, we use equation (7), removing from it the numerical constant and substituting for R_2 the resistivity at the above-mentioned limit $T = \tau$.

The resistivities caused by electron-electron collisions vary in the ratio $(\tau/T_1)^3$, and bearing in mind that $R_{2el} \sim R_2$, we see that the resistivities again become of the same order of magnitude at low temperatures, if

$$T_1 \sim \frac{\Theta^2}{\tau}.$$

The Debye temperature Θ is, however, usually small (it contains the nuclear mass in the denominator as $\hbar \omega_0 \sim k \Theta$, and is of the order $(\alpha/M)^{1/2}$, where α is an elastic constant and M the nuclear mass). On the other hand τ is very large, larger than 10,000°K. The two resistivities therefore become of the same order of magnitude at a relatively low temperature T_1 . It is impossible to say anything more definitely, as the equation $R_{el} \sim T^3$ contains an unknown numerical coefficient.

APPENDIX B

PAPERS NOT INCLUDED IN THE COLLECTED PAPERS

Apart from the papers which are contained in the main body of this Collection, there are a number of papers which are either completely out-of-date or contain slight errors. As they are sometimes quoted, we are giving a list of them here, and also give a brief abstract of them. We have excluded, though, various contributions made by Landau to conferences, abstracts of which appeared in the *J. Phys. U.S.S.R.* 4, 277, 278, 279, 380 (1941); 6, 225, 229 (1942); and in the *J. Exp. Theor. Physics*, 11, 195, 574 (1942), apart from one (*J. Phys. U.S.S.R.* 4, 284 (1941)).

On the Derivation of the Klein-Fock Equation

A derivation analogous to Schrödinger's derivation of the non-relativistic wave equation is given of the Klein-Gordon equation starting from a Hamiltonian equation.

D. Iwanenko und L. Landau, Zur Ableitung der Klein-Fockschen Gleichung, *Zs. Physik*, 40, 161 (1926).

Note on Quantum Statistics

A note objecting to Dirac's suggestion that for electrons only totally anti-symmetric wave functions should be allowed.

D. Iwanenko und L. Landau, Bemerkungen über Quantenstatistik, *Zs. Physik*, 42, 562 (1927).

The Connection between Wave Mechanics and Classical Mechanics

Various aspects of the connection between the Hamilton-Jacobi equation and the Schrödinger or the Klein-Gordon equation are discussed, and the similarity between wave- and geometrical optics, on the one hand, and wave- and classical mechanics, on the other hand, is stressed.

Д. Иваницко и Л. Ландау, Связь волновой механики с классической, *Журнал Русского Физико-Химического Общества, физический отдел*, 59, 255 (1927).

On the Theory of the Magnetic Electron

An attempt to introduce the fourth quantum number of the electron through considering antisymmetric tensors of different rank. A generalisation is made to many-electron systems.

D. Iwanenko and L. Landau, Zur Theorie des magnetischen Elektrons, *Zs. Physik*, 48, 340 (1928).

Universal Constants and Limiting Transitions

A general discussion of the question of dimensions and of the choice of independent units.

Г. Гамов, Д. Иваненко, и Л. Ландау, Мировые постоянные и предельный переход, *Журнал Русского Физико-Химического Общества, физический отдел*, 60, 13 (1928).

On the Spin-effect in the Many-body Problem

A discussion is given of possible generalisations of the Dirac equation to many-electron systems.

L. Landau, Zum spin-Effekt im Mehrkörperproblem, *Physik. Zs.* 30, 654 (1929).

On Ioffe's Theory of Electrical Discharge

A critique of a paper by Ioffe (*Physik. Zs. der Sowjetunion*, 1, 155 (1932)).

L. Landau und L. Rosenkewitsch, Über die Theorie des elektrischen Durchschlages von A. Joffe, *Zs. Physik*, 78, 847 (1932).

L. Landau und L. Rosenkewitsch, Über die Theorie des elektrischen Durchschlages von A. Joffe, *Physik. Zs. der Sowjetunion*, 2, 200 (1932).

On the Theory of Superconductivity

Thermodynamic arguments are applied to a theory where superconductivity is based upon a saturation current within any small volume element.

L. Landau, Zur Theorie der Supraleitfähigkeit, *Physik. Zs. der Sowjetunion*, 4, 43 (1933).

On Deviations from Ohm's Law for Semiconductors in Strong Electric fields

A consideration of possible deviations from Ohm's law in strong fields.†

L. Landau und A. Kompanejev, Über die Abweichung der Halbleiter vom Ohmschen Gesetz in starken elektrischen Feldern, *Physik. Zs. der Sowjetunion*, 6, 163 (1934).

Л. Ландау и А. Компанеев, Об отклонениях полупроводников от закона Ома в сильных электрических полях, *Журнал Экспериментальной и Теоретической Физики*, 5, 276 (1935).

† An erratum to this paper was published in *Physik. Zs. Sowjetunion*, 9, 479 (1936).

The Theory of Phase Transitions

Essentially an abstract of the two papers on phase transitions which appeared in the *Physik. Zs. der Sowjetunion*, 11, 26, 545 (1937); Collected Papers No. 29, p. 193.

L. Landau, The theory of phase transitions, *Nature*, 138, 840 (1936).

Theory of the Superfluidity of Helium II

Essentially an abstract of the classical paper on liquid helium which appeared in the *J. Physics U.S.S.R.* 5, 71, 1941; Collected Papers No. 46, p. 301.

L. Landau, Theory of the superfluidity of helium II, *Phys. Rev.* 60, 356 (1941).

Resonance Scattering of Neutrons on Light and Heavy Nuclei

A conference contribution discussing the applicability of the Breit-Wigner formula to resonance neutron scattering.

L. Landau, Resonance scattering of neutrons on light and heavy nuclei, *J. Physics U.S.S.R.* 4, 284 (1941).

Stability of Tangential Discontinuities in Compressible Fluids

An (erroneous) discussion of the stability of tangential discontinuities in compressible fluids.

L. Landau, Stability of tangential discontinuities in compressible fluids, *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS*, 44, 139 (1944).

On the Theory of The Viscosity of Helium II

Essentially an abstract of the two papers in the *J. Exp. Theor. Physics U.S.S.R.* 19, 637, 709 (1949); Collected Papers Nos. 69 and 70, pp. 494 and 511.

Л. Д. Ландау и И. М. Халатник, К теории вязкости гелия II, *Известия Академии Наук СССР, серия физическая*, 12, 216 (1948).

Concerning W. P. Allis' Criticism of My Paper on Coulomb Interactions in a Plasma

It is shown that Allis' criticism of Landau's paper on the transport equation of a plasma (*Phys. Z. Sowjet Un.* 10, 154 (1936); Collected Papers No. 24, p. 163) was based upon an incorrect expansion of the collision integral.

L. Landau, Concerning W. P. Allis' criticism of my paper on Coulomb interactions in a plasma, *Physical Review*, 77, 567 (1950).